Multivariable Nevanlinna-class functions: positive kernels and transfer-function realizations

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Outline of topics

Kinds of functions

Schur-Agler class over $\mathbb{D}^d$
Herglotz-Agler class over $\mathbb{D}^d$
Schur-Agler class over $(\mathbb{C}_+)^d$
Bessmertnyi class over $\cup_{\lambda \in \mathbb{T}} (\lambda \mathbb{C}_+)^d$
Herglotz-Agler class over $(\mathbb{C}_+)^d$

Goal:

Definitions and realization theory, general and rational case
Flavors of realization: conservative, energy-conserving, co-energy-conserving; dissipative

Two types of domains:

$\Omega = \mathbb{D}^d$: discrete-time linear systems
$\Omega = (\mathbb{C}_+)^d$: continuous-time linear systems
$d > 1 \Rightarrow$ multidimensional linear systems

Schur-Agler class over $\mathbb{D}^d$

Theorem (Agler 1990):

Given $S \colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$, T.F.A.E.:

1. $S \in \mathcal{SA}_d(\mathcal{U}, \mathcal{Y})$: $S \colon \mathbb{D}^d \underset{holo}{\to} \mathcal{L}(\mathcal{U}, \mathcal{Y})$ &
   $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{CBL}(\mathcal{K})^d \Rightarrow \|S(\mathbf{T})\| \le 1$.
2. S has an Agler decomposition: there exist positive kernels $K_k(z, w)$ on $\mathbb{D}^d$ with
   $I - S(z)S(w)^* = \sum_{k=1}^d (1 - z_k \overline{w_k}) K_k(z, w)$
3. S has a unitary/conservative Givone-Roesser realization: there exist state space $\mathcal{X}$ with spectral resolution $\{P_1, \dots, P_d\}$, and unitary $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X} \oplus \mathcal{Y}$ so that

$$S(z) = D + C(I - Z(z)A)^{-1} Z(z) B$$

where $Z(z) = z_1 P_1 + \cdots + z_d P_d$

Connection with multidimensional system theory

Transfer function of Givone-Roesser system

$S(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ = transfer function of GR system

$$\begin{bmatrix} x_1(n+\mathbf{e}_1) \\ \vdots \\ x_d(n+\mathbf{e}_d) \end{bmatrix} = A \begin{bmatrix} x_1(n) \\ \vdots \\ x_d(n) \end{bmatrix} + Bu(n)$$

$$y(n) = C \begin{bmatrix} x_1(n) \\ \vdots \\ x_d(n) \end{bmatrix} + Du(n)$$

where $n \in \mathbb{Z}_+^d$. $\mathbf{e}_k = (0, \dots, 1, \dots, 0)$ standard basis vectors $(k = 1, \dots, d)$

Sketch of proof

(1) $\Rightarrow$ (2): Cone-separation argument

(2) $\Rightarrow$ (3): Lurking-isometry argument

Write $K_k(z,w) = H_k(z)H_k(w)^*$ so Agler decomposition becomes
$I - S(z)S(w)^* = \sum_{k=1}^d (1 - z_k \overline{w_k}) H_k(z) H_k(w)^* \Leftrightarrow$
$\langle \sum_{k=1}^d z_k \overline{w_k} H_k(z) H_k(w)^* y_w, y_z \rangle + \langle y_w, y_z \rangle =$
$\langle \sum_{k=1}^d H_k(z) H_k(w)^* y_w, y_z \rangle + \langle S(z)S(w)^* y_w, y_z \rangle \Leftrightarrow$
$\left\langle \begin{bmatrix} Z(w)^* H(w)^* \\ I \end{bmatrix} y_w, \begin{bmatrix} Z(z)^* H(z)^* \\ I \end{bmatrix} y_z \right\rangle = \left\langle \begin{bmatrix} H(w)^* \\ S(w)^* \end{bmatrix} y_w, \begin{bmatrix} H(z)^* \\ S(z)^* \end{bmatrix} y_z \right\rangle$
so $V \colon \begin{bmatrix} Z(w)^* H(w)^* \\ I \end{bmatrix} y_w \mapsto \begin{bmatrix} H(w)^* \\ S(w)^* \end{bmatrix} y_w$ extends to isometry from
$\mathcal{D} := \overline{\text{span}} \left\{ \begin{bmatrix} Z(w)^* H(w)^* \\ I \end{bmatrix} y_w \right\}$ onto $\mathcal{R} := \left\{ \overline{\text{span}} \begin{bmatrix} H(w)^* \\ S(w)^* \end{bmatrix} y_w \right\}$

