The quaternionic functional calculus and the evolution operator

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Motivations

Evolution equation (in the abstract setting)

Consider $u : [0, T] \to X$ and the evolution equation

$$u'(t) = Bu(t) + f(t), \quad t > 0, \qquad u(0) = u_0$$

where $B : \mathcal{D}(B) \subset X \to X$ is a suitable closed operator in a Banach space X, where $u_0 \in X$.

Motivations

Evolution operator

Under suitable assumptions on the spectrum of B and on the resolvent operator $(\lambda I - B)^{-1}$, by the Dunford integral we have

$$e^{tB} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - B)^{-1} \, e^{t\lambda} \, d\lambda$$

where Γ is a suitable curve that surrounds the spectrum of B. Under appropriate hypothesis the unique solution is given by (the so called variation of constants formula)

$$u(t) = e^{tB} u_0 + \int_0^t e^{(t-s)B} f(s) ds.$$

Motivations

A nonlinear equation

The study of the semigroup e^{tB} and the variation of constants formula are important to face nonlinear problems

$$u'(t) = Bu(t) + g(u(t)), \quad t > 0, \qquad u(0) = u_0$$

by fixed point arguments:

$$u(t) = G(u(t))$$

where

$$G(u(t)) := e^{tB}u_0 + \int_0^t e^{(t-s)B} g(u(s))ds$$

we can obtain existence and uniqueness of the solution of the nonlinear problem.

Motivations

The mathematical motivation is to extend the classical functional calculus to quaternionic operators:

Functional calculus in the complex case

$$f(B) = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathcal{I} - B)^{-1} f(\lambda)\, d\lambda,$$

where Γ is a rectifiable Jordan curve that surrounds the spectrum of B.

Evolution operator

$$e^{tB} = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathcal{I} - B)^{-1} e^{t\lambda}\, d\lambda,$$

where Γ is a rectifiable Jordan curve that surrounds the spectrum of B.

Motivations

Quaternionic Schrödinger equation

1. For the quaternionic Schrödinger equation

$$\frac{d}{dt}\psi = \mathcal{H}\psi, \qquad \psi(0) = \psi_0$$

   where $\mathcal{H}$ is a quaternionic linear operator and ψ is the quaternionic wave function, we have the problem to define $e^{t\mathcal{H}}$.

2. For quaternionic quantum mechanics see: S. Adler, *Quaternionic Quantum Field Theory*, Oxford University Press, 1995.

Motivations

Strongly continuous and uniformly continuous semigroup

A family $\{E(t)\}_{t\geq 0}$ of bounded linear operators on a complex Banach space X is called strongly continuous semigroup if:

1. $$E(t+\tau)=E(t)E(\tau), \quad \textit{for} \quad t,\tau\geq 0,$$
2. $$E(0)=\mathcal{I},$$
3. and for every $x\in X$, $E(t)x$ is continuous in $t\in[0,\infty]$.
4. If, in addition, the map
$$t\mapsto E(t)$$
is continuous in the uniform operator topology, the family $\{E(t)\}_{t\geq 0}$ is called a uniformly continuous semigroup in the space of all bounded linear operators.

Motivations

Theorem ($E(t)$ continuous in the uniformly operator topology)

- If $E(t)$ is continuous in the uniform operator topology, then there exists a bounded linear operator B such that $E(t) = e^{tB}$.
- Conversely, (from the Riesz-Dunford functional calculus) for any bounded linear operator B, the operator e^{tB} is a uniformly continuous semigroup.

Moreover, the generator

$$B = \lim_{h \to 0} (E(h) - I)/h$$

and the Laplace transform of the semigroup, for $Re[\lambda]$ suitable large is

$$(\lambda I - B)^{-1} = \int_0^\infty e^{-\lambda t} \ E(t) dt.$$

Motivations

If $E(t)$ continuous in the strong operator topology, then the problem becomes more difficult to study. In an appropriate sense e^{tB} still equals $E(t)$, but now B is an unbounded linear operator.

