State feedback for overdetermined 2D systems: Pole placement for bundle maps over an algebraic curve.

Liran Shaul

Department of Mathematics
Ben-Gurion University of the Negev

Joint work with Victor Vinnikov

30 June 2011

Here is the plan of my lecture:

1. Pole placement - classical case
2. The definition of operator vessels
3. Transfer functions of operator vessels
4. The discriminant curve of a vessel
5. Meromorphic bundle maps
6. State feedback and the pole placement problem
7. The solution
8. Conclusions

Recall that given a linear system

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

its tranfer function is given by $T(\lambda) = D + C(\lambda I - A)^{-1}B$, so its poles are (contained in) the spectrum of A.

Recall that given a linear system

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

its tranfer function is given by $T(\lambda) = D + C(\lambda I - A)^{-1}B$, so its poles are (contained in) the spectrum of A.
The pole placement problem asks if by state feedback one can change the poles of the system arbitrarily.

Recall that given a linear system

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

its tranfer function is given by $T(\lambda) = D + C(\lambda I - A)^{-1}B$, so its poles are (contained in) the spectrum of A.
The pole placement problem asks if by state feedback one can change the poles of the system arbitrarily.
Algebraically this means: can we find a feedback operator F, acting from the state space to the input space, such that the spectrum of $A + BF$ will be equal to a given set.

Recall that given a linear system

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

its tranfer function is given by $T(\lambda) = D + C(\lambda I - A)^{-1}B$, so its poles are (contained in) the spectrum of A.
The pole placement problem asks if by state feedback one can change the poles of the system arbitrarily.
Algebraically this means: can we find a feedback operator F, acting from the state space to the input space, such that the spectrum of $A + BF$ will be equal to a given set.
It is well known that

Theorem 1.1. *Arbitrary pole placement is possible if and only if the system is controllable.*

Recall that given a linear system

$$\begin{cases} \frac{dx}{dt}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

its tranfer function is given by $T(\lambda) = D + C(\lambda I - A)^{-1}B$, so its poles are (contained in) the spectrum of A.
The pole placement problem asks if by state feedback one can change the poles of the system arbitrarily.
Algebraically this means: can we find a feedback operator F, acting from the state space to the input space, such that the spectrum of $A + BF$ will be equal to a given set.
It is well known that

Theorem 1.1. *Arbitrary pole placement is possible if and only if the system is controllable.*

The aim of this talk is to discuss this problem for a particular kind of overdetermined 2D continuous-time time-invariant systems.

The definition of operator vessels

We start with an overdetermined 2D continuous-time time-invariant linear i/s/o system Σ

$$
\begin{cases}
\frac{\partial x}{\partial t_1}(t_1, t_2) = A_1 x(t_1, t_2) + B_1 u(t_1, t_2) \\
\frac{\partial x}{\partial t_2}(t_1, t_2) = A_2 x(t_1, t_2) + B_2 u(t_1, t_2) \\
y(t_1, t_2) = C x(t_1, t_2) + D u(t_1, t_2)
\end{cases}
$$

The definition of operator vessels

We start with an overdetermined 2D continuous-time time-invariant linear i/s/o system Σ

$$\begin{cases} \frac{\partial x}{\partial t_1}(t_1, t_2) = A_1 x(t_1, t_2) + B_1 u(t_1, t_2) \\ \\ \frac{\partial x}{\partial t_2}(t_1, t_2) = A_2 x(t_1, t_2) + B_2 u(t_1, t_2) \\ \\ y(t_1, t_2) = C x(t_1, t_2) + D u(t_1, t_2) \end{cases}$$

$u(t_1, t_2) \in \mathcal{E}$ - input space. $x(t_1, t_2) \in \mathcal{H}$ - state space. $y(t_1, t_2) \in \mathcal{E}_*$ - output space.

The definition of operator vessels

We start with an overdetermined 2D continuous-time time-invariant linear i/s/o system Σ

$$\begin{cases} \frac{\partial x}{\partial t_1}(t_1,t_2) = A_1 x(t_1,t_2) + B_1 u(t_1,t_2) \\ \\ \frac{\partial x}{\partial t_2}(t_1,t_2) = A_2 x(t_1,t_2) + B_2 u(t_1,t_2) \\ \\ y(t_1,t_2) = C x(t_1,t_2) + D u(t_1,t_2) \end{cases}$$

$u(t_1,t_2) \in \mathcal{E}$ - input space. $x(t_1,t_2) \in \mathcal{H}$ - state space. $y(t_1,t_2) \in \mathcal{E}_*$ - output space.
All spaces are finite dimensional over the complex numbers.

The definition of operator vessels

We start with an overdetermined 2D continuous-time time-invariant linear i/s/o system Σ

$$\begin{cases} \frac{\partial x}{\partial t_1}(t_1,t_2) = A_1 x(t_1,t_2) + B_1 u(t_1,t_2) \\ \frac{\partial x}{\partial t_2}(t_1,t_2) = A_2 x(t_1,t_2) + B_2 u(t_1,t_2) \\ y(t_1,t_2) = C x(t_1,t_2) + D u(t_1,t_2) \end{cases}$$

$u(t_1,t_2) \in \mathcal{E}$ - input space. $x(t_1,t_2) \in \mathcal{H}$ - state space. $y(t_1,t_2) \in \mathcal{E}_*$ - output space.
All spaces are finite dimensional over the complex numbers.
$A_1, A_2 : \mathcal{H} \to \mathcal{H}$. $B_1, B_2 : \mathcal{E} \to \mathcal{H}$. $C : \mathcal{H} \to \mathcal{E}_*$. $D : \mathcal{E} \to \mathcal{E}_*$.