Proof of (2) $\Rightarrow$ (3) continued

Extend V to a unitary $\mathbf{U}^* = \left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U}$.

Then $\left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right] \left[\begin{smallmatrix} Z(w)^* H(w)^* \\ I \end{smallmatrix}\right] y_w = \left[\begin{smallmatrix} H(w)^* \\ S(w)^* \end{smallmatrix}\right] y_w \Rightarrow$

$H(w)^* y_w = (I - A^* Z(w)^*)^{-1} C^* y_w$ and then

$(B^* Z(w)^* (I - A^* Z(w)^*)^{-1} C^* + D^*) y_w = S(w)^* y_w \Rightarrow$

$S(z) = D + C(I - Z(z)A)^{-1} Z(z) B$ with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ unitary.

Canonical functional model version (B.-Bolotnikov 2010)

Use $\widehat{K}_k(z,w) = e_k(z) e_k(w)^*$ where $e_k \colon \mathcal{H}(\widehat{K}_k) \to \mathcal{Y} \oplus \mathcal{U}$

evaluation maps: $e_k(z) \colon \left[\begin{smallmatrix} f_k \\ g_k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} f_k(z) \\ g_k(z) \end{smallmatrix}\right]$

with $\widehat{K}_k = \left[\begin{smallmatrix} K_k^L & K_k^{LR} \\ K_k^{RL} & K_k^R \end{smallmatrix}\right]$ positive kernels giving 2-sided Agler decom.

$$\begin{bmatrix} I - S(z)S(w)^* & S(z) - S(\overline{w}) \\ S(\overline{z})^* - S(w)^* & I - S(\overline{z})^* S(\overline{w}) \end{bmatrix} = \sum \begin{bmatrix} (1 - z_k \overline{w_k}) K_k^L(z,w) & (z_k - \overline{w_k}) K_k^{LR}(z,w) \\ (z_k - \overline{w_k}) K_k^{RL}(z,w) & (1 - z_k \overline{w_k}) K_k^R(z,w) \end{bmatrix}$$

Then $\mathcal{X}_k = \mathcal{H}_k(\widehat{K}_k) \Rightarrow$ de Branges-Rovnyak functional model

The rational inner case

Assume $\mathcal{U} = \mathcal{Y} = \mathbb{C}^N$.
S inner: $S(z)$ unitary values a.e. for $z \in \mathbb{T}^d$

Theorem:
Given $S \colon \mathbb{D}^d \to \mathbb{C}^{N \times N}$, T.F.A.E.:

1. S is rational inner in $\mathcal{S}A_d(\mathbb{C}^N)$.
2. $S(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ with $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathbb{C}^N \to \mathcal{X} \oplus \mathbb{C}^N$ unitary with $\dim \mathcal{X} < \infty$

History:

- Cole-Wermer (1999): $N = 1$, $d = 2$
- B.-Sadosky-Vinnikov (2005): general N, $d = 2$
- Kummert (1989): general N, $d = 2$
- Knese (2010): $N = 1$, general d
- B.–Kaliuzhnyi-Verbovetskyi (2011): general N, general d