Theorem Hille-Phillips-Yosida ($E(t)$ continuous in the strong operator topology)

A closed linear operator B with dense domain is the infinitesimal generator of a strongly continuous semigroup
if and only if
there exist real numbers $M > 0$ and ω such that for every real number $\lambda > \omega$, λ is in the resolvent set of B and

$$\|(\lambda I - B)^{-n}\| \leq M(\lambda - \omega)^{-n},$$

for $n \in \mathbb{N}$.

Motivations

Problems related to the Fueter's theory

- The Fueter operator

$$\frac{\partial}{\partial \overline{q}} = \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right).$$

- The exponential function

$$f(q) = e^q$$

is not Fueter regular.

- The Cauchy-Fueter kernel

$$\mathcal{G}(q) = \frac{\bar{q}}{|q|^4} = \frac{q^{-1}}{|q|^2} = q^{-2}\overline{q}^{-1}.$$

Motivations

Problems associated to the Fueter's theory

- Power series expansion of the kernel (holds for $|p| < |q|$) is

$$\mathcal{G}(q,p) := \mathcal{G}(q-p) = \sum_{n\geq 0} \sum_{\nu\in\sigma_n} P_\nu(p)\mathcal{G}_\nu(q).$$

- In the case in which the components of T commute, the sum $\mathcal{G}(q,T)$ is

$$\mathcal{G}(q,T) = (q\mathcal{I} - T)^{-2}(\overline{q\mathcal{I} - T})^{-1}.$$

- What is the sum $\mathcal{G}(q,T)$ of

$$\mathcal{G}(q,p) = \sum_{n\geq 0} \sum_{\nu\in\sigma_n} P_\nu(p)\mathcal{G}_\nu(q)$$

  when one replaces p by operator T with noncommuting components?

The Cauchy formulas for slice regular functions

- Denote by $\mathbb{H}$ the algebra of real quaternions.
- $\mathbb{S}$ is the sphere of unit purely imaginary quaternions, i.e.

$$\mathbb{S} = \{q = x_1 i + j x_2 + k x_3 \in \mathbb{H} \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

- $L_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I.

Axially symmetric domain

We say that $U \subseteq \mathbb{H}$ is an *axially symmetric* domain if, for all $u + Iv \in U$, the whole 2-sphere $u + v\mathbb{S}$ is contained in U.

Slice domain

Let $U \subseteq \mathbb{H}$ be a domain. We say that U is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is non empty and if $L_I \cap U$ is a domain in L_I for all $I \in \mathbb{S}$.

The Cauchy formulas for slice regular functions

Definition of slice regular functions

Let $U \subseteq \mathbb{H}$, $f: \ U \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $L_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element on L_I.
We say that f is a left slice regular function if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

We say that f is right slice regular function if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy) I\right) = 0.$$

The Cauchy formulas for slice regular functions

The Cauchy formula (the left case)

Let $\overline{U} \subset W$ be an axially symmetric s-domain, and let $\partial(U \cap L_I)$ be the union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Set $ds_I = ds/I$. Let f be a left regular function on $W \subset \mathbb{H}$. Then, if $q \in U$, we have

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S_L^{-1}(s,q) ds_I f(s)$$

and the integrals do not depend on the choice of the imaginary unit $I \in \mathbb{S}$ and on U, where

$$S_L^{-1}(s,q) := -(q^2 - 2q\mathrm{Re}[s] + |s|^2)^{-1}(q - \bar{s}).$$

The Cauchy formulas for slice regular functions

The Cauchy formula (the right case)

Let f be a right regular function on $W \subset \mathbb{H}$. Then, if $q \in U$, we have

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} f(s) ds_I S_R^{-1}(s,q)$$

and the integrals do not depend on the choice of the imaginary unit $I \in \mathbb{S}$ and on U, where

$$S_R^{-1}(s,q) := -(q-\bar{s})(q^2 - 2q\mathrm{Re}[s] + |s|^2)^{-1}.$$

The quaternionic functional calculus for bounded operators: preliminaries

Left and right slice regular functions and left and right linear quaternionic operators

- There are four case of interest: left and right slice regular functions and left and right linear quaternionic operators.
- We will consider only two sided vector spaces V.
- A map $T : V \to V$ is said to be a **right linear** operator if $T(u+v) = T(u) + T(v)$, $T(us) = T(u)s$, for all $s \in \mathbb{H}$ and for all $u, v \in V$.
- A map $T : V \to V$ is said to be a **left linear** operator if $T(u+v) = T(u) + T(v)$, $T(su) = sT(u)$, for all $s \in \mathbb{H}$ and for all $u, v \in V$.

