Hilbert subspaces meet Chu spaces

Xavier MARY
MODAL'X
Université Paris-Ouest Nanterre-La Défense, France

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Introduction

From the 50's: generalizations of Hilbert spaces

- Quadratic spaces (Gross '64);
- Orthomodular spaces/Form Hilbert spaces (Piron '64, Araki '66, Keller '80, Soler '95);
- Bilinear modules over commutative rings (Knebusch '69);
- Hilbert C^*-modules (Kaplansly '53, Paschke '73, Rieffel '74);
- Hilbertian Operator spaces (Pisier '96).

Questions:

1. Global common theory ?
2. How to generalize Hilbert subspaces ?

E locally convex vector space (l.c.s.) on $\mathbb{R}$ or $\mathbb{C}$.

Hilbert subspaces (Aronszajn, Schwartz,...)

H is a Hilbert subspace of E ($H \hookrightarrow E$) if H is a Hilbert space, **continuously** embedded in E.

Reproducing Kernel Hilbert Spaces

RKHS are Hilbert subspaces of a product space $\mathbb{C}^X$.

- The Hardy space, the Bergman space are RKHS.
- Hilbert subspaces of the space of holomorphic functions, or distributions.

We need analogs of:

- Locally convex spaces and continuous maps
- Prehilbert spaces
  - symmetry
  - positivity
- Completeness

And also, what about reproducing kernels ?

Possible answer : **the Chu category** (Barr and Chu '79).

Chu categories (or Chu spaces) are linked with:

- Linear logic (Girard, Seely) / Dialectica (Hyland, de Paiva)
- Theoretical computer science (Pratt)
- Functional analysis: Pairs of TVS, "“two-norm”" spaces (Barr)
- Topology, algebraic geometry (Giuli, Tholen) / Homotophy (Egger)

I. What are Chu spaces ?

A glimpse of category theory

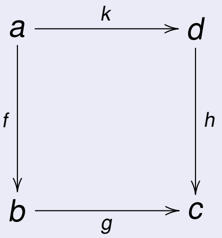
A category $\mathcal{C}$ is a collection

- Of objects $a, b, c, \ldots \in Obj(\mathcal{C})$
- Of arrows (morphisms) between objects $a \xrightarrow{f} b$

Plus a composition law $g \circ f : a \xrightarrow{f} b \xrightarrow{g} c$

Commutative diagrams

$g \circ f = h \circ k$



Monoidal Category

$\mathcal{C}$ is monoidal if

- It has a "tensor product" $\otimes$.
- It has an identity object I s.t. $X \otimes I \sim X \sim I \otimes X$.

Closed Monoidal category

- It has an "internal hom": $\mathcal{C}(X, Y)$ can be interpreted as an object in $\mathcal{C}$ (sometimes denoted by $X \multimap Y$).
- Adjointness conditions between the tensor product and the internal hom.

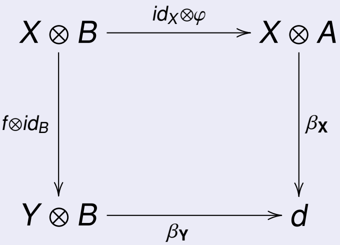
Examples are:

- Sets with product $\times$ of sets. $I = \{1\}$.
- Modules over a commutative ring R with classical tensor product $\otimes$. $I = R$.

Chu spaces (Barr and Chu '79)

$(\mathcal{C}, \otimes)$ = closed monoidal category, d = dualizing object. Category $Chu(\mathcal{C}, \otimes, d)$ has

- for <u>objects</u> triples $\mathbf{X} = (X, A, \beta : X \otimes A \to d)$ $(\mathbf{X} = (X \otimes A \xrightarrow{\beta} d))$
- for <u>arrows</u> $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $\mathbf{f} = (\mathrm{f}, \varphi)$, $f : X \to Y$, $\varphi : B \to A$ such that



Example 1: Boolean Chu spaces

Boolean Chu spaces in Logic, Computer science, Topology...