The definition of operator vessels - A1

Assuming x is smooth, we have $\frac{\partial}{\partial t_1}\frac{\partial x}{t_2} = \frac{\partial}{\partial t_2}\frac{\partial x}{t_1}$, so that from Σ, we have:

$$A_1\frac{\partial x}{\partial t_2} + B_1\frac{\partial u}{\partial t_2} = A_2\frac{\partial x}{\partial t_1} + B_2\frac{\partial u}{\partial t_1}$$

The definition of operator vessels - A1

Assuming x is smooth, we have $\frac{\partial}{\partial t_1}\frac{\partial x}{t_2} = \frac{\partial}{\partial t_2}\frac{\partial x}{t_1}$, so that from Σ, we have:

$$A_1\frac{\partial x}{\partial t_2} + B_1\frac{\partial u}{\partial t_2} = A_2\frac{\partial x}{\partial t_1} + B_2\frac{\partial u}{\partial t_1}$$

Replacing $\frac{\partial x}{\partial t_i}$ with the terms in Σ, we obtain

$$A_1[A_2x + B_2u] + B_1\frac{\partial u}{\partial t_2} = A_2[A_1x + B_1u] + B_2\frac{\partial u}{\partial t_1} \tag{2.1}$$

The definition of operator vessels - A1

Assuming x is smooth, we have $\frac{\partial}{\partial t_1}\frac{\partial x}{t_2} = \frac{\partial}{\partial t_2}\frac{\partial x}{t_1}$, so that from Σ, we have:

$$A_1 \frac{\partial x}{\partial t_2} + B_1 \frac{\partial u}{\partial t_2} = A_2 \frac{\partial x}{\partial t_1} + B_2 \frac{\partial u}{\partial t_1}$$

Replacing $\frac{\partial x}{\partial t_i}$ with the terms in Σ, we obtain

$$A_1[A_2 x + B_2 u] + B_1 \frac{\partial u}{\partial t_2} = A_2[A_1 x + B_1 u] + B_2 \frac{\partial u}{\partial t_1} \tag{2.1}$$

Setting $u = 0$, we see that we must have $A_1 A_2 = A_2 A_1$. Hence, we require our systems to satisfy this compatibility condition:

$$(A1) \qquad A_1 A_2 = A_2 A_1$$

The definition of operator vessels - A2

Under the assumption (A1), (2.1) becomes

$$B_2 \frac{\partial u}{\partial t_1} - B_1 \frac{\partial u}{\partial t_2} + (A_2 B_1 - A_1 B_2) u = 0 \tag{2.2}$$

The definition of operator vessels - A2

Under the assumption (A1), (2.1) becomes

$$B_2 \frac{\partial u}{\partial t_1} - B_1 \frac{\partial u}{\partial t_2} + (A_2 B_1 - A_1 B_2) u = 0 \tag{2.2}$$

We now take an auxiliary Hilbert space $\widetilde{\mathcal{E}}$ and a factorization

$$B_2 = \widetilde{B}\sigma_2 \quad B_1 = \widetilde{B}\sigma_1 \quad A_2 B_1 - A_1 B_2 = \widetilde{B}\gamma \tag{2.3}$$

where

$$\widetilde{B} : \widetilde{\mathcal{E}} \to \mathcal{H} \quad \sigma_1 : \mathcal{E} \to \widetilde{\mathcal{E}} \quad \sigma_2 : \mathcal{E} \to \widetilde{\mathcal{E}} \quad \gamma : \mathcal{E} \to \widetilde{\mathcal{E}}$$

The definition of operator vessels - A2

Under the assumption (A1), (2.1) becomes

$$B_2 \frac{\partial u}{\partial t_1} - B_1 \frac{\partial u}{\partial t_2} + (A_2 B_1 - A_1 B_2) u = 0 \qquad (2.2)$$

We now take an auxiliary Hilbert space $\widetilde{\mathcal{E}}$ and a factorization

$$B_2 = \widetilde{B} \sigma_2 \quad B_1 = \widetilde{B} \sigma_1 \quad A_2 B_1 - A_1 B_2 = \widetilde{B} \gamma \qquad (2.3)$$

where

$$\widetilde{B} : \widetilde{\mathcal{E}} \to \mathcal{H} \quad \sigma_1 : \mathcal{E} \to \widetilde{\mathcal{E}} \quad \sigma_2 : \mathcal{E} \to \widetilde{\mathcal{E}} \quad \gamma : \mathcal{E} \to \widetilde{\mathcal{E}}$$

In terms of this factorization, (2.3) becomes our second compatibility condition:

$$(A2) \quad A_2 \widetilde{B} \sigma_1 - A_1 \widetilde{B} \sigma_2 = \widetilde{B} \gamma$$

The definition of operator vessels

Using this factorization, our equation (2.2) becomes

$$\widetilde{B}[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma]u(t_1, t_2) = 0 \tag{2.4}$$

The definition of operator vessels

Using this factorization, our equation (2.2) becomes

$$\widetilde{B}[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0 \tag{2.4}$$

A sufficient condition for this to hold is

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0 \tag{2.5}$$

which we shall assume our input signal satisfies.

The definition of operator vessels

Using this factorization, our equation (2.2) becomes

$$\widetilde{B}[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma]u(t_1, t_2) = 0 \tag{2.4}$$

A sufficient condition for this to hold is

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma]u(t_1, t_2) = 0 \tag{2.5}$$

which we shall assume our input signal satisfies.
Such a signal will be called an **admissible input signal**.

The definition of operator vessels

To force symmetry (for example, to allow the construction of an inverse system), we require the output signal $y(t_1, t_2)$ to satisfy a similar PDE

$$[\sigma_{2*}\frac{\partial}{\partial t_1} - \sigma_{1*}\frac{\partial}{\partial t_2} + \gamma_*]y(t_1, t_2) = 0 \tag{2.6}$$

where

$$\sigma_{1*} : \mathcal{E}_* \to \widetilde{\mathcal{E}}_* \quad \sigma_{2*} : \mathcal{E}_* \to \widetilde{\mathcal{E}}_* \quad \gamma_* : \mathcal{E}_* \to \widetilde{\mathcal{E}}_*$$

here, $\widetilde{\mathcal{E}}_*$ is another auxiliary Hilbert space.