The quaternionic functional calculus for bounded operators: preliminaries

Left and right linear operators act in an opposite way

- The composition of left and right linear operators act in an opposite way with respect to the composition of maps.
- If T and T' are right linear

$$TT'v = T(T'v).$$

- If T and T' are left linear

$$vTT' = (vT)T'.$$

- This has important consequences for unbounded operators in the definition of the S-resolvent operators.

The quaternionic functional calculus for bounded operators: preliminaries

Quaternionic functional spaces

- V is a bilateral quaternionic Banach space,
- $\mathcal{B}^R(V)$ right linear bounded operators on V,
- $\mathcal{B}^L(V)$ left linear bounded operators on V,
- $\mathcal{B}(V)$ when we do not specify between left or right linear bounded operators.

The Cauchy kernel operator series

Definition of the Cauchy kernel operator series

Let $T \in \mathcal{B}(V)$ and assume for $\|T\| < |s|$.
The left Cauchy kernel operator series is

$$S_L^{-1}(s, T) = \sum_{n \geq 0} T^n s^{-1-n}.$$

The right Cauchy kernel operator series is

$$S_R^{-1}(s, T) = \sum_{n \geq 0} s^{-1-n} T^n.$$

The Cauchy kernel operator series

- It is important to note the $S_L^{-1}(s,T)$ and $S_R^{-1}(s,T)$ are formally the same operators used for right linear operators or for left linear operators. They simply act in a different way.
- We do not require that the components of T commute.

Theorem (The sum of the series)

Let $T \in \mathcal{B}(V)$ and assume $\|T\| < |s|$. Then we have

$$\sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}),$$

and

$$\sum_{n\geq 0} s^{-1-n} T^n = -(T - \bar{s}\mathcal{I})(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}.$$

The Cauchy kernel operator series

Definition of the S-spectrum $\sigma_S(T)$

Let $T \in \mathcal{B}(V)$. The S-spectrum $\sigma_S(T)$ of T is

$$\sigma_S(T) = \{s \in \mathbb{H} \quad : \quad T^2 - 2\, Re[s] T + |s|^2 \mathcal{I} \quad \text{is not invertible}\}.$$

The S-resolvent set $\rho_S(T)$

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

The Cauchy kernel operator series

Definition of the S-resolvent operators

For $s \in \rho_S(T)$ we define the left S-resolvent operator

$$S_L^{-1}(s, T) := -(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}),$$

and the right S-resolvent operator as

$$S_R^{-1}(s, T) := -(T - \bar{s}\mathcal{I})(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}.$$

The Cauchy kernel operator series

The S-resolvent equations

The left S-resolvent operator satisfies

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I},$$

and the right S-resolvent operator satisfies

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}.$$

The Cauchy kernel operator series

Admissible domains

- Let $T \in \mathcal{B}(V)$ and $U \subset \mathbb{H}$ be an axially symmetric s-domain that contains the S-spectrum $\sigma_S(T)$ such that $\partial(U \cap L_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.

The Cauchy kernel operator series

Locally regular functions on $\sigma_S(T)$

- We denote by $\mathcal{R}^L_{\sigma_S(T)}$ the set of functions locally left regular on $\sigma_S(T)$.
- We denote by $\mathcal{R}^R_{\sigma_S(T)}$ the set of functions locally right regular on $\sigma_S(T)$.