$Chu_2(Set) = Chu(Set, \times, \{0,1\})$

An object $\mathbf{X} = (X \times A \xrightarrow{\beta} \{0,1\})$ is equivalently defined as

- a FUNCTION $\beta : X \times A \to \{0,1\}$
- a RELATION $\mathcal{R}$ on $X \times A$, $\beta(x,a) = 1 \iff x\mathcal{R}a$
- a $0-1$-VALUED MATRIX

An arrow $\mathbf{f} = (f, \varphi) \in \mathbf{Chu}(\mathbf{X}, \mathbf{Y})$ is a pair a functions $f : X \to Y$, $\varphi : B \to A$ such that

$$\forall x \in X, \forall b \in B,\; f(x)\mathcal{R}_{\mathbf{Y}}b \iff x\mathcal{R}_{\mathbf{X}}\varphi(b)$$

Example 1: Boolean Chu spaces

Let $X = \{x, y, z\},\ A = \{a, b\}$. Then

$$\mathbf{X} = (X, A, \beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}) \text{ and } \mathbf{X}' = (X, A, \beta' = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix})$$

are Chu spaces.
$\mathbf{f} = (f, \varphi) : \mathbf{X}' \to \mathbf{X}$ defined by

$$f(x) = f(z) = z,\ f(y) = x,\ \ \varphi(a) = \varphi(b) = b$$

is a morphism of Chu spaces, but there is no morphism from $\mathbf{X}$ to $\mathbf{X}'$.

Example 2: kernel functions

Dualizing object $d = \mathbb{C}$ instead of $\{0,1\}$ of the previous slide.

$Chu_{\mathbb{C}}(Set) = Chu(Set, \times, \mathbb{C})$

An object $\mathbf{X} = (X \times A \overset{K}{\to} \mathbb{C})$ is a two-variable complex function $K(x,a)$ on $X \times A$.
An arrow $\mathbf{f} = (f, \varphi) \in \mathbf{Chu}\,(\mathbf{X}, \mathbf{Y})$ is a pair a functions $f : X \to Y$, $\varphi : B \to A$ such that

$$\forall x \in X, \forall b \in B,\ K_{\mathbf{Y}}\left(f\left(x\right), b\right) = K_{\mathbf{X}}\left(x, \varphi\left(a\right)\right)$$

Example: $\mathbf{X} = (\mathbb{R}, \mathbb{R}, K(x,y) = x^2 + iy)$.

Example 3: $R-$modules

R =commutative ring, d fixed $R-$module.

$Chu_d(R - mod) = Chu(R - mod, \otimes, d)$

An object $\mathbf{X} = (X \otimes A \xrightarrow{\beta} d)$ is a $R-$bilinear map $\beta : X \times A \to d$.
An arrow $\mathbf{f} = (f, \varphi) \in Chu(\mathbf{X}, \mathbf{Y})$ is a pair of "adjoint" maps $f : X \to Y$, $\varphi : B \to A$ such that

$$\forall x \in X, \forall b \in B,\ \beta_{\mathbf{Y}}(f(x), b) = \beta_{\mathbf{X}}(x, \varphi(b))$$

Remark: works also for non-commutative rings with involution / needs either bimodules or a generalized notion of Chu spaces.

Abelian Groups ($\longleftrightarrow \mathbb{Z}$–modules)

1. $R = d = \mathbb{Z}$.

$Chu_{\mathbb{Z}}(Ab) = Chu(Ab, \otimes, \mathbb{Z})$

An object $\mathbf{X} = (X \otimes A \xrightarrow{\beta} \mathbb{Z})$ is a pair of of Abelian groups X and A together with a bi-additive form $\beta : X \times A \to \mathbb{Z}$, *i.e.* an integral bilinear form. Such pairings occur for instance in the theory of Unimodular Lattices.

2. $R = \mathbb{Z}$, $d = \mathbb{R}/\mathbb{Z}$

$Chu_{\mathbb{R}/\mathbb{Z}}(Ab) = Chu(Ab, \otimes, \mathbb{R}/\mathbb{Z})$

An object $\mathbf{X} = (X \otimes A \xrightarrow{\beta} \mathbb{R}/\mathbb{Z})$ is a pair of Abelian groups X and A together with a bi-additive map $\beta : X \times A \to \mathbb{R}/\mathbb{Z}$.
Such pairings occur for instance in the theory of Elliptic Curves.

Example 4: real or complex vector spaces

$R = d = \mathbb{K}$ is a field.

$Chu_{\mathbb{K}}(\mathcal{V}) = Chu(\mathcal{V}, \otimes, \mathbb{K})$

An object $\mathbf{X} = (X \otimes A \xrightarrow{\beta} R)$ is a pair of vector spaces together with a bilinear form β on $X \times A$.

