Slice monogenic functions and a functional calculus for n-tuples of operators

Irene Sabadini

Politecnico di Milano

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Motivations

Let T be a bounded operator on a Banach space, we define

$$f(T) = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathcal{I} - T)^{-1} f(\lambda)\, d\lambda,$$

where Γ is a rectifiable Jordan curve that surrounds the spectrum of T.

Problems

- How to generalize the definition to quaternionic operators? Quaternionic Quantum Mechanics and Quantum Fields (Adler, Oxford University Press 1995)
- How to define a function of n-tuples of operators?

- H.Weyl (1930): The Theory of Groups and Quantum Mechanics
- R.F. Anderson (J. Funct. Anal., 1969): functional calculus uses the Fourier transform and holds for self-adjoint operators $A_1, ..., A_n$.
- J. L. Taylor (Acta Math., 1970): uses holomorphic functions in several variables. The most explicit calculus is for commuting operators. Algebraic treatment.
- B. Jefferies, A. McIntosh and coauthors (1987-2004) use monogenic functions with values in a Clifford algebra for commuting operators or noncommuting operators satisfying additional hypothesis on their spectrum.

Slice monogenic functions

Notations

Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \dots, e_n$ satisfying the relations

$$e_i e_j + e_j e_i = 0,\ i \neq j \qquad e_i^2 = -1.$$

An element in the Clifford algebra will be denoted by

$$\sum_A e_A x_A$$

where

$$A = \{i_1 \dots i_r\} \in \mathcal{P}\{1, 2, \dots, n\},\quad i_1 < \dots < i_r$$

is a multi-index and $e_A = e_{i_1} e_{i_2} \dots e_{i_r}$, $e_\emptyset = 1$.

Slice monogenic functions

Example. $\mathbb{R}_2 = \mathbb{H}$

$$q = x_0 + e_1 x_1 + e_2 x_2 + e_1 e_2 x_{12},$$

$e_1 = i,\ e_2 = j,\ e_1 e_2 = k.$
An element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element

$$x = x_0 + \underline{x} = x_0 + \sum_{j=1}^{n} x_j e_j \in \mathbb{R}_n$$

called, in short, paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as

$$|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2,$$

$\underline{x}$ is the 1-vector part of x;
the conjugate of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{j=1}^{n} x_j e_j$.

Slice monogenic functions

The sphere $\mathbb{S}$

$\mathbb{S} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n \mid x_1^2 + \ldots + x_n^2 = 1\}$ $I \in \mathbb{S}$, $I^2 = -1$

Imaginary unit associated to $x \in \mathbb{R}^{n+1}$

Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set

$$I_x = \begin{cases} \frac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S} \text{ otherwise.} \end{cases}$$

The $(n-1)$-sphere $[x]$

Given an element $x \in \mathbb{R}^{n+1}$, we define

$$[x] = \{y \in \mathbb{R}^{n+1} \ : \ y = x_0 + I|\underline{x}|, \quad I \in \mathbb{S}\}.$$

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Slice monogenic functions

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The complex plane $\mathbb{C}_I$

The vector space $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by $\mathbb{C}_I$, while an element belonging to $\mathbb{C}_I$ will be denoted by $u + Iv$, for u, $v \in \mathbb{R}$. $\mathbb{C}_I$ can be identified with a complex plane.

Slice monogenic functions

Slice monogenic functions

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f: \ U \to \mathbb{R}_n$ be a real differentiable function.
Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$.
We say that f is a (left) slice monogenic function, or s-monogenic function, if for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + Iv) = 0.$$

We say that f is a right slice monogenic function, or right s-monogenic function, if for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_I(u + Iv) + \frac{\partial}{\partial v} f_I(u + Iv) I\right) = 0.$$

The Cauchy formula with slice-monogenic kernel

Theorem

Let x, $s \in \mathbb{R}^{n+1}$. Then

$$\sum_{n \geq 0} x^n s^{-1-n} = -(x^2 - 2xs_0 + |s|^2)^{-1}(x - \overline{s})$$

for $|x| < |s|$, where $\overline{s} = s_0 - \underline{s}$.