The quaternionic functional calculus for bounded operators

Crucial theorem

Let $U \subset \mathbb{H}$ be a domain as above and set $ds_I = ds/I$. Then

$$\int_{\partial(U\cap L_I)} S_L^{-1}(s,T)\ ds_I\ f(s), \quad for \quad f \in \mathcal{R}^L_{\sigma_S(T)},$$

and

$$\int_{\partial(U\cap L_I)} f(s)\ ds_I\ S_R^{-1}(s,T), \quad for \quad f \in \mathcal{R}^R_{\sigma_S(T)}.$$

do not depend on the open set U nor on the imaginary unit $I \in \mathbb{S}$.

Proof

It is based on the Cauchy formulas and on the quaternionic version of the Hahn-Banach theorem.

The quaternionic functional calculus for bounded operators

Definition of the quaternionic functional calculi (bounded operators)

Let $U \subset \mathbb{H}$ be a domain as above and set $ds_I = ds/I$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S_L^{-1}(s,T)\, ds_I\, f(s), \quad \textit{for} \quad f \in \mathcal{R}^L_{\sigma_S(T)},$$

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} f(s)\, ds_I\, S_R^{-1}(s,T), \quad \textit{for} \quad f \in \mathcal{R}^R_{\sigma_S(T)}.$$

When $T \in \mathcal{B}^L(V)$ we have $f(T)(v) = vf(T)$ while if $T \in \mathcal{B}^R(V)$ we have $f(T)(v) = f(T)v$.

Functional calculus for unbounded linear quaternionic operators

The class $\mathcal{K}(V)$ of closed operators with dense domain

- We denote by $\mathcal{K}^R(V)$ the set of right linear closed densely defined operators $T : \mathcal{D}(T) \subset V \to V$, such that:
  1. $\mathcal{D}(T)$ is dense in V,
  2. $\mathcal{D}(T^2) \subset \mathcal{D}(T)$ is dense in V.
- We denote by $\mathcal{K}^L(V)$ the analogous set of densely defined left linear closed operators.
- We use the symbol $\mathcal{K}(V)$ when we do not distinguish between $\mathcal{K}^L(V)$ and $\mathcal{K}^R(V)$.

Functional calculus for unbounded linear quaternionic operators

The S-resolvent and the S-spectrum sets

- We denote by $\rho_S(T)$ the S-resolvent set of T as

$$\rho_S(T) = \{ s \in \mathbb{H} \ : \ (T^2 - 2T\mathrm{Re}[s] + |s|^2\mathcal{I})^{-1} \in \mathcal{B}(V) \ \ \}.$$

- We define the S-spectrum $\sigma_S(T)$ of T as

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$

Functional calculus for unbounded linear quaternionic operators

The problem related to the resolvent operators

The definition of the S-resolvent operators S_L^{-1}, S_R^{-1} relies on a deep difference between the case of left and right unbounded linear operators.

- Let $T \in \mathcal{K}(V)$ and $s \in \rho_S(T)$. We denote by $Q_s(T)$ the operator:

$$Q_s(T) := (T^2 - 2T\mathrm{Re}[s] + |s|^2\mathcal{I})^{-1} : \quad V \to \mathcal{D}(T^2).$$

- For $s \in \rho_S(T)$, the left S-resolvent operator used in the bounded case, that is:

$$S_L^{-1}(s, T) = -Q_s(T)(T - \bar{s}\mathcal{I}),$$

and observe that in the case of right linear unbounded operators turns out to be defined only on $\mathcal{D}(T)$ while in the case of left linear unbounded operators it is defined on V.

Functional calculus for unbounded linear quaternionic operators

The S-resolvent on V

- This fact is due to the presence of the term $Q_s(T)T$.
- However, for $T \in \mathcal{K}^R(V)$, observe that the operator $Q_s(T)T$ is the restriction to the dense subspace $\mathcal{D}(T)$ of V of a bounded linear operator defined on V.
- This fact follows by the commutation relation $Q_s(T)Tv = TQ_s(T)v$ which holds for all $v \in \mathcal{D}(T)$ since the polynomial operator $T^2 - 2T\mathrm{Re}[s] + |s|^2\mathcal{I} : \mathcal{D}(T^2) \to V$ has real coefficients.
- More precisely, for $T \in \mathcal{K}^R(V)$, we have $TQ_s(T) : V \to \mathcal{D}(T)$ and it is continuous for $s \in \rho_S(T)$.