Realization of rational inner: idea of the proof

Given S rational, $S(z) = S(z)^{*-1}$ for $z \in \mathbb{T}^d$ & Agler decom
$I - S(z)S(w)^* = \sum_{k=1}^d (1 - z_k \overline{w_k}) K_k(z,w)$. Then:
$\frac{1}{\prod_{j=1}^d (1 - z_j \overline{w_j})} I \succeq \frac{I - S(z)S(w)^*}{\prod_{j=1}^d (1 - z_j \overline{w_j})} \succeq \frac{K_k(z,w)}{\prod_{j \neq k}^d (1 - z_j \overline{w_j})} \succeq K_k(z,w) \Rightarrow$
$\iota^* \colon k_{Sz}(\cdot, w) y \mapsto K_k(\cdot, w) y$ is contractive from $H^2_{\mathcal{Y}}(\mathbb{D}^d)$ to $\mathcal{H}(K_k)$
with adjoint $\iota \colon f \mapsto f$ equal to the inclusion map $\Rightarrow$
elements of $\mathcal{H}(K_k)$ are holomorphic on $\mathbb{D}^d \Rightarrow$
$K_k(z,w) = H_k(z) H_k(w)^*$ with $H \colon \mathbb{D}^d \to \mathcal{L}(\mathcal{X}_k, \mathbb{C}^N)$ holo.

Agler decom with $S(z) = D(z)^{-1} N(z)$ (D and N matrix polys) $\Rightarrow$
$D(z)D(w)^* - N(z)N(w)^* = \sum_{k=1}^d (1 - z_k \overline{w_k}) \widetilde{H}_k(z) \widetilde{H}_k(w)^*$
Take $z = w = t\mu$ with $t \in \mathbb{D}$, $\mu \in \mathbb{T}^d$ to get
$\frac{D(t\mu)D(t\mu)^* - N(t\mu)N(t\mu)^*}{1 - |t|^2} = \sum_{k=1}^d \widetilde{H}_k(t\mu) \widetilde{H}_k(t\mu)^*$

LHS = poly in $t, \overline{t}$ (since $D(\mu)D(\mu)^* = N(\mu)N(\mu)^*$ for $\mu \in \mathbb{T}^d$) $\Rightarrow$
[Fourier series computation $\Rightarrow$ $\widetilde{H}_k$ = polynomial] $\Rightarrow$
$\dim \mathcal{X}_k = \dim \mathcal{H}(K_k) < \infty$ as claimed.

Herglotz-Agler class on $\mathbb{D}^d$

Theorem

Given $\Phi \colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U})$, T.F.A.E.:

1. $\Phi \in \mathcal{HA}_d(\mathcal{U})$, i.e., $\Phi \colon \mathbb{D}^d \underset{holo}{\to} \mathcal{L}(\mathcal{U})$ &
   $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{CBL}(\mathcal{K})^d \Rightarrow \Phi(\mathbf{T}) \in \overline{\mathcal{A}}\mathcal{L}(\mathcal{U} \otimes \mathcal{K})$
2. Φ has an Agler decomposition, i.e., there exist positive kernels $K_k(z, w)$ on $\mathbb{D}^d$ so that
   $\Phi(z) + \Phi(w)^* = \sum_{k=1}^d (1 - z_k \overline{w_k}) K_k(z, w)$
3. Φ has DT-impedance-conservative realization, i.e., there exists state space $\mathcal{X}$, unitary U on $\mathcal{X}$, $B \colon \mathcal{U} \to \mathcal{X}$, $\alpha = \alpha^*$ on $\mathcal{U}$, so that $\Phi(z) = i\alpha + \frac{B^*B}{2} + B^*U^*(I - Z(z)U^*)^{-1}Z(z)B$
   $= i\alpha + \frac{1}{2}B^*(I + U^*Z(z))(I - U^*Z(z))^{-1}B$
   where $Z(z) = z_1 P_1 + \cdots + z_d P_d$
   ($\{P_1, \dots, P_d\}$ = spectral resolution not necessarily commuting with U)