The Cauchy formula with slice-monogenic kernel

Definition of noncommutative Cauchy kernel

We will call the expression

$$S^{-1}(s,x) = -(x^2 - 2s_0 x + |s|^2)^{-1}(x - \bar{s}), \tag{1}$$

defined for $x^2 - 2s_0 x + |s|^2 \neq 0$, noncommutative Cauchy kernel.

The Cauchy formula with slice-monogenic kernel

Proposition

$x^2 - 2s_0x + |s|^2$ vanishes on the $(n-1)$-sphere

$$[s] = \{y = s_0 + I|\underline{s}|,\ \ I \in \mathbb{S}\}$$

Theorem

The function $S^{-1}(s,x)$ is left s-monogenic in the variable x and right s-monogenic in the variable s in its domain of definition.

The Cauchy formula with slice-monogenic kernel

Let $U \subseteq \mathbb{R}^{n+1}$ be a domain.

- U is an s-domain if $U \cap \mathbb{R}$ is non empty and if $\mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.
- U is axially symmetric if, for all $u + Iv \in U$, the whole (n-1)-sphere $[u + Iv]$ is contained in U.

Representation formula

Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Let f be an s-monogenic function on U. For any $x = x_0 + I_x|\underline{x}| \in U$ the following formula holds:

$$f(x) = \frac{1}{2}\Big[1 - I_x I\Big] f(x_0 + I|\underline{x}|) + \frac{1}{2}\Big[1 + I_x I\Big] f(x_0 - I|\underline{x}|).$$

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The Cauchy formula with slice-monogenic kernel

Cauchy formula with slice monogenic kernel

Let $U \subset \mathbb{R}^{n+1}$ be a bounded axially symmetric s-domain such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Let f be a left s-monogenic function on $W \supset U$, $x \in U$ and set $ds_I = ds/I$, $ds = du + Idv$. Then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S^{-1}(s,x) ds_I f(s) \tag{2}$$

where

$$S^{-1}(s,x) = -(x^2 - 2s_0 x + |s|^2)^{-1}(x - \bar{s})$$

and the integral does not depend on U and on the imaginary unit $I \in \mathbb{S}$.

The Cauchy formula with slice-monogenic kernel

1. G. Gentili, D.C. Struppa, *A new approach to Cullen-regular functions of a quaternionic variable*, C. R. Acad. Sci. Paris, Ser. I, 2006.
2. F.Colombo, I. S., D.C. Struppa, *Slice monogenic functions*, Israel Journal of Mathematics, 2009.
3. F.Colombo, I. S., *A structure formula for slice monogenic functions and some of its consequences* , Hypercomplex Analysis, Trends in Mathematics, Birkhäuser, 2009.
4. F.Colombo, I. S., D.C. Struppa, *Extension properties for slice monogenic functions*, Israel Journal of Mathematics, 2010.
5. F.Colombo, I. S., D. C. Struppa *Duality theorems for slice hyperholomorphic functions*, J. Reine Angew. Math., 2010.
6. R. Ghiloni, A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math. (2011).

The Cauchy formula with slice-monogenic kernel

1. F.Colombo, I. S., D.C. Struppa, *Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions*, Progress in Mathematics, Birkhäuser, Basel, 2011.

Functional calculus for n-tuples of operators

Notation for several non commuting operators

- By V we denote a Banach space over $\mathbb{R}$ with norm $\|\cdot\|$.
- By V_n we denote the two-sided Banach module over $\mathbb{R}_n$ corresponding to $V \otimes \mathbb{R}_n$.
- An element in V_n is of the type $\sum_A v_A \otimes e_A = \sum_A v_A e_A$ (where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index).
- We define $\|v\| = \sum_A \|v_A\|$.

Functional calculus for n-tuples of operators

- $\mathcal{B}(V)$ is the space of bounded $\mathbb{R}$-linear operators from the Banach space V to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$.
- Let $T_j \in \mathcal{B}(V)$, $j = 0, 1, \ldots, n$.
- We define an operator $T = T_0 + \sum_{j=1}^{n} T_j e_j$ and its action on $v = \sum v_B e_B \in V_n$ as $T(v) = \sum_{i,B} T_i(v_B) e_i e_B$.
- The operator T is right-linear and bounded on V_n. We define $\|T\| = \sum_i \|T_i\|_{\mathcal{B}(V)}$.
- $\mathcal{B}^{0,1}(V_n) = \{T \mid T = T_0 + \sum_{j=1}^{n} T_j e_j \mid T_j \in \mathcal{B}(V)\}$.