Functional calculus for unbounded linear quaternionic operators

Definition (The S-resolvent operators for $T \in \mathcal{K}^R(V)$)

- Definition of the left S-resolvent operator

$$S_L^{-1}(s,T)v := -Q_s(T)(T-\bar{s}\mathcal{I})v, \quad \textit{for all} \quad v \in \mathcal{D}(T),$$

- and we will call

$$\hat{S}_L^{-1}(s,T)v = Q_s(T)\bar{s}v - TQ_s(T)v, \quad \textit{for all} \quad v \in V,$$

  the extended left S-resolvent operator.

- Definition of the right S-resolvent operator

$$S_R^{-1}(s,T)v := -(T-\mathcal{I}\bar{s})Q_s(T)v, \quad \textit{for all} \quad v \in V.$$

Functional calculus for unbounded linear quaternionic operators

Definition (The S-resolvent operators for $T \in \mathcal{K}^L(V)$)

- We define the left S-resolvent operator as

$$vS_L^{-1}(s,T) := -vQ_s(T)(T - \overline{s}\mathcal{I}), \quad \textit{for all} \quad v \in V.$$

- We define the right S-resolvent operator as

$$vS_R^{-1}(s,T) := -v(T - \mathcal{I}\overline{s})Q_s(T), \quad \textit{for all} \quad v \in \mathcal{D}(T),$$

- and we will call

$$v\hat{S}_R^{-1}(s,T) = vQ_s(T)\overline{s} - vQ_s(T)T, \quad \textit{for all} \quad v \in V,$$

the extended right S-resolvent operator.

Functional calculus for unbounded linear quaternionic operators

Definition

Let A be an operator containing the term $Q_s(T)T$ (resp. $TQ_s(T)$). We define $\hat{A}$ to be the operator obtained from A by substituting each occurrence of $Q_s(T)T$ (resp. $TQ_s(T)$) by $TQ_s(T)$ (resp. $Q_s(T)T$).

A second difference between the left and the right functional calculus are the S-resolvent equations which, to hold on V, need different extensions of the operators involved.

Functional calculus for unbounded linear quaternionic operators

Theorem (The S-resolvent equations)

If $T \in \mathcal{K}^R(V)$ and $s \in \rho_S(T)$, then the left S-resolvent operator satisfies the equation

$$\hat{S}_L^{-1}(s,T)sv - T\hat{S}_L^{-1}(s,T)v = \mathcal{I}v, \quad \textit{for all} \quad v \in V.$$

$$sS_R^{-1}(s,T)v - (S_R^{-1}\widehat{(s,T)}T)v = \mathcal{I}v, \quad \textit{for all} \quad v \in V.$$

If $T \in \mathcal{K}^L(V)$ and $s \in \rho_S(T)$, then the left S-resolvent operator satisfies the equation

$$v\hat{S}_L^{-1}(s,T)s - v\widehat{TS_L^{-1}}(s,T) = v\mathcal{I}, \quad \textit{for all} \quad v \in V.$$

$$vs\hat{S}_R^{-1}(s,T) - v(\hat{S}_R^{-1}(s,T)T) = v\mathcal{I}, \quad \textit{for all} \quad v \in V.$$

Functional calculus for unbounded linear quaternionic operators

Definition (The functional calculus for linear closed quaternionic operators)

- Let $T : \mathcal{D}(T) \to V$ be a linear closed densely defined operator.
- Assume f defined on the extended S-spectrum $\overline{\sigma}_S(T) := \sigma_S(T) \cup \{\infty\}$.
- $p = \Phi(s) = (s-k)^{-1}$, $\Phi(\infty) = 0$, $\Phi(k) = \infty$.
- $\phi(p) := f(\Phi^{-1}(p))$ and

$$A := (T - k\mathcal{I})^{-1}, for \ \ some \ \ k \in \rho_S(T) \cap \mathbb{R} \neq 0.$$