The definition of operator vessels

To force symmetry (for example, to allow the construction of an inverse system), we require the output signal $y(t_1, t_2)$ to satisfy a similar PDE

$$[\sigma_{2*}\frac{\partial}{\partial t_1} - \sigma_{1*}\frac{\partial}{\partial t_2} + \gamma_*]y(t_1, t_2) = 0 \tag{2.6}$$

where

$$\sigma_{1*} : \mathcal{E}_* \to \widetilde{\mathcal{E}}_* \quad \sigma_{2*} : \mathcal{E}_* \to \widetilde{\mathcal{E}}_* \quad \gamma_* : \mathcal{E}_* \to \widetilde{\mathcal{E}}_*$$

here, $\widetilde{\mathcal{E}}_*$ is another auxiliary Hilbert space.
Assuming this PDE holds, and, again, assuming no input, one may check that

$$(A3) \quad \sigma_{2*}CA_1 - \sigma_{1*}CA_2 + \gamma_*C = 0$$

The definition of operator vessels

This, in turn, forces the input signal to satisfy another PDE

$$[\sigma_{2*}D\frac{\partial}{\partial t_1} - \sigma_{1*}D\frac{\partial}{\partial t_2} + \sigma_{2*}C\widetilde{B}\sigma_1 - \sigma_{1*}C\widetilde{B}\sigma_2 + \gamma_* D]u(t_1, t_2) = 0 \qquad (2.7)$$

The definition of operator vessels

This, in turn, forces the input signal to satisfy another PDE

$$[\sigma_{2*}D\frac{\partial}{\partial t_1} - \sigma_{1*}D\frac{\partial}{\partial t_2} + \sigma_{2*}C\widetilde{B}\sigma_1 - \sigma_{1*}C\widetilde{B}\sigma_2 + \gamma_* D]u(t_1,t_2) = 0 \qquad (2.7)$$

To make sure any admissible input signal satisfies this equation, we assume there is an operator $\widetilde{D} : \mathcal{E}_* \to \widetilde{\mathcal{E}}_*$ such that

$$(A4) \quad \sigma_{1*}D = \widetilde{D}\sigma_1 \quad \sigma_{2*}D = \widetilde{D}\sigma_2 \quad \gamma_* D = \widetilde{D}\gamma + \sigma_{1*}C\widetilde{B}\sigma_2 - \sigma_{2*}C\widetilde{B}\sigma_1.$$

The definition of operator vessels

This, in turn, forces the input signal to satisfy another PDE

$$[\sigma_{2*} D \frac{\partial}{\partial t_1} - \sigma_{1*} D \frac{\partial}{\partial t_2} + \sigma_{2*} C\widetilde{B}\sigma_1 - \sigma_{1*} C\widetilde{B}\sigma_2 + \gamma_* D] u(t_1, t_2) = 0 \quad (2.7)$$

To make sure any admissible input signal satisfies this equation, we assume there is an operator $\widetilde{D} : \mathcal{E}_* \to \widetilde{\mathcal{E}}_*$ such that

$$(A4) \quad \sigma_{1*} D = \widetilde{D}\sigma_1 \quad \sigma_{2*} D = \widetilde{D}\sigma_2 \quad \gamma_* D = \widetilde{D}\gamma + \sigma_{1*} C\widetilde{B}\sigma_2 - \sigma_{2*} C\widetilde{B}\sigma_1.$$

assuming such a $\widetilde{D}$ exists, any admissible input signal satisfies (2.7).

The definition of operator vessels

The above discussion leads us to the notion of a **Livsic-Kravitsky commutative operator vessel**:

The definition of operator vessels

The above discussion leads us to the notion of a **Livsic-Kravitsky commutative operator vessel**:
An operator vessel $\mathcal{B}$ is a collection of operators and spaces

$$\mathcal{B} = (A_1, A_2, \widetilde{B}, C, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*; \mathcal{H}, \mathcal{E}, \mathcal{E}_*, \widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}_*)$$

The definition of operator vessels

The above discussion leads us to the notion of a **Livsic-Kravitsky commutative operator vessel**:
An operator vessel $\mathcal{B}$ is a collection of operators and spaces

$$\mathcal{B} = (A_1, A_2, \widetilde{B}, C, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*; \mathcal{H}, \mathcal{E}, \mathcal{E}_*, \widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}_*)$$

satisfying:

$$\begin{array}{ll}
(A1) & A_1A_2 = A_2A_1 \\
(A2) & A_2\widetilde{B}\sigma_1 - A_1\widetilde{B}\sigma_2 = \widetilde{B}\gamma \\
(A3) & \sigma_{2*}CA_1 - \sigma_{1*}CA_2 + \gamma_*C = 0 \\
(A4) & \sigma_{1*}D = \widetilde{D}\sigma_1 \quad \sigma_{2*}D = \widetilde{D}\sigma_2 \\
 & \gamma_*D = \widetilde{D}\gamma + \sigma_{1*}C\widetilde{B}\sigma_2 - \sigma_{2*}C\widetilde{B}\sigma_1
\end{array}$$

The definition of operator vessels

we also require the input signals u to be admissible input signals:

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0$$

The definition of operator vessels

we also require the input signals u to be admissible input signals:

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0$$

which implies that the output signals y satisfy

$$[\sigma_{2*} \frac{\partial}{\partial t_1} - \sigma_{1*} \frac{\partial}{\partial t_2} + \gamma_*] y(t_1, t_2) = 0$$

The definition of operator vessels

we also require the input signals u to be admissible input signals:

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0$$

which implies that the output signals y satisfy

$$[\sigma_{2*} \frac{\partial}{\partial t_1} - \sigma_{1*} \frac{\partial}{\partial t_2} + \gamma_*] y(t_1, t_2) = 0$$

The system of equations associated to a vessel $\mathcal{B}$ is

$$\begin{cases} \frac{\partial x}{\partial t_1}(t_1, t_2) = A_1 x(t_1, t_2) + \widetilde{B} \sigma_1 u(t_1, t_2) \\ \frac{\partial x}{\partial t_2}(t_1, t_2) = A_2 x(t_1, t_2) + \widetilde{B} \sigma_2 u(t_1, t_2) \\ y(t_1, t_2) = C x(t_1, t_2) + D u(t_1, t_2) \end{cases}$$

State space isomorphism

Given a vessel $\mathcal{B}$ and an isomorphism of the state space $N : \mathcal{H} \to \mathcal{H}$, we may perform state space isomorphism on $\mathcal{B}$, and obtain a new vessel $\mathcal{B}^N =$

$$(N^{-1}A_1N, N^{-1}A_2N, N^{-1}\widetilde{B}, CN, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*; \mathcal{H}, \mathcal{E}, \mathcal{E}_*, \widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}_*)$$

the new vessel $\mathcal{B}^N$ shares with $\mathcal{B}$ all its intrinsic properties.

Transfer functions of operator vessels

We construct the transfer function of a vessel $\mathcal{B}$ using frequency domain analysis.