Functional calculus for n-tuples of operators

Definition

Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \mathbb{R}^{n+1}$. We define the S-resolvent operator series as

$$S^{-1}(s,T) := \sum_{n\geq 0} T^n s^{-1-n} \tag{3}$$

for $\|T\| < |s|$.

Theorem

Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \mathbb{R}^{n+1}$. Then

$$\sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2T\mathrm{Re}[s] + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}), \tag{4}$$

for $\|T\| < |s|$.

Functional calculus for n-tuples of operators

Definition (The S-spectrum and the S-resolvent set)

Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \mathbb{R}^{n+1}$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} \quad : \quad T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I} \quad \textit{is not invertible}\}.$$

The S-resolvent set $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

Definition (The S-resolvent operator)

Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. We define the S-resolvent operator as

$$S^{-1}(s,T) := -(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}).$$

Functional calculus for n-tuples of operators

Theorem

Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. Let $S^{-1}(s, T)$ be the S-resolvent operator. Then $S^{-1}(s, T)$ satisfies the (S-resolvent) equation

$$S^{-1}(s, T)s - TS^{-1}(s, T) = \mathcal{I}.$$

Having in mind the definition of $\sigma_S(T)$ we can state the following result:

Theorem (Structure of the S-spectrum)

Let $T \in \mathcal{B}^{0,1}(V_n)$ and suppose that $p = \mathrm{Re}[p] + \underline{p}$ belongs $\sigma_S(T)$ with $\underline{p} \neq 0$. Then all the elements of the $(n-1)$-sphere $[p]$ belong to $\sigma_S(T)$.

This result implies that if $p \in \sigma_S(T)$ then either p is a real point or the whole $(n-1)$-sphere $[p]$ belongs to $\sigma_S(T)$.

Functional calculus for n-tuples of operators

Theorem (Compactness of S-spectrum)

Let $T \in \mathcal{B}^{0,1}(V_n)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set. Moreover, $\sigma_S(T)$ is contained in $\{s \in \mathbb{R}^{n+1} \ : \ |s| \leq \|T\| \ \}$.

Functional calculus for n-tuples of operators

Definition (Admissible sets U)

We say that $U \subset \mathbb{R}^{n+1}$ is an admissible set if

- it is an axially symmetric s-domain that contains the S-spectrum $\sigma_S(T)$ of $T \in \mathcal{B}^{0,1}(V_n)$,
- $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.

Definition (Locally s-monogenic on $\sigma_S(T)$)

Suppose that U is admissible and $\overline{U}$ is contained in a domain of s-monogenicity of a function f. Then such a function f is said to be locally s-monogenic on $\sigma_S(T)$.
We will denote by $\mathcal{M}_{\sigma_S(T)}$ the set of locally s-monogenic functions on $\sigma_S(T)$.

Functional calculus for n-tuples of operators

Theorem

Let $T \in \mathcal{B}^{0,1}(V_n)$ and $f \in \mathcal{M}_{\sigma_S(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be an admissible set and let $ds_I = ds/I$ for $I \in \mathbb{S}$. Then the integral

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S^{-1}(s, T) \; ds_I \; f(s)$$

does not depend on the open set U nor on the choice of the imaginary unit $I \in \mathbb{S}$.

Functional calculus for n-tuples of operators

Definition (of the S-functional calculus)

Let $T \in \mathcal{B}^{0,1}(V_n)$ and $f \in \mathcal{M}_{\sigma_S(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be an admissible set and let $ds_I = ds/I$ for $I \in \mathbb{S}$. We define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S^{-1}(s, T)\, ds_I\, f(s). \tag{5}$$

Functional calculus for n-tuples of operators

1. F. Colombo, I. S., D.C. Struppa, *A new functional calculus for noncommuting operators*, J. Funct. Anal., **254** (2008), 2255–2274.
2. F. Colombo, I. S., *The Cauchy formula with s-monogenic kernel and a functional calculus for noncommuting operators*, J. Math. Anal. Appl. **373** (2011), 655-679.
   $\Rightarrow$ Cauchy formula allowing to define $f(T)$ in the general case.
3. F. Colombo, I. S. *A structure formula for slice monogenic functions and some of its consequences*, Hypercomplex Analysis, Trends in Mathematics, Birkhäuser, 2009, 69–99.