IT IS NOT a dual pair of vector spaces, for the latter verify a separation (non-degeneracy) property!

Fortunately... There exists a notion of nondegeneracy for Chu spaces!

Idea

Use the arrow β to identify A with arrows from X to d (the "internal hom").

Precisely, exists $(X \multimap d) \in Obj(\mathcal{C})$, **a special class of maps** from X to d and $(d \multimapinv A)$ (maps from A to d) such that the morphism $\beta : X \otimes A \to d$ defines:

- An arrow $\beta_r : A \longrightarrow (X \multimap d)$;
- An arrow ${}_l\beta : X \longrightarrow (d \multimapinv A)$.

Remark:

- These maps exists $\Longleftrightarrow$ the category is closed.
- For symmetric (or braided) tensor products, $(X \multimap d) \sim (d \multimapinv X)$;

Extensional Chu spaces

- $\mathbf{X} = (X \otimes A \xrightarrow{\beta} d)$ is extensional if $\beta_r : A \to (X \multimap d)$ is monic ("injective");
- $\mathbf{X} = (X \otimes A \xrightarrow{\beta} d)$ is coextensional if ${}_l\beta : X \to (d \multimapinv A)$ is monic.

Extensional-coextensional (biextensional) Chu spaces are denoted by

$$chu_d(\mathcal{C})$$

(instead of $Chu_d(\mathcal{C})$).

Example 1: Boolean Chu spaces

$Chu_2(Set) = Chu(Set, \times, \{0,1\})$

We interpret $X \multimap \{0,1\}$ as column vectors (of length X) and $\{0,1\} \multimapinv A$ as row vectors.

Let $\mathbf{X} = (X \times A \xrightarrow{\beta} \{0,1\})$ with β interpreted as a boolean matrix. The map β defines

- a map $\beta_r : A \to (X \multimap \{0,1\})$ that associates to a the COLUMN WITH LABEL a;
- a map ${}_l\beta : X \to (\{0,1\} \multimapinv A)$ that associates to x the ROW WITH LABEL X.

$\mathbf{X}$ is extensional $\iff$ columns are disctinct.

Example 1: Boolean Chu spaces

Let $X = \{x, y, z\}, A = \{a, b\}, Z = \{z\}$. Then

$$\mathbf{X} = (X, A, \beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix})$$

is both extensional and coextensional, whereas

$$\mathbf{X}' = (X, A, \beta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix})$$

is extensional but **not** coextensional ($R_x = R_z$).

Example 2: kernel functions

$Chu_{\mathbb{C}}(Set) = Chu(Set, \times, \mathbb{C})$

Let $\mathbf{X} = (X \times A \xrightarrow{K} \mathbb{C})$. The two-variable function K defines

- a map $K_r : A \to (X \multimap \mathbb{C}) = \mathbb{C}^X$ defined by $K_r(a) = K(., a)$;
- a map ${}_lK : X \to (\mathbb{C} \multimapinv A) = \mathbb{C}^A$ defined by ${}_lK(x) = K(x, .)$.

$\mathbf{X}$ is extensional if K_r is injective:

$$K(., a) = K(., b) \iff a = b$$

or equivalently (separation property):

$$a \neq b, \Rightarrow \exists x \in X,\ K(x, a) \neq K(x, b)$$

$(\mathbb{R}, \mathbb{R}, K(x, y) = x^2 + iy)$ is coextensional, but not extensional $(K(1, .) = K(-1, .))$.

Example 3: $R-$modules, $d = R$

$Chu_R(R - mod) = Chu(R - mod, \otimes, R)$

Let $\mathbf{X} = (X \otimes A \xrightarrow{\beta} R)$. The maps are

- $\beta_r : A \to (X \multimap R) = L(X, R)$ defined by $\beta_r(a) = \beta(. \otimes a)$;
- ${}_l\beta : X \to (R \multimapinv A) = L(A, R)$ defined by ${}_l\beta(x) = \beta(x \otimes .)$.