Herglotz-Agler decomposition/realization: proofs

Proof 1:
Given $\Phi \in \mathcal{HA}_d(\mathcal{U})$ with $\Phi(0) > 0$, set $S = (\Phi - I)(\Phi + I)^{-1}$. At the system level, Staffans diagonal transform:
$\widehat{u}(n) = \frac{1}{\sqrt{2}}(u(n) + y(n))$, $\widehat{y}(n) = \frac{1}{\sqrt{2}}(u(n) - y(n))$.
Then check: $S \in \mathcal{SA}_d(\mathcal{U})$. Transform back to see the Schur-Agler-class results imply Herglotz-Agler results ($d = 1$: Staffans (MTNS 2002))

Proof 2: Direct lurking-isotropic-subspace argument

$\Phi(z) + \Phi(w)^* = \sum_{k=1}^d (1 - z_k \overline{w_k}) K_k(z, w)$ with
$K_k(z, w) = H_k(z) H_k(w)^* \Rightarrow$

$$\mathcal{V} := \left\{ \begin{bmatrix} H(z) \\ \Phi(z) \\ Z(z)\Phi(z) \\ I \end{bmatrix} u \colon z \in \mathbb{D}^d,\ u \in \mathcal{U} \right\} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$$ is $\mathcal{J}$-isotropic

where $\mathcal{J} = \begin{bmatrix} -I_{\mathcal{X}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\mathcal{U}} \\ 0 & 0 & I_{\mathcal{X}} & 0 \\ 0 & I_{\mathcal{X}} & 0 & 0 \end{bmatrix}$.

Argue that $\mathcal{V}$ can be embedded inside a $\mathcal{J}$-isotropic graph space:

$$\mathcal{V} = \begin{bmatrix} A & B \\ C & D \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$$

$\mathcal{V}$ $\mathcal{J}$-isotropic forces $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U^* & B \\ B^* U^* & i\alpha + \frac{B^* B}{2} \end{bmatrix}$ with U unitary.

Solve for $\Phi(z) = D + C(I - Z(z)A)^{-1} Z(z) B$ as before.

(B.-Staffans 2006)

Proof 3 via integral representation formula ($d = 1$ only):

$d = 1$: Herglotz integral representation formula
$\Phi(z) = i\alpha + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)$ for positive operator measure μ

Operator-theoretic interpretation:

Define $B \colon \mathcal{U} \to L^{\mu}$ via $B \colon u \mapsto \sqrt{2}\mu \cdot u$ and $U = M_{\zeta}$ on $\mathcal{X} := L^{\mu}$
where L^{μ} is the Hellinger space:
$\langle \mu \cdot f, \mu \cdot g \rangle_{L^{\mu}} = \int_{\mathbb{T}} \langle d\mu(\zeta) f(\zeta), g(\zeta) \rangle_{\mathcal{U}}$

Then:
$\Phi(z) = i\alpha + \frac{1}{2} B^{*}(U + zI)(U - zI)^{-1} B$
$= i\alpha + \frac{1}{2} B^{*}(I + 2z(U - zI)^{-1} B$
$= (i\alpha + \frac{B^{*}B}{2}) + zB^{*}U^{*}(I - zU^{*})^{-1} B$
$=$ DT-impedance-conservative realization

DT-Herglotz-Agler class: the rational case

Impedance-inner

Φ is DT-impedance-inner/positive-real-lossless means
$\Phi(z) = -\Phi(z)^*$ for a.e. $z \in \mathbb{T}^d$

Theorem:

Given $\Phi\colon \mathbb{D}^d \to \mathbb{C}^{N\times N}$, T.F.A.E.:

1. $\Phi \in \mathcal{HA}_d(\mathbb{C}^N)$ over $\mathbb{D}^d$, Φ is a rational, and Φ is DT-impedance-inner/positive-real-lossless
2. Φ has a finite-dimensional DT-impedance-conservative realization, i.e., there is an $M \times M$ unitary matrix U, an $M \times N$ matrix B, an $N \times N$ self-adjoint matrix α so that $\Phi(z) = i\alpha + \frac{B^*B}{2} + B^*U^*(I - Z(z)U^*)^{-1}Z(z)B$

Proof:

Apply diagonal transform to reduce to results for the rational Schur-Agler class.