Some properties of the functional calculus

Theorem

Let $T \in \mathcal{B}^{0,1}(V_n)$.

(a) If f and $g \in \mathcal{M}_{\sigma_S(T)}$ then

$$(f+g)(T) = f(T)+g(T), \quad (f\lambda)(T) = f(T)\lambda, \quad \text{for all } \lambda \in \mathbb{R}_n.$$

(b) If $\phi \in \mathcal{N}_{\sigma(T)}$ and $g \in \mathcal{M}_{\sigma_S(T)}$. Then

$$(\phi g)(T) = \phi(T)g(T).$$

where

$$\mathcal{N}(U) = \{f \in \mathcal{M}(U) \mid f(\mathbb{C}_I) \subseteq \mathbb{C}_I, \forall I \in \mathbb{S}\}.$$

(c) If $f(s) = \sum_{n\geq 0} s^n p_n$, $p_n \in \mathbb{R}_n$, belongs to $\mathcal{M}_{\sigma_S(T)}$, then

$$f(T) = \sum_{n\geq 0} T^n p_n.$$

Some properties of the functional calculus

Theorem (Composition)

Let $T \in \mathcal{B}^{0,1}(V_n)$, $f \in \mathcal{N}_{\sigma_S(T)}$, $\phi \in \mathcal{N}_{\sigma_S(f(T))}$ and let $F(s) = \phi(f(s))$. Then $F \in \mathcal{M}_{\sigma_S(T)}$ and $F(T) = \phi(f(T))$.

The S-Spectral Mapping Theorem

Let $T \in \mathcal{B}^{0,1}(V_n)$, $f \in \mathcal{N}_{\sigma(T)}$, and $\lambda \in \sigma_S(T)$. Then

$$\sigma_S(f(T)) = f(\sigma_S(T)) = \{f(s) : s \in \sigma_S(T)\}.$$

Some properties of the functional calculus

Definition (S-spectral radius of T)

Let $T \in \mathcal{B}^{0,1}(V_n)$. We call S-spectral radius of T the real nonnegative number

$$r_S(T) := \sup\{\, |s| \; : \; s \in \sigma_S(T)\,\}.$$

The S-spectral radius theorem

Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $r_S(T)$ be the S-spectral radius of T. Then

$$r_S(T) = \lim_{m\to\infty} \|T^m\|^{1/m}.$$

Functional calculus for n-tuples of operators

A functional calculus for unbounded operators

Let V be a Banach space and $T = T_0 + \sum_{j=1}^{m} e_j T_j$ where $T_\mu : \mathcal{D}(T_\mu) \to V$ are linear operators for $\mu = 0, 1, ..., n$ where at least one of the T_j's is an unbounded operator. Define the extended S-spectrum as

$$\overline{\sigma}_S(T) := \sigma_S(T) \cup \{\infty\}.$$

Functional calculus for n-tuples of operators

Definition

Let V be a Banach space and V_n be the two-sided Banach module over $\mathbb{R}_n$ corresponding to $V \otimes \mathbb{R}_n$. Let $T_\mu : \mathcal{D}(T_\mu) \subset V \to V$ be linear closed densely defined operators for $\mu = 0, 1, ..., n$. Let

$$\mathcal{D}(T) = \{v \in V_n \ : \ v = \sum_B v_B e_B, \quad v_B \in \bigcap_{\mu=0}^{n} \mathcal{D}(T_\mu) \ \} \tag{6}$$

be the domain of the operator $T = T_0 + \sum_{j=1}^n e_j T_j, \quad T : \mathcal{D}(T) \subset V_n \to V_n$. Let us assume that

1) $\bigcap_{\mu=0}^n \mathcal{D}(T_\mu)$ is dense in V_n,
2) $\mathcal{D}(T^2) \subset \mathcal{D}(T)$ is dense in V_n,

Some properties of the functional calculus

Definition

Consider $k \in \mathbb{R}^{n+1}$ and the homeomorphism

$$\Phi : \overline{\mathbb{R}}^{n+1} \to \overline{\mathbb{R}}^{n+1} \quad \text{for} \quad k \in \mathbb{R}^{n+1}$$

defined by

$$p = \Phi(s) = (s-k)^{-1}, \quad \Phi(\infty) = 0, \quad \Phi(k) = \infty.$$

Some properties of the functional calculus

Definition

Let $T : \mathcal{D}(T) \to V_n$ be a linear closed operator as above with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{M}_{\overline{\sigma}_S(T)}$. Let us consider

$$\phi(p) := f(\Phi^{-1}(p))$$

and the operator

$$A := (T - k\mathcal{I})^{-1}, \quad \text{for some} \quad k \in \rho_S(T) \cap \mathbb{R}.$$

We define

$$f(T) = \phi(A). \tag{7}$$

Preliminaries for the $\mathcal{SC}$-functional calculus

Definition

Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$.

- We say that $S^{-1}(s,x)$ is written in the form I if

$$S^{-1}(s,x) := -(x^2 - 2x\mathrm{Re}[s] + |s|^2)^{-1}(x - \bar{s}).$$

- We say that $S^{-1}(s,x)$ is written in the form II if

$$S^{-1}(s,x) := (s - \bar{x})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-1}.$$

Preliminaries for the $\mathcal{SC}$-functional calculus

Can we substitute T for x in $\mathcal{S}^{-1}(s,x)$ written in the form II? Yes, but we must require that the components of the operator T commute among themselves.

We denote by $\mathcal{BC}^{0,1}(V_n)$ the space of bounded paravector operators with commuting components.

Theorem

Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $s \in \mathbb{R}^{n+1}$. Then

$$\sum_{m\geq 0} T^m s^{-1-m} = (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-1},$$

for $\|T\| < |s|$.

Preliminaries for the $\mathcal{SC}$-functional calculus

Definition (The $\mathcal{F}$-spectrum and the $\mathcal{F}$-resolvent sets)

Let $T \in \mathcal{BC}^{0,1}(V_n)$. We define the $\mathcal{F}$-spectrum of T as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} \quad : \quad s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T} \quad \textit{is not invertible}\}.$$

The $\mathcal{F}$-resolvent set of T is defined by

$$\rho_{\mathcal{F}}(T) = \mathbb{R}^{n+1} \setminus \sigma_{\mathcal{F}}(T).$$

Definition (The $\mathcal{S}_{\mathcal{C}}$-resolvent operator)

Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $s \in \rho_{\mathcal{F}}(T)$. We define the $\mathcal{S}_{\mathcal{C}}$-resolvent operator as

$$\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T) := (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-1}.$$

Definition of the $\mathcal{SC}$-functional calculus

Theorem

Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be admissible domain and let $ds_I = ds/I$ for $I \in \mathbb{S}$. Then the integral

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} {\mathcal{S}_{\mathcal{C}}}^{-1}(s, T)\, ds_I\, f(s) \tag{8}$$

does not depend on the open set U nor on the choice of the imaginary unit $I \in \mathbb{S}$.

Definition of the $\mathcal{SC}$-functional calculus

Definition (of the $\mathcal{S}_{\mathcal{C}}$-functional calculus)

Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be admissible domain and let $ds_I = ds/I$ for $I \in \mathbb{S}$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S}_{\mathcal{C}}{}^{-1}(s, T)\ ds_I\ f(s). \tag{9}$$

Theorem

Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then $\sigma_{\mathcal{F}}(T) = \sigma_S(T)$.

Definition of the $\mathcal{SC}$-functional calculus

Advantage: the $\mathcal{F}$-spectrum is easier to compute than the $\mathcal{S}$-spectrum. $\mathcal{F}$-spectrum takes into account the commutativity of the operators while the $\mathcal{S}$-spectrum does not.
The $\mathcal{F}$-spectrum is related to the $\mathcal{F}$-functional calculus.

- F. Colombo, I.S., *The $\mathcal{F}$-spectrum and the $\mathcal{SC}$-functional calculus*, Proceedings of the Royal Society of Edinburgh, Section A, (2012).

The $\mathcal{SC}$-functional calculus

Proposition

Let f be an holomorphic function in an open set of the upper half complex plane

$$f(x+\iota y) = u(x,y) + \iota v(x,y)$$

$$q = x_0 + ix_1 + jx_2 + kx_3 := x_0 + \underline{q}$$

Then

$$\Delta_4(u(x_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} v(x_0, |\underline{q}|))$$

is Fueter regular, while when $x_0 + \underline{x} \in \mathbb{R}^{n+1}$

$$\Delta_{n+1}^{\frac{n-1}{2}}(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|))$$

is in the kernel of Dirac operator.

The $\mathcal{SC}$-functional calculus

Remark

Even though $S^{-1}(s,x)$ written in the form I is more suitable for several applications, for example for the definition of a functional calculus, it does not allow easy computation of the powers of the Laplacian

$$\Delta = \partial_{x_0}^2 + \sum_{j=1}^{n} \partial_{x_j}^2$$

applied to it. The form II is the one that allows, by iteration, the computation of $\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$.

The $\mathcal{SC}$-functional calculus

Theorem (Explicit computation of $\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$)

Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let $S^{-1}(s,x) = (s-\bar{x})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-1}$ be the slice-monogenic Cauchy kernel and let $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator in the variable x. Then, for $h \geq 1$, we have:

$$\Delta^h S^{-1}(s,x) = C_{n,h}(s-\bar{x})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-(h+1)}.$$

where

$$C_{n,h} := (-1)^h \prod_{\ell=1}^{h}(2\ell) \prod_{\ell=1}^{h}(n-(2\ell-1)).$$

The $\mathcal{SC}$-functional calculus

Theorem

Let n be an odd number and let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function $\Delta^h S^{-1}(s,x)$ is a right s-monogenic function in the variable s, for any h.

Theorem

Let n be an odd number and let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function $\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$ is a monogenic function in the variable x.

The $\mathcal{SC}$-functional calculus

Definition (The $\mathcal{F}_n$-kernel)

Let n be an odd number. Let $x, s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the $\mathcal{F}_n$-kernel as

$$\mathcal{F}_n(s,x) := \Delta^{\frac{n-1}{2}} S^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} (n-1)! \left(\frac{n-1}{2}\right)!.$$

The $\mathcal{SC}$-functional calculus

Theorem (The Fueter mapping theorem in integral form)

Let n be an odd number. Let $W \subset \mathbb{R}^{n+1}$ be an axially symmetric open set and let $f \in \mathcal{M}(W)$. Let U be a bounded axially symmetric open set such that $\overline{U} \subset W$. Suppose that the boundary of $U \cap L_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $x \in U$, the function $\breve{f}(x)$ given by

$$\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$$

is monogenic and it admits the integral representation

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap L_I)} \mathcal{F}_n(s,x) ds_I f(s), \quad ds_I = ds/I, \tag{10}$$

where the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}$.

The $\mathcal{SC}$-functional calculus

Definition ($\mathcal{F}$-resolvent operator)

Let n be an odd number, $s \in \mathbb{R}^{n+1}$ and let $T \in \mathcal{BC}_n^{0,1}(V_n)$. For $s \in \rho_{\mathcal{F}}(T)$ we define the $\mathcal{F}$-resolvent operator by

$$\mathcal{F}_n(s, T) := \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}}.$$

The $\mathcal{SC}$-functional calculus

Definition

Let $T \in \mathcal{BC}^{0,1}(V_n)$ and let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain that contains the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$, such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Let W be an open set in $\mathbb{R}^{n+1}$. A function $f \in \mathcal{SM}(W)$ is said to be locally s-monogenic on $\sigma_{\mathcal{F}}(T)$ if there exists a domain $U \subset \mathbb{R}^{n+1}$, as above and such that $\overline{U} \subset W$, on which f is s-monogenic. We will denote by $\mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ the set of locally s-monogenic functions on $\sigma_{\mathcal{F}}(T)$.

The $\mathcal{SC}$-functional calculus

Definition (The $\mathcal{F}$-functional calculus)

Let n be an odd number, $T \in \mathcal{BC}^{0,1}(V_n)$. Let U be an open set, containing $\sigma_{\mathcal{F}}(T)$, as above. Suppose that $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. We define the $\mathcal{F}$-functional calculus as

$$\breve{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n(s, T)\, ds_I\, f(s).$$

F. Colombo, I.S., F. Sommen *The Fueter mapping theorem in integral form and the $\mathcal{F}$-functional calculus*, Mathematical Methods in the Applied Sciences, (2010).