$\mathbf{X}$ is extensive if for all $a \in A$

$$(\forall x \in X, \beta(x \otimes a) = 0) \Rightarrow a = 0$$

It is coextensive if for all $x \in X$

$$(\forall a \in A, \beta(x \otimes a) = 0) \Rightarrow x = 0$$

Example 4: Vector spaces

1. Let E be a l.c.s, E' its continuous dual. Then $\mathbf{E} = (E, E', \langle ., . \rangle)$ with evaluation pairing is extensional by the Hahn-Banach theorem, and coextensional by construction.
2. Hilbert spaces are extensional and coextensional (biextensional).
3. Let $\mathcal{P}$ be the space of polynomials, then $\mathbf{X} = (\mathcal{P} \otimes \mathcal{P} \to \mathbb{C})$ with bilinear map $\beta(P, Q) = \int_{[0,1]} P'(x)Q(x)dx$ is extensional but **not** coextensional ($\beta(1, .) = 0$).

Example 5: Abelian groups

$R = d = \mathbb{Z}$

Let $X = p\mathbb{Z}$, $A = \mathbb{Z}$, with biaddive map $\beta(x, a) = \frac{xa}{p}$). Then $\mathbf{X} = (X, A, \beta)$ is extensional, ($\beta(., a) = 0 \iff a = 0$) and coextensional.

$R = \mathbb{Z}$, $d = \mathbb{R}/\mathbb{Z}$

- Let $X = A = \mathbb{Z}/n\mathbb{Z}$, with biaddive map $\beta(x, a) = \frac{xa}{n}$). Then $\mathbf{X} = (X, A, \beta)$ is biextensional.
- Let $Y = \mathbb{R}$, $B = \mathbb{Z}$ with biaddive map $\gamma(y, b) = yb$). $\mathbf{Y} = (Y, b, \gamma)$ is extensional but not coextensional $(\gamma(y, .) = \gamma(y + 1, .))$.

II. Chu spaces as analogs of locally convex spaces ?

Definition (Weak and Mackey topologies (Mackey, Grothendieck, Treves))

*Let E be a l.c.s., with (continuous) dual E'. A topology τ on E is **polar** if $(E, \tau)' = E'$.*

- *The **weak** (initial) topology is the coarsest polar topology.*
- *The **Mackey** topology is the finest polar topology.*

E_w (resp. E_m) $=$ E with the weak (resp. Mackey) topology.

Theorem (L. Schwartz)

$$H \hookrightarrow E \iff H \hookrightarrow E_w \iff H \hookrightarrow E_m$$

Weak and Mackey spaces are Chu spaces

Let $chu_{\mathbb{C}}(\mathcal{V})$ be the category of extensional-coextensional Chu vector spaces. Then

Theorem (Barr 2000)

$$chu_{\mathbb{C}}(\mathcal{V}) \sim \textit{Weak spaces} \sim \textit{Mackey spaces}$$

III. Symmetry ?

Symmetry

In classical cases, obvious natural bijection s between maps $X \otimes A \to d$ and maps $A \otimes X \to d$.

1. Define an involution $(-)^*$

$$(X, A, X \otimes A \stackrel{\beta}{\to} d)^* = (A, X, A \otimes X \stackrel{s(\beta)}{\to} d)$$

2. Define symmetric objects as $\mathbf{X} = (X \otimes X \stackrel{\beta}{\to} d)$ such that $s(\beta) = \beta$, that is as **Fixed Points** of the involution $(.)^*$.

Examples

1. Boolean Chu spaces with β interpreted as a boolean matrix. Then s associates to the matrix β is transpose β^T.
2. Complex kernel functions.
   - $[s(K)](a, x) = K(x, a)$ (classical symmetry)
   - $[s(K)](a, x) = \overline{K(x, a)}$ (hermitian symmetry)
3. Chu R–modules.
   - $[s(\beta)](a \otimes x) = \beta(x \otimes a)$ (classical symmetry)
   - If R is involutive, $[s(\beta)](a \otimes x) = \overline{\beta(x \otimes a)}$ (hermitian symmetry)

Examples

Consider Boolean Chu spaces. Then the involution $(-)^*$ works as follows:

$$(X, A, \beta)^* = (A, X, \beta^T)$$

Symmetric objects are of the form $\mathbf{X} = (X \times X \xrightarrow{\beta} \{0, 1\})$ with $\beta = \beta^T$.

IV. Positivity ?

Positivity

No general defintion at this time...