Transfer functions of operator vessels

We construct the transfer function of a vessel $\mathcal{B}$ using frequency domain analysis.
We let

$$u(t_1, t_2) = e^{\lambda_1 t_1 + \lambda_2 t_2} u_0$$

$$x(t_1, t_2) = e^{\lambda_1 t_1 + \lambda_2 t_2} x_0$$

and

$$y(t_1, t_2) = e^{\lambda_1 t_1 + \lambda_2 t_2} y_0$$

for some $u_0 \in \mathcal{E}$, $x_0 \in \mathcal{H}$ and $y_0 \in \mathcal{E}_*$.

Transfer functions of operator vessels

In this case, the input signal u is an admissible input signal if and only if u_0 satisfies the following algebraic equation

$$[\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma]u_0 = 0$$

Transfer functions of operator vessels

In this case, the input signal u is an admissible input signal if and only if u_0 satisfies the following algebraic equation

$$[\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma]u_0 = 0$$

and similarly, for the output

$$[\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*]y_0 = 0$$

Transfer functions of operator vessels

In this case, the input signal u is an admissible input signal if and only if u_0 satisfies the following algebraic equation

$$[\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma]u_0 = 0$$

and similarly, for the output

$$[\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*]y_0 = 0$$

Plugging u, x and y into Σ, we obtain the following system of equations

$$\begin{aligned} \lambda_1 x_0 &= A_1 x_0 + \widetilde{B}\sigma_1 u_0 \\ \lambda_2 x_0 &= A_2 x_0 + \widetilde{B}\sigma_2 u_0 \\ y_0 &= C x_0 + D u_0 \end{aligned}$$

Transfer functions of operator vessels

To solve this system (i.e, to find y_0 in terms of u_0), we multiply the first equation by $\xi_1 \in \mathbb{C}$, the second by $\xi_2 \in \mathbb{C}$. Adding the resulting equations, we obtain

$$(\xi_1\lambda_1 + \xi_2\lambda_2)x_0 = (\xi_1 A_1 + \xi_2 A_2)x_0 + \widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0$$

Transfer functions of operator vessels

To solve this system (i.e, to find y_0 in terms of u_0), we multiply the first equation by $\xi_1 \in \mathbb{C}$, the second by $\xi_2 \in \mathbb{C}$. Adding the resulting equations, we obtain

$$(\xi_1\lambda_1 + \xi_2\lambda_2)x_0 = (\xi_1 A_1 + \xi_2 A_2)x_0 + \widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0$$

Recall that the joint spectrum of a pair of commuting square matrices $A, B \in M_n(\mathbb{C})$, denoted by $Spec(A, B)$, is the set of pairs (λ, μ) which have a common eigenvector $\exists 0 \neq v \in \mathbb{C}^n$, $Av = \lambda v$ and $Bv = \mu v$.

Transfer functions of operator vessels

To solve this system (i.e, to find y_0 in terms of u_0), we multiply the first equation by $\xi_1 \in \mathbb{C}$, the second by $\xi_2 \in \mathbb{C}$. Adding the resulting equations, we obtain

$$(\xi_1\lambda_1 + \xi_2\lambda_2)x_0 = (\xi_1 A_1 + \xi_2 A_2)x_0 + \widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0$$

Recall that the joint spectrum of a pair of commuting square matrices $A, B \in M_n(\mathbb{C})$, denoted by $Spec(A, B)$, is the set of pairs (λ, μ) which have a common eigenvector $\exists 0 \neq v \in \mathbb{C}^n$, $Av = \lambda v$ and $Bv = \mu v$.

Lemma 3.1. *Given $A, B \in M_n(\mathbb{C})$, such that $AB = BA$, there exist $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1 A + \xi_2 B$ is invertible, if and only if $(0, 0) \notin Spec(A, B)$.*

Transfer functions of operator vessels

To solve this system (i.e, to find y_0 in terms of u_0), we multiply the first equation by $\xi_1 \in \mathbb{C}$, the second by $\xi_2 \in \mathbb{C}$. Adding the resulting equations, we obtain

$$(\xi_1\lambda_1 + \xi_2\lambda_2)x_0 = (\xi_1 A_1 + \xi_2 A_2)x_0 + \widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0$$

Recall that the joint spectrum of a pair of commuting square matrices $A, B \in M_n(\mathbb{C})$, denoted by $Spec(A, B)$, is the set of pairs (λ, μ) which have a common eigenvector $\exists 0 \neq v \in \mathbb{C}^n$, $Av = \lambda v$ and $Bv = \mu v$.

Lemma 3.1. *Given $A, B \in M_n(\mathbb{C})$, such that $AB = BA$, there exist $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1 A + \xi_2 B$ is invertible, if and only if $(0, 0) \notin Spec(A, B)$.*

Hence, assuming $(\lambda_1, \lambda_2) \notin Spec(A_1, A_2)$, we may choose ξ_1, ξ_2 such that $\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2)$ is invertible, so that $x_0 = (\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2))^{-1}\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0$.

Transfer functions of operator vessels

This implies that
$y_0 = (D + C(\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2))^{-1}\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2))u_0$. Hence, the transfer function of a vessel $\mathcal{B}$ is given by

$$S_{\mathcal{B}}(\lambda_1, \lambda_2) = D + C(\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2))^{-1}\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)$$

Note that for a u_0 with u admissible input signal, this is independent of ξ_1, ξ_2.

Controllability, observability and minimal vessels

Let $\mathcal{C}$ denote the controllable subspace, i.e the space of all vectors $h \in \mathcal{H}$ such that there exist an admissible input u for which the state function x satisfies, $x(0,0) = 0$, and $x(t_1,t_2) = h$ for some $(t_1,t_2) \in \mathbb{R}^2$. A vessel $\mathcal{B}$ is called controllable if $\mathcal{C} = \mathcal{H}$.

Controllability, observability and minimal vessels

Let $\mathcal{C}$ denote the controllable subspace, i.e the space of all vectors $h \in \mathcal{H}$ such that there exist an admissible input u for which the state function x satisfies, $x(0,0) = 0$, and $x(t_1, t_2) = h$ for some $(t_1, t_2) \in \mathbb{R}^2$. A vessel $\mathcal{B}$ is called controllable if $\mathcal{C} = \mathcal{H}$.
Similarly, let $\mathcal{O}^{\perp}$ denote the unobservable subspace, i.e the subspace of all vectors $h \in \mathcal{H}$ such that the unique solution (u, x, y) of the system of equations associated to the vessel, with $x(0,0) = h$, and $u \equiv 0$ has $y \equiv 0$. A vessel $\mathcal{B}$ is called observable if $\mathcal{O}^{\perp} = 0$.