Schur-Agler class over $(\mathbb{C}_+)^d$

Theorem:
Given $W\colon (\mathbb{C}_+)^d \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$, T.F.A.E.:

1. $W \in CT - \mathcal{SA}_d(\mathcal{U}, \mathcal{Y})$, i.e., $W\colon \underset{holo}{\to} \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{CAL}(\mathcal{K})^d \Rightarrow \|W(\mathbf{T})\| \le 1$
2. W has an CT-scattering-conservative Agler decomposition, i.e. there exist positive kernels $K_k(z, w)$ over $(\mathbb{C}_+)^d$ so that $I - W(z)W(w)^* = \sum_{k=1}^d (z_k + \overline{w_k})K_k(z, w)$
3. W ha a CT-scattering-conservative Staffans realization: $W(z) = C\&D \begin{bmatrix} (zI - A|_{\mathcal{X}})^{-1}B \\ I \end{bmatrix}$ where $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is a CT-scattering-conservative operator node:

CT-scattering-conservative operator node:

$\mathbf{U} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ has domain $\mathcal{D}$ dense in $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ whose graph
$\mathcal{G}(\mathbf{U}) := \begin{bmatrix} A\&B \\ C\&D \\ I \; 0 \\ 0 \; I \end{bmatrix}$ is also $\mathcal{J}$-Lagrangian as a subspace of $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$
where $\mathcal{J} = \begin{bmatrix} 0 & 0 & I_{\mathcal{X}} & 0 \\ 0 & I_{\mathcal{Y}} & 0 & 0 \\ I_{\mathcal{X}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{\mathcal{U}} \end{bmatrix}$.

Consequence: $\mathbf{U}$ is a system node:
A generates a C_0 semigroup (in fact A is maximal dissipative) &
$\mathcal{D}(\mathbf{U}) = \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] : A|_{\mathcal{X}}x + Bu \in \mathcal{X}\}$

The bounded case: In case $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is bounded, $\mathcal{J}$-Lagragian graph means $A^* + A + C^*C = 0$, $C^*D + B = 0$, $D^*D + I = 0$ with similar relations connected with the adjoint system.

History ($d = 1$): Arov-Nudelman (1996), Staffans (MTNS 2002), B-Staffans (2006) and Smuljan (1986), Salamon (1987)

Proof of realization

1. Use Cayley transform to reduce to the Schur-Agler class over $\mathbb{D}^d$ (Staffans MTNS 2002)
2. Use lurking isotropic subspace argument (B.-Staffans 2006)

The rational case:

Theorem: Given $W\colon (\mathbb{C}_+)^d \to \mathbb{C}^{N\times N}$, T.F.A.E.:

1. $W \in \mathcal{SA}_d(\mathbb{C}^N)$ over $(\mathbb{C}_+)^d$ is rational and unitary on $(i\mathbb{R})^d$
2. W has a finite-dimensional CT-scattering-conservative realization: there is block matrix $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with diagonal blocks of sizes $M\times M$ and $N\times N$ respectively with entries satisfying $A + A^* + C^*C = 0$, $C^*D + B = 0$, $D^*D + I = 0$ together with a spectral resolution on $\{P_1,\dots,P_d\}$ on $\mathbb{C}^M$ such that $W(z) = D + C(Z(z) - A)^{-1}B$ with $Z(z) = z_1P_1 + \cdots + z_dP_d$.