- The functional calculus $f(T)$ is defined as follows:

$$f(T) = \phi(A).$$

Functional calculus for unbounded linear quaternionic operators

Theorem (for $T \in \mathcal{K}^R(V)$)

- If $T \in \mathcal{K}^R(V)$ with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. Then the operator $f(T)$, is independent of $k \in \rho_S(T) \cap \mathbb{R}$, and, for $f \in \mathcal{R}^L_{\sigma_S(T)}$ and $v \in V$, we have

$$f(T)v = f(\infty)\mathcal{I}v + \frac{1}{2\pi} \int_{\partial(W\cap L_I)} \hat{S}_L^{-1}(s,T)\, ds_I\, f(s)v,$$

and for $f \in \mathcal{R}^R_{\sigma_S(T)}$ and $v \in V$, we have

$$f(T)v = f(\infty)\mathcal{I}v + \frac{1}{2\pi} \int_{\partial(W\cap L_I)} f(s)\, ds_I\, S_R^{-1}(s,T)v.$$

Functional calculus for unbounded linear quaternionic operators

Theorem (for $T \in \mathcal{K}^L(V)$)

- If $T \in \mathcal{K}^L(V)$ with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. Then the operator $f(T)$ is independent of $k \in \rho_S(T) \cap \mathbb{R}$, and, for $f \in \mathcal{R}^L_{\sigma_S(T)}$ and $v \in V$, we have

$$vf(T) = vf(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(W \cap L_I)} v\, S_L^{-1}(s,T)\; ds_I\; f(s),$$

and for $f \in \mathcal{R}^R_{\sigma_S(T)}$ and $v \in V$, we have

$$vf(T) = vf(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(W \cap L_I)} v\, f(s)\; ds_I\; \hat{S}_R^{-1}(s,T).$$

Quaternionic semigroups

Quaternionic semigroup

A family $\{\mathcal{U}(t)\}_{t\geq 0}$ of linear bounded quaternionic operators in V will be called a *strongly continuous quaternionic semigroup* if

1. $\mathcal{U}(t+\tau) = \mathcal{U}(t)\mathcal{U}(\tau), \qquad t, \tau \geq 0,$
2. $\mathcal{U}(0) = \mathcal{I},$
3. for every $v \in V$, $\mathcal{U}(t)v$ is continuous in $t \in [0, \infty]$.
4. If, in addition, the map $t \to \mathcal{U}(t)$ is continuous in the uniform operator topology, then the family $\{\mathcal{U}(t)\}_{t\geq 0}$ is called a *uniformly continuous quaternionic semigroup* in $\mathcal{B}(V)$.

Quaternionic semigroups

Theorem (uniformly continuous quaternionic semigroup)

Let $\{\mathcal{U}(t)\}_{t\geq 0}$ be a uniformly continuous quaternionic semigroup in $\mathcal{B}(V)$. Then:

1. there exists a bounded linear quaternionic operator T such that $\mathcal{U}(t) = e^{t\,T}$;
2. the quaternionic operator T is given by the formula

$$T = \lim_{h\to 0} \frac{\mathcal{U}(h) - \mathcal{U}(0)}{h};$$

3. we have the relation:

$$\frac{d}{dt} e^{t\,T} = T\, e^{t\,T} = e^{t\,T}\, T.$$

Quaternionic semigroups

Theorem: Laplace transform for T bounded

Let $T \in \mathcal{B}(V)$ and let $s_0 > \|T\|$. Then the right S-resolvent operator $S_R^{-1}(s,T)$ is given by

$$S_R^{-1}(s,T) = \int_0^{+\infty} e^{-t\,s}\, e^{t\,T}\, dt.$$

Let $T \in \mathcal{B}(V)$ and let $s_0 > \|T\|$. Then the left S-resolvent operator $S_L^{-1}(s,T)$ is given by

$$S_L^{-1}(s,T) = \int_0^{+\infty} e^{tT}\, e^{-ts}\, dt.$$

Characterization results: Hille-Phillips-Yosida in the quaternionic setting

Theorem: Characterization result

Let $\mathcal{U}(t)$ be a quaternionic semigroup on a quaternionic Banach space V. Then $\mathcal{U}(t)$ has a bounded infinitesimal quaternionic generator if and only if it is uniformly continuous.