Controllability, observability and minimal vessels

Let $\mathcal{C}$ denote the controllable subspace, i.e the space of all vectors $h \in \mathcal{H}$ such that there exist an admissible input u for which the state function x satisfies, $x(0,0) = 0$, and $x(t_1, t_2) = h$ for some $(t_1, t_2) \in \mathbb{R}^2$. A vessel $\mathcal{B}$ is called controllable if $\mathcal{C} = \mathcal{H}$.
Similarly, let $\mathcal{O}^\perp$ denote the unobservable subspace, i.e the subspace of all vectors $h \in \mathcal{H}$ such that the unique solution (u, x, y) of the system of equations associated to the vessel, with $x(0,0) = h$, and $u \equiv 0$ has $y \equiv 0$. A vessel $\mathcal{B}$ is called observable if $\mathcal{O}^\perp = 0$.
A vessel is called minimal if it is both controllable and observable.

The discriminant curve of a vessel

Assume now that $\dim \mathcal{E} = \dim \widetilde{\mathcal{E}}$, and $\dim \mathcal{E}_* = \dim \widetilde{\mathcal{E}}_*$.

The discriminant curve of a vessel

Assume now that $\dim \mathcal{E} = \dim \widetilde{\mathcal{E}}$, and $\dim \mathcal{E}_* = \dim \widetilde{\mathcal{E}}_*$.
We have seen that a frequency function u satisfies the compatibility PDE if and only if $(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma) u_0 = 0$.

The discriminant curve of a vessel

Assume now that $\dim \mathcal{E} = \dim \widetilde{\mathcal{E}}$, and $\dim \mathcal{E}_* = \dim \widetilde{\mathcal{E}}_*$.
We have seen that a frequency function u satisfies the compatibility PDE if and only if $(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)u_0 = 0$.
We define two kernel bundles

$$E_{in}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) \quad E_{out}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*)$$

The discriminant curve of a vessel

Assume now that $\dim \mathcal{E} = \dim \widetilde{\mathcal{E}}$, and $\dim \mathcal{E}_* = \dim \widetilde{\mathcal{E}}_*$.
We have seen that a frequency function u satisfies the compatibility PDE if and only if $(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)u_0 = 0$.
We define two kernel bundles

$$E_{in}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) \quad E_{out}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*)$$

Let

$$p_{in}(\lambda_1, \lambda_2) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) \quad p_{out}(\lambda_1, \lambda_2) = \det(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*)$$

The discriminant curve of a vessel

Assume now that $\dim \mathcal{E} = \dim \widetilde{\mathcal{E}}$, and $\dim \mathcal{E}_* = \dim \widetilde{\mathcal{E}}_*$.
We have seen that a frequency function u satisfies the compatibility PDE if and only if $(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)u_0 = 0$.
We define two kernel bundles

$$E_{in}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) \quad E_{out}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*)$$

Let

$$p_{in}(\lambda_1, \lambda_2) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) \quad p_{out}(\lambda_1, \lambda_2) = \det(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*)$$

Theorem 4.1. *(Livsic-Kravitsky): $p_{in} \equiv \lambda p_{out}$ for some $\lambda \in \mathbb{C}^\times$.*

The discriminant curve of a vessel

Let $p = p_{in} = \lambda p_{out}$.

The discriminant curve of a vessel

Let $p = p_{in} = \lambda p_{out}$.
Let $C_0 = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : p(\lambda_1, \lambda_2) = 0\}$. The curve C_0 (and the associated projective plane curve C) is called the discriminant curve of the vessel $\mathcal{B}$.

The discriminant curve of a vessel

Let $p = p_{in} = \lambda p_{out}$.
Let $C_0 = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : p(\lambda_1, \lambda_2) = 0\}$. The curve C_0 (and the associated projective plane curve C) is called the discriminant curve of the vessel $\mathcal{B}$.
Since whenever u is an admissible input signal, the resulting output signal y is also admissible, we see that the natural domain of definition of the transfer function $S_{\mathcal{B}}$ is the input bundle E_{in}, and that for any $(\lambda_1, \lambda_2) \in C$, $S_{\mathcal{B}}(\lambda_1, \lambda_2)$ maps $E_{in}(\lambda_1, \lambda_2)$ into $E_{out}(\lambda_1, \lambda_2)$.

Cayley-Hamilton theorem for vessels

Theorem 4.2. *(Livsic-Kravitsky): The generalized Cayley-Hamilton theorem for vessels: For a minimal vessel* $\mathcal{B}$*,* $p(A_1, A_2) = 0$*.*

Cayley-Hamilton theorem for vessels

Theorem 4.2. *(Livsic-Kravitsky): The generalized Cayley-Hamilton theorem for vessels: For a minimal vessel* $\mathcal{B}$*,* $p(A_1, A_2) = 0$*.*

Corollary 4.3. $Spec(A_1, A_2) \subseteq C$*.*

The discriminant curve of a vessel

The input and output bundles

In all that follows we make the following assumptions on the curve C : C is smooth of degree m, intersects the line at infinity at m distinct points, and $p(\lambda_1, \lambda_2) = (p'(\lambda_1, \lambda_2))^r$, where p' is irreducible.

The discriminant curve of a vessel

The input and output bundles

In all that follows we make the following assumptions on the curve C : C is smooth of degree m, intersects the line at infinity at m distinct points, and $p(\lambda_1, \lambda_2) = (p'(\lambda_1, \lambda_2))^r$, where p' is irreducible.

Theorem 4.4. *If $r = 1$, for each $\lambda = (\lambda_1, \lambda_2) \in C$,*
$\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = 1.$

The discriminant curve of a vessel

The input and output bundles

In all that follows we make the following assumptions on the curve C : C is smooth of degree m, intersects the line at infinity at m distinct points, and $p(\lambda_1, \lambda_2) = (p'(\lambda_1, \lambda_2))^r$, where p' is irreducible.