But assume $d = I$. Then exists a “canonical map”

$$\pi : d \otimes d \to d$$

We say that $\mathbf{X} = (X \otimes A \stackrel{\beta}{\to} d)$ is positive if forall $f : d \to X$ monic, exists $h : d \to d$ monic s.t.

$$\begin{array}{ccc} d \otimes d & \xrightarrow{h \otimes h} & d \otimes d \\ {\scriptstyle f \otimes f}\downarrow & & \downarrow{\scriptstyle \pi} \\ X \otimes X & \xrightarrow[\beta]{} & d \end{array}$$

Example: R–modules, $d = R$

$Chu_R(R - mod) = Chu(R - mod, \otimes, R)$

Let $\mathbf{X} = (X \otimes X \xrightarrow{\beta} R)$. Then $\mathbf{X}$ is positive if and only if

$$\forall x \in X, \exists r \in R, \beta(x,x) = r^2$$

(or $r\bar{r}$ if we consider involutive rings).

V. Completeness ?

Completeness

- Ideas from algebraic geometry.
- Complete object $=$ object closed in any extension.
- Needs a closure operator that **keep symmetric objects symmetric !**
- Question 1: is the closure of extensional objects extensional ?
- Question 2: how is this **categorical completion** related to the **metric completion** ?

One sided Completion (Giuli, Tholen)

- Subobject: $\mathbf{X} = (X \otimes A \stackrel{\beta}{\to} d) \leq \mathbf{Y} = (Y \otimes B \stackrel{\beta'}{\to} d)$ if exists $\mathbf{f} = (f, \varphi) : \mathbf{X} \to \mathbf{Y}$ with f monic (embedding), φ epi.
- Closure: defined by equalizer of functions $\beta_Y(., y)$ that coincide on X.

1. Equalizer of f,g = "largest subset" on which f and g agree.
2. clearly does not keep symmetry.

Example: real vector spaces

$Chu_{\mathbb{R}}(\mathcal{V})$

- The Closure operator $\longleftrightarrow$ double orthogonal.
- $\mathbf{X} = (X \otimes A \xrightarrow{\beta} \mathbb{R})$ is complete iff $X = A^*$ **algebraic dual** iff X is **weakly complete.**

Two sided completion

- Subobject: f monic (emmbedding)

$$\begin{array}{ccc} X \otimes X & & \\ \downarrow f\otimes f & \searrow \beta_{\mathbf{X}} & \\ Y \otimes Y & \xrightarrow{\beta_{\mathbf{Y}}} & d \end{array}$$

- Closure: same as previously.

Subobjects of positive ones are positive.

Example: real vector spaces

$Chu_{\mathbb{R}}(\mathcal{V})$

- Closure operator $\longleftrightarrow$ double orthogonal.
- $\mathbf{X} = (X \otimes X \stackrel{\beta_X}{\rightarrow} \mathbb{R}) \leq \mathbf{Y} = (Y \otimes Y \stackrel{\beta_Y}{\rightarrow} \mathbb{R})$ if $f(X) \subset Y$, with f unitary.

Closure and biextensionality

Let $Y = l^2$, $X = Span\{e_0 + e_1 + \frac{1}{n}e_n, n \geq 2\} \subset Y = l^2$.
Define

$$\beta_Y(u, v) = \sum_{n \in \mathbb{N}} (-1)^n u_n v_n$$

on $Y \times Y$, and let β_X be its restriction to $X \times X$. They both define extensional-coextensional chu spaces.
But $X^{\perp} = \mathbb{R}.(e_0 + e_1)$, $X^{\perp\perp} = \{u \in l^2, u_0 = u_1\} \supset X^{\perp}$, and $X^{\perp\perp}$ is **not extensional**.

If positivity implies biextensionality, completion of positive objects will be positive hence biextensional.

Example: real Hilbert spaces

Theorem (Categorical completion / metric completion)

Let $\mathbf{X} = (X, X, \beta)$ *be a* ***positive*** *Chu space. Then it is complete (in our categorical sense) if and only if* (X, β) *is a Hilbert space.*

Hilbert C^*-modules

If a positive Chu space $\mathbf{X} = (X, X, \beta)$ is complete, then it is a Hilbert C^*-module, but the converse is not true in general.

Let $\mathcal{A} = C([0, 1])$ and $X = C_0([0, 1])$. Then X is a Hilbert C^*-module but $\mathbf{X}$ is not closed in $\mathbf{Y} = (\mathcal{A}, \mathcal{A}, \beta)$ (with β the product).

VI. Kernels ?

Kernels

Any positive chu space **X** embedded in a Chu space **Y** defines a **positive** Chu morphism

$$\varkappa : \mathbf{Y}^* \to \mathbf{Y}$$

THE END