The proof is based on the principle of uniform boundedness.

Characterization results: Hille-Phillips-Yosida in the quaternionic setting

The Laplace transforms are an important tool to prove the Characterization result.

Theorem: Laplace transform for T unbounded

Let $\mathcal{U}(t)$ be a strongly continuous quaternionic semigroup and set

$$\omega_0 := \lim_{t \to +\infty} \frac{1}{t} \ln \|\mathcal{U}(t)\|.$$

Assume that $\mathcal{U}(t)$ is generated by a linear quaternionic operator T and take $s \in \mathbb{H}$ such that $Re[s] > \omega_0$. Then we have that $s \in \rho_S(T)$ and the left extended S-resolvent operator is given by

$$\widehat{S}_L^{-1}(s, T)v = \int_0^{\infty} \mathcal{U}(t)\, e^{-ts}\, v\, dt, \quad v \in V.$$

Characterization results: Hille-Phillips-Yosida in the quaternionic setting

Theorem (Hille-Yosida-Phillips: necessary condition)

T is a closed linear quaternionic operator with dense domain whose S-spectrum lies in the half space $Re[s] \leq \omega$, where $\omega \in \mathbb{R}$. $\mathcal{U}(t)$, for $t \geq 0$, is a strongly continuous semigroup. Assume that there exist $M > 0$ and $\omega \in \mathbb{R}$ such that: $\|\mathcal{U}(t)\| \leq Me^{\omega t}, \quad t \geq 0$, and

$$\widehat{S}_L^{-1}(s, T)v = \int_0^\infty \mathcal{U}(t)\, e^{-st}\, v\, dt, \quad v \in V.$$

Then we have the following estimate

$$\|\sum_{n=0}^{n} \binom{n}{k} T^{n-k} Q_s(T)^n (\overline{s})^k\| \leq \frac{M}{(Re[s] - \omega)^n}, \quad n \in \mathbb{N}.$$

Characterization results: Hille-Phillips-Yosida in the quaternionic setting

Theorem (Hille-Yosida-Phillips: sufficient condition)

If there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for every real number $s_0 > \omega$, with $s_0 \in \rho_S(T)$, we have

$$\|(s_0\mathcal{I} - T)^{-n}\| \leq \frac{M}{(s_0 - \omega)^n}, \quad n \in \mathbb{N},$$

then the closed linear quaternionic operator T, with dense domain, is the infinitesimal generator of a strongly continuous semigroup.

Concluding remarks

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In the classical case of a complex unbounded linear operator $B : \mathcal{D}(B) \subset X \to X$, where X is a complex Banach space, the resolvent operator

$$R(\lambda, B) := (\lambda\mathcal{I} - B)^{-1}, \quad \text{for} \quad \lambda \in \rho(B),$$

satisfies the following relations:

$$(\lambda\mathcal{I} - B)R(\lambda, B)x = x, \quad \text{for all} \quad x \in X,$$

$$R(\lambda, B)(\lambda\mathcal{I} - B)x = x, \quad \text{for all} \quad x \in \mathcal{D}(B).$$

Concluding remarks

Operators $S_L(s,T)$, $S_L^{-1}(s,T)$ and $S_R(s,T)$, $S_R^{-1}(s,T)$ on V and on $\mathcal{D}(T)$

We study what happens in the quaternionic case for unbounded operators. The analogue of $\lambda\mathcal{I} - B$, associated to the left S-resolvent operator, is defined by

$$S_L(s,T) = (T - \bar{s}\mathcal{I})^{-1}\, s\, (T - \bar{s}\mathcal{I}) - T$$

for those $\bar{s} \in \mathbb{H}$ such that $(T - \bar{s}\mathcal{I})^{-1}$ is a bounded operator. Observe that for the operator $S_L(s,T)$ the following identity

$$(T - \bar{s}\mathcal{I})^{-1}\, s\, (T - \bar{s}\mathcal{I}) - T = -(T - \bar{s}\mathcal{I})^{-1}(T^2 - 2s_0 T + |s|^2\mathcal{I})$$

holds for bounded operators.