Theorem 4.4. *If $r = 1$, for each $\lambda = (\lambda_1, \lambda_2) \in C$,* $\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = 1.$

If $r > 1$ it may be the case that at some points the dimension of E_{in} and E_{out} drops. To avoid this, we shall make an assumption in the sequel that for each $\lambda = (\lambda_1, \lambda_2) \in C$, $\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = r$.

The discriminant curve of a vessel

The input and output bundles

In all that follows we make the following assumptions on the curve C : C is smooth of degree m, intersects the line at infinity at m distinct points, and $p(\lambda_1, \lambda_2) = (p'(\lambda_1, \lambda_2))^r$, where p' is irreducible.

Theorem 4.4. *If $r = 1$, for each $\lambda = (\lambda_1, \lambda_2) \in C$,* $\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = 1$.

If $r > 1$ it may be the case that at some points the dimension of E_{in} and E_{out} drops. To avoid this, we shall make an assumption in the sequel that for each $\lambda = (\lambda_1, \lambda_2) \in C$, $\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = r$.
It follows that E_{in} and E_{out} are actually holomorphic vector bundles of rank r over the curve C.

The discriminant curve of a vessel

The input and output bundles

In all that follows we make the following assumptions on the curve C : C is smooth of degree m, intersects the line at infinity at m distinct points, and $p(\lambda_1, \lambda_2) = (p'(\lambda_1, \lambda_2))^r$, where p' is irreducible.

Theorem 4.4. *If $r = 1$, for each $\lambda = (\lambda_1, \lambda_2) \in C$,* $\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = 1.$

If $r > 1$ it may be the case that at some points the dimension of E_{in} and E_{out} drops. To avoid this, we shall make an assumption in the sequel that for each $\lambda = (\lambda_1, \lambda_2) \in C$, $\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = r$.
It follows that E_{in} and E_{out} are actually holomorphic vector bundles of rank r over the curve C.
Furthermore, our construction of the transfer function, shows that $S_{\mathcal{B}}$ is actually a bundle map. In summary, we have:

Theorem 4.5. *For a vessel $\mathcal{B}$, the transfer function $S_{\mathcal{B}} : E_{in} \to E_{out}$ is a meromorphic bundle map, with* $\textbf{\textit{poles}}(S_{\mathcal{B}}) \subseteq \textbf{\textit{Spec}}(A_1, A_2)$.

Meromorphic bundle maps and their divisor data

Given a compact Riemann surface X, and two holomorphic vector bundles E, F over X, a bundle map $T : E \to F$ which is meromorphic as a map between the complex manifolds E and F will be called a meromorphic bundle map.

Meromorphic bundle maps and their divisor data

Given a compact Riemann surface X, and two holomorphic vector bundles E, F over X, a bundle map $T : E \to F$ which is meromorphic as a map between the complex manifolds E and F will be called a meromorphic bundle map.
Given a point $p \in X$, and a local section ϕ at a neighborhood of p, with $\phi(p) \neq 0$, we may write in local coordinates $T(z)\phi(z) = z^k\psi(z)$ where ψ is a local section of F with $\psi(p) \neq 0$. If $k > 0$ we say that T has a right zero of order k at direction ϕ. If $k < 0$ we say that ψ has a right pole of order k at direction ψ.

Meromorphic bundle maps and their divisor data

Given a compact Riemann surface X, and two holomorphic vector bundles E, F over X, a bundle map $T : E \to F$ which is meromorphic as a map between the complex manifolds E and F will be called a meromorphic bundle map.

Given a point $p \in X$, and a local section ϕ at a neighborhood of p, with $\phi(p) \neq 0$, we may write in local coordinates $T(z)\phi(z) = z^k\psi(z)$ where ψ is a local section of F with $\psi(p) \neq 0$. If $k > 0$ we say that T has a right zero of order k at direction ϕ. If $k < 0$ we say that ψ has a right pole of order k at direction ψ.

Left zeros and poles are defined similarly.

Meromorphic bundle maps and their divisor data

Given a compact Riemann surface X, and two holomorphic vector bundles E, F over X, a bundle map $T : E \to F$ which is meromorphic as a map between the complex manifolds E and F will be called a meromorphic bundle map.

Given a point $p \in X$, and a local section ϕ at a neighborhood of p, with $\phi(p) \neq 0$, we may write in local coordinates $T(z)\phi(z) = z^k\psi(z)$ where ψ is a local section of F with $\psi(p) \neq 0$. If $k > 0$ we say that T has a right zero of order k at direction ϕ. If $k < 0$ we say that ψ has a right pole of order k at direction ψ.

Left zeros and poles are defined similarly.

In the matrix case over sub-bundles of a trivial bundle (which is really what we have in mind), such a theory of divisors has been developed extensively in the book "Interpolation of rational matrix functions" by Ball, Gohberg and Rodman.

Using the local theory presented in the book of Ball, Gohberg and Rodman one may show that

Theorem 5.1. *Given a minimal vessel*

$$\mathcal{B} = (A_1, A_2, \widetilde{B}, C, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*; \mathcal{H}, \mathcal{E}, \mathcal{E}_*, \widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}_*)$$

its left pole data is determined up to state space similarity by the triple $(A_1, A_2, \widetilde{B})$.

State feedback

We now formulate state feedback for vessels. Let $\mathcal{B}$ be a vessel, and suppose that $F : \mathcal{H} \to \mathcal{E}$. We may form a new collection

$$\mathcal{B}_F^{\textbf{Closed loop}} = (A_1 + \widetilde{B}\sigma_1 F, A_2 + \widetilde{B}\sigma_2 F, \widetilde{B}, C + DF, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

State feedback

We now formulate state feedback for vessels. Let $\mathcal{B}$ be a vessel, and suppose that $F : \mathcal{H} \to \mathcal{E}$. We may form a new collection

$$\mathcal{B}_F^{\textbf{Closed loop}} = (A_1 + \widetilde{B}\sigma_1 F, A_2 + \widetilde{B}\sigma_2 F, \widetilde{B}, C + DF, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

A small calculation shows that

Proposition 6.1. *The collection $\mathcal{B}_F^{\textit{Closed loop}}$ is an operator vessel if and only if F satisfies the following 2 equations:*

$$\sigma_2 F A_1 - \sigma_1 F A_2 + \gamma F = 0$$
$$\sigma_1 F \widetilde{B} \sigma_2 - \sigma_2 F \widetilde{B} \sigma_1 = 0$$

State feedback

We now formulate state feedback for vessels. Let $\mathcal{B}$ be a vessel, and suppose that $F : \mathcal{H} \to \mathcal{E}$. We may form a new collection

$$\mathcal{B}_F^{\textbf{Closed loop}} = (A_1 + \widetilde{B}\sigma_1 F, A_2 + \widetilde{B}\sigma_2 F, \widetilde{B}, C + DF, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

A small calculation shows that

Proposition 6.1. *The collection $\mathcal{B}_F^{\textit{Closed loop}}$ is an operator vessel if and only if F satisfies the following 2 equations:*

$$\sigma_2 F A_1 - \sigma_1 F A_2 + \gamma F = 0$$
$$\sigma_1 F \widetilde{B} \sigma_2 - \sigma_2 F \widetilde{B} \sigma_1 = 0$$

Such an F is called an admissible feedback for $\mathcal{B}$.