Herglotz-Agler over $(\mathbb{C}_+)^d$ and Bessmertnyi class

Herglotz-Agler-class over $(\mathbb{C}_+)^d$:

$F\colon (\mathbb{C}_+)^d \underset{holo}{\to} \mathcal{L}(\mathcal{U})$ in $\mathcal{HA}_d(\mathcal{U})$ means

$\mathbf{R} = (R_1, \dots, R_d) \in \mathcal{CAL}(\mathcal{K})^d \Rightarrow F(\mathbf{R}) \in \mathcal{AL}(\mathcal{U} \otimes \mathcal{K})$

Define $\Omega_d = \cup_{\lambda \in \mathbb{T}} (\lambda \mathbb{C}_+)^d$

$F \in \mathcal{B}_d(\mathcal{U})$ means all the following hold:

- $F\colon \Omega_d \underset{holo}{\to} \mathcal{L}(\mathcal{U})$
- $F|_{(\mathbb{C}_+)^d} \in \mathcal{HA}_d(\mathcal{U})$
- $F(\lambda z_1, \dots, \lambda z_d) = \lambda F(z_1, \dots, z_d)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$
- $F(\overline{z_1}, \dots, \overline{z_d}) = F(z_1, \dots, z_d)^*$

Remark:

For $d = 1$,

$\mathcal{B}_d(\mathcal{U}) = \{F\colon F(z) = \alpha z$ for some $\alpha = \alpha^* \geq 0$ in $\mathcal{L}(\mathcal{U})\}$

Characterization of Bessmertnyi class

Theorem (Bessmertnyi, Kaliuzhnyi-Verbovetskyi (2004))

Given $F\colon \Omega_d \to \mathcal{L}(\mathcal{U})$, T.F.A.E.:

1. $F \in \mathcal{B}_d(\mathcal{U})$
2. There are holomorphic positive kernels $K_1(z,w), \dots, K_d(z,w)$ on $(\mathbb{C}_+)^d$ so that $F(z) = \sum_{k=1}^d z_k K_k(z,w)$ for $z, w \in (\mathbb{C}_+)^d$ $\Rightarrow$ each K_k extends uniquely to holomorphic positive kernels on Ω_d with $K_k(\lambda z, \lambda w) = K_k(z,w)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$
3. $F(z) = D(z) - C(z)A(z)^{-1}B(z)$ where $\mathbf{U}(z) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}$ has the form $\mathbf{U}(z) = \sum_{k=1}^d z_k \mathbf{U}_k$ with $\mathbf{U}_k = \mathbf{U}_k^* \geq 0$ (so $\mathbf{U}_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$, $A(z) = \sum_{k=1}^d z_k A_k$, etc.)

Alternative formulation of Bessmertnyi Agler decomposition

Condition (2) holds $\Leftrightarrow$ there are positive kernels $K_1, \dots, K_d$ on $(\mathbb{C}_+)^d$ so that the two decompositions

$$F(z) + F(w)^* = \sum_{k=1}^{d} (z_k + \overline{w_k}) K_k(z, w)$$

$$F(z) - F(w)^* = \sum_{k=1}^{d} (z_k - \overline{w_k}) K_k(z, w)$$

hold simultaneously.

The rational Bessmertnyi class

Theorem (B.–Kaliuzhnyi-Verbovetskyi (2011))

Given $F\colon \Omega_d \to \mathcal{L}(\mathcal{U})$, T.F.A.E.:

1. F is rational and in $\mathcal{B}_d(\mathbb{C}^N)$
2. F has a finite-dimensional Bessmertnyi realization, i.e., there is a homogeneous matrix pencil $\mathbf{U}(z) = \left[\begin{smallmatrix} A(z) & B(z) \\ C(z) & D(z) \end{smallmatrix}\right] = \sum_{k=1}^d z_k \mathbf{U}_k$ with $\mathbf{U}_k = \mathbf{U}_k^* \geq 0$ so that $F(z) = D(z) - C(z)A(z)^{-1}B(z)$

Idea of proof:

Apply diagonal transform and Cayley transform of variables to reduce to Schur-Agler class on the disk. Bessmertnyi version of Schur-Agler class consists of $S(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ where $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies $U = U^{*-1} = U^{-1}$
Check finitely many positive squares property of Agler decompositions preserved by these transforms

A larger subclass of $\mathcal{HA}_d(\mathcal{U})$

An enlarged Bessmertnyi class

Fact: Suppose that $F(z) = D(z) - C(z)A(z)^{-1}B(z)$ where $\mathbf{U}(z) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}$ has the form $\mathbf{U}(z) = \mathbf{U}_0 + \sum_{k=1}^{d} z_k \mathbf{U}_k$ subject to (1) $\mathbf{U}_0 + \mathbf{U}_0^* = 0$ and (2) $\mathbf{U}_k = \mathbf{U}_k^* \geq 0$ for $k = 1, \dots, d$. Then $F \in \mathcal{HA}_d(\mathcal{U})$, i.e., $\mathcal{B}_{d,\text{enlarged}}(\mathcal{U}) \subset \mathcal{HA}_d(\mathcal{U})$

The single-variable case:

$d = 1 \Rightarrow$ new kind of realization for Herglotz class over $\mathbb{C}_+$ (and hence also for Nevanlinna/Weyl class over Π_+ (upper half plane))

Herglotz class over $\mathbb{C}_+$ (single-variable case)

Herglotz representation

Given $F \colon \mathbb{C}_+ \underset{holo}{\to} \mathcal{L}(\mathcal{U})$ with $F(z) + F(z)^* \geq 0$ for $z \in \mathbb{C}_+$, then there exists $\alpha = -\alpha^*$ and $\beta = \beta^* \geq 0$ in $\mathcal{L}(\mathcal{U})$ so that $F(z) = \alpha + z\beta + \int_{i\mathbb{R}} \frac{1+zs}{s+z} d\mu(s)$ where μ is a finite positive operator measure

Nevanlinna representation

If $\int_{i\mathbb{R}} (1-s^2) d\mu(s) < \infty$ and $d\nu(s) = (1-s^2)d\mu(s)$, then
$F(z) = \left[\alpha + \int_{i\mathbb{R}} \frac{s}{1-s^2} d\nu(s)\right] + z\beta + \int_{i\mathbb{R}} \frac{d\nu(s)}{s+z}$
Hypothesis to eliminate $\alpha + \beta z$ term and to guarantee $\int_{i\mathbb{R}} (1-s^2) d\mu(s) < \infty$:
(H1): $\lim_{x \to \infty} x |\langle F(x)u, u\rangle| < \infty$ for each $u \in \mathcal{U}$
Then $F(z) = \int_{i\mathbb{R}} \frac{d\nu(s)}{s+z}$

Realization from Herglotz and Nevanlinna representatons

Herglotz representation

$F(z) = \alpha + z\beta + \int_{i\mathbb{R}} \frac{1+zs}{s+z} d\mu(s) \Rightarrow$

$F(z) = \alpha + z\beta + B^*(I + zX)(X + zI)^{-1}B$ where

$X = -X^* = M_s$ on L^μ, B: $u \mapsto \mu \cdot u$ in L^μ

Drawback

No useful generalization to $\mathcal{HA}_d(\mathcal{U})$:

($Z(z)$ and X do not commute)

Hypothesis to eliminate $z\beta$ term:

(H2): $\lim_{x \to \infty} \frac{1}{x} F(x) u = 0$ for each $u \in \mathcal{U}$

Realization for $\mathcal{HA}_d(\mathcal{U})$ with no $z\beta$ term

Theorem

Given $F\colon (\mathbb{C}_+)^d \to \mathcal{L}(\mathcal{U})$ such that
d-variable (H2): $\lim_{x\to\infty} \frac{1}{x} F(x\mathbf{1})u = 0$ for each $u \in \mathcal{U}$
$\Rightarrow$ T.F.A.E.:

1. $F \in \mathcal{HA