Concluding remarks

Definition for $T \in \mathcal{K}^R(V)$

Take $\bar{s} \in \mathbb{H}$ such that $(T - \bar{s}\mathcal{I})^{-1}$ is a bounded operator. Let $T \in \mathcal{K}^R(V)$. Then we define

$$S_L(s,T)v := -(T - \bar{s}\mathcal{I})^{-1}(T^2 - 2s_0 T + |s|^2\mathcal{I})v : \quad v \in \mathcal{D}(T^2)$$

$$\hat{S}_L(s,T)v := [(T - \bar{s}\mathcal{I})^{-1} s\, (T - \bar{s}\mathcal{I}) - T]v : \quad v \in \mathcal{D}(T)$$

where, with an abuse of notation we have denoted by $\hat{S}_L(s,T)$ the extension of $S_L(s,T)$ on $\mathcal{D}(T)$. Moreover, we set

$$S_R(s,T)v := [(T - \mathcal{I}\bar{s})\, s\, (T - \mathcal{I}\bar{s})^{-1} - T]v, \quad v \in \mathcal{D}(T).$$

Concluding remarks

Definition for $T \in \mathcal{K}^L(V)$.

Let $T \in \mathcal{K}^L(V)$. Take $\overline{s} \in \mathbb{H}$ such that $(T - \overline{s}\mathcal{I})^{-1}$ is a bounded operator.
Then we define

$$vS_L(s, T) := v[(T - \overline{s}\mathcal{I})^{-1} s (T - \overline{s}\mathcal{I}) - T], \quad v \in \mathcal{D}(T).$$

Moreover, we set

$$v\hat{S}_R(s, T)v := v[(T - \mathcal{I}\overline{s}) s (T - \mathcal{I}\overline{s})^{-1} - T], \quad v \in \mathcal{D}(T)$$

where, with an abuse of notation we have denoted by $\hat{S}_R(s, T)$ the extension of $S_R(s, T)$ on $\mathcal{D}(T)$.

Concluding remarks

Theorem

Take $\bar{s} \in \mathbb{H}$ such that $(T - \bar{s}\mathcal{I})^{-1}$ is a bounded operator and $s \in \rho_S(T)$.

- Let $T \in \mathcal{K}^R(V)$. Then we have

$$\hat{S}_L(s,T)\hat{S}_L^{-1}(s,T)v = \mathcal{I}v, \quad \textit{for all} \quad v \in V,$$

  and

$$S_R(s,T)S_R^{-1}(s,T)v = \mathcal{I}v, \quad \textit{for all} \quad v \in V,$$

- Let $T \in \mathcal{K}^L(V)$. Then we have

$$vS_L(s,T)S_L^{-1}(s,T) = v\mathcal{I}, \quad \textit{for all} \quad v \in V,$$

  and

$$v\hat{S}_R(s,T)\hat{S}_R^{-1}(s,T) = v\mathcal{I}, \quad \textit{for all} \quad v \in V,$$

References on which my talk is based

- F.C., G. Gentili, I. Sabadini, D.C. Struppa, *Non commutative functional calculus: bounded operators*, Complex Analysis and Operator Theory, **4** (2010), 821–843.
- F. C., I. Sabadini, *On some properties of the quaternionic functional calculus*, Journal Geometric Analysis, **19** (2009), 601-627.
- F. C., I. Sabadini, *On the formulations of the quaternionic functional calculus*, Journal of Geometry and Physics, **60** (2010), 1490–1508
- F. C., I. Sabadini, *The quaternionic evolution operator*, Advances in Mathematics, **227** (2011), 1772–1805.
- F. C. I. Sabadini, D. C. Struppa, *Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions*, Progress in Mathematics, Vol. 289, Birkhäuser, 2011.