The pole placement problem

We may now formulate the pole placement problem for operator vessels:

The pole placement problem

We may now formulate the pole placement problem for operator vessels:

Problem 6.2. *Given a vessel $\mathcal{B}$, find an admissible feedback F such that the left pole data of the transfer function of the closed loop system $S_{\mathcal{B}_F^{\text{Closed loop}}}$ is a prescribed pole data on the discriminant curve C.*

The factorization

As might be expected, it turns out that the effect of state feedback takes place in the input space. In fact, for every controllable vessel $\mathcal{B}$ and any admissible feedback F, the transfer function of the closed loop vessel $\mathcal{B}_F^{\textbf{Closed loop}}$ factors

The factorization

As might be expected, it turns out that the effect of state feedback takes place in the input space. In fact, for every controllable vessel $\mathcal{B}$ and any admissible feedback F, the transfer function of the closed loop vessel $\mathcal{B}_F^{\textbf{Closed loop}}$ factors

$$E_{\text{in}} \xrightarrow{R^{-1}} E_{\text{in}} \xrightarrow{S} E_{\text{out}}$$

where S is the transfer function of the open loop vessel, and R is the transfer function of a vessel $\mathcal{B}_F^{\textbf{Controller}}$ whose construction will be explained in the sequel.

The Ball-Vinnikov realization theorem

Suppose $\sigma_1, \sigma_2, \gamma, \gamma_* \in M_n(\mathbb{C})$ are given matrices, and suppose that $\det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma_*)$ is a polynomial defining a smooth irreducible curve C which intersects the line at infinity at ($m = \deg C$) distinct points. Let $E_{\text{in}}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)$ and $E_{\text{out}}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma_*)$ be vector bundles over C.

The Ball-Vinnikov realization theorem

Suppose $\sigma_1, \sigma_2, \gamma, \gamma_* \in M_n(\mathbb{C})$ are given matrices, and suppose that $\det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma_*)$ is a polynomial defining a smooth irreducible curve C which intersects the line at infinity at ($m = \deg C$) distinct points. Let $E_{\text{in}}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)$ and $E_{\text{out}}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma_*)$ be vector bundles over C.

Theorem 7.1. *(Ball-Vinnikov): Given any meromorphic bundle map $S : E_{in} \to E_{out}$ such that S acts as the identity operator at all points of C at infinity. Then there exist a unique (up to state space isomorphism) minimal vessel $\mathcal{B}$ with the same determinantal representations and with $D = \widetilde{D} = I$ such that $S_{\mathcal{B}} = S$.*

The vessel $\mathcal{B}_F^{\textbf{Controller}}$

In order to solve the pole placement problem, we must analyze 2 different algebraic questions:

1. Determine what are the admissible feedbacks (if any)
2. Learn to control the joint spectrum $\mathrm{Spec}(A_1 + \widetilde{B}\sigma_1 F, A_2 + \widetilde{B}\sigma_2 F)$.

The answer to both of these problems is given by the following object:

The vessel $\mathcal{B}_F^{\textbf{Controller}}$

In order to solve the pole placement problem, we must analyze 2 different algebraic questions:

1. Determine what are the admissible feedbacks (if any)
2. Learn to control the joint spectrum $\text{Spec}(A_1 + \widetilde{B}\sigma_1 F, A_2 + \widetilde{B}\sigma_2 F)$.

The answer to both of these problems is given by the following object: Given a vessel $\mathcal{B}$ and any operator (not necessarily admissible) $F : \mathcal{H} \to \mathcal{E}$, define a collection

$$\mathcal{B}_F^{\textbf{Controller}} = (A_1, A_2, \widetilde{B}, -F, I, I, \sigma_1, \sigma_2, \gamma, \sigma_1, \sigma_2, \gamma)$$

The vessel $\mathcal{B}_F^{\textbf{Controller}}$

Theorem 7.2. *An operator F is an admissible feedback for $\mathcal{B}$ if and only if $\mathcal{B}_F^{\textit{Controller}}$ is a vessel*

The proof is an easy verification.

The vessel $\mathcal{B}_F^{\textbf{Controller}}$

Theorem 7.2. *An operator F is an admissible feedback for $\mathcal{B}$ if and only if $\mathcal{B}_F^{\textit{Controller}}$ is a vessel*

The proof is an easy verification.

Theorem 7.3. *For any admissible feedback F, the transfer function of the closed loop system $S_{\mathcal{B}_F^{\textit{Closed loop}}}$ factors as*

$$S_{\mathcal{B}_F^{\textit{Closed loop}}} = S_{\mathcal{B}} \circ (S_{\mathcal{B}_F^{\textit{Controller}}})^{-1}$$

again, the proof is by calculation.

The pole placement theorem

Theorem 7.4. *Let $\mathcal{B}$ be a minimal vessel. Given a left pole data D on the discriminant curve C, there exist an admissible feedback F such that* $\textit{left poles}(\mathcal{B}_F^{\textit{Closed loop}}) = D$ *if and only if there is a meromorphic bundle map* $T : E_{in} \to E_{out}$ *with the same left zero data as the left zero data of* $S_{\mathcal{B}}$ *and with the same behavior at all points at infinity as* $S_{\mathcal{B}}$.

The pole placement theorem

Theorem 7.4. *Let $\mathcal{B}$ be a minimal vessel. Given a left pole data D on the discriminant curve C, there exist an admissible feedback F such that left poles$(\mathcal{B}_F^{\textit{Closed loop}}) = D$ if and only if there is a meromorphic bundle map $T : E_{in} \to E_{out}$ with the same left zero data as the left zero data of $S_{\mathcal{B}}$ and with the same behavior at all points at infinity as $S_{\mathcal{B}}$.*

One direction of the theorem is almost immediate. If such an F is given, by calculating the inverse system of the closed loop system, one sees immediately that both the open loop and the closed loop transfer functions share the same left zero data, and the same behavior at infinity.

The pole placement theorem

For the converse, suppose such a T is given. Let $R = T^{-1} \circ S_{\mathcal{B}}$, $R : E_{in} \to E_{in}$. Then $R|_{\infty} = I$, and has the same left pole data as S does.

The pole placement theorem

For the converse, suppose such a T is given. Let $R = T^{-1} \circ S_{\mathcal{B}}$, $R : E_{in} \to E_{in}$. Then $R|_{\infty} = I$, and has the same left pole data as S does.
Using the Ball-Vinnikov realization theorem, we may construct a vessel $\mathcal{V}$ with $S_{\mathcal{V}} = R$. Then, since $\mathcal{B}$ and $\mathcal{V}$ share the same pole data, it follows that (up to state space isomorphism which does not affect pole placement), they share the same $(A_1, A_2, \widetilde{B})$. It follows that $\mathcal{V} = \mathcal{B}_F^{\textbf{Controller}}$ for $F = -C_{\mathcal{V}}$, and thus, $T = S_{\mathcal{B}_F^{\textbf{Closed loop}}}$, which completes the proof. □

The case of line bundles

To discuss some examples, let us, for simplicity, restrict ourselves to the $r = 1$ case. Thus, E_{in} and E_{out} are now assumed to be line bundles. It is then may be verified that

The case of line bundles

To discuss some examples, let us, for simplicity, restrict ourselves to the $r = 1$ case. Thus, E_{in} and E_{out} are now assumed to be line bundles. It is then may be verified that

Theorem 8.1. *To give a meromorphic bundle map $T : E_{in} \to E_{out}$ with the same left zero data as the left zero data of $S_{\mathcal{B}}$ and with the same behavior at all points at infinity as $S_{\mathcal{B}}$ is equivalent to construct a rational function $f \in K(C)$ whose zero divisor is equal to the poles of T, its pole divisor is equal to the poles of $S_{\mathcal{B}}$, and its value at all points at infinity is* 1.

The case of line bundles

To discuss some examples, let us, for simplicity, restrict ourselves to the $r = 1$ case. Thus, E_{in} and E_{out} are now assumed to be line bundles. It is then may be verified that

Theorem 8.1. *To give a meromorphic bundle map $T : E_{in} \to E_{out}$ with the same left zero data as the left zero data of $S_{\mathcal{B}}$ and with the same behavior at all points at infinity as $S_{\mathcal{B}}$ is equivalent to construct a rational function $f \in K(C)$ whose zero divisor is equal to the poles of T, its pole divisor is equal to the poles of $S_{\mathcal{B}}$, and its value at all points at infinity is* 1.

Thus, in this case we see that the main obstruction for pole placement is the genus of the curve.

Genus 0 case

We now show how to obtain information about pole placement in more specific examples (again, $r = 1$).

Genus 0 case

We now show how to obtain information about pole placement in more specific examples (again, $r = 1$).

Example 8.2. Suppose $\mathcal{B}$ is a vessel, such that the discriminant curve C is a smooth curve of genus 0. Let $m = \deg C$. One has $m = 1$ or $m = 2$. Suppose $m = 1$. Then C is actually a line. Hence, C intersects the line at infinity at precisely one point. In this case, the interpolation problem of finding a rational function which is 1 at infinity, having prescribed zeros and poles may always be solved, so we can place poles one the curve arbitrarily. This is not surprising, as vessels with $m = 0$ are the same as classical linear systems, so our theorem is indeed a generalization of the classical pole placement.

Genus 0 case

Example 8.3. Suppose now $g = 0$ and $m = 2$. In this case, C is a conic, and it intersects the line at infinity at 2 distinct points. In this case, one can always place exactly $n - 1$ poles, the last pole is then determined from the other $n - 1$.

General conclusions

Recall that $L(D) = \{f \in K(C) : (f) + D \geq 0\}$, and that $l(D) = \dim L(D)$. Let $P = \mathrm{Spec}(A_1, A_2)$.

Corollary 8.4. *Let $l = l(P - D_\infty)$. If $l = 0$ then no poles could be placed. If $l \geq 1$, then generically, one may place exactly l poles. Moreover, generically, once the l poles were chosen, the rest $n - l$ poles of the system are determined uniquely.*

General conclusions

Corollary 8.5. *Let* $m = \deg C$, $n = \dim \mathcal{H}$. *If* $n < m$, *no poles could be placed. If* $n - m > 2g - 2$, *then* $l = n - m + 1 - g$, *so that* $n - m + 1 - g$ *poles could be placed generically.*

Elliptic curves

Example 8.6. As a final example, suppose C is an elliptic curve. In this case, $g = 1$, and $m = 3$. For elliptic curves, the Riemann-Roch theorem implies that $l(D) = \deg(D)$ for all D with $\deg(D) \geq 1$. Hence, if $n \geq 4$, one can always place poles, and, generically, one can place $n - 3$ poles. If $n \leq 2$, no place could be placed.

Elliptic curves

Example 8.6. As a final example, suppose C is an elliptic curve. In this case, $g = 1$, and $m = 3$. For elliptic curves, the Riemann-Roch theorem implies that $l(D) = \deg(D)$ for all D with $\deg(D) \geq 1$. Hence, if $n \geq 4$, one can always place poles, and, generically, one can place $n - 3$ poles. If $n \leq 2$, no place could be placed.
The case where $n = 3$ is a special case. In this case, the possibility of placing a single pole depends on the points of p. The theory of special divisors on elliptic curves then shows:

Theorem 8.7. *If $g = 1$ and $n = 3$, then $l > 0$ (so that one can place poles) if and only if for $p = p_1 + p_2 + p_3$, the points p_1, p_2 and p_3 lie on one line. (In terms of the group of points of the elliptic curve, this just means that $p_1 + p_2 + p_3 = 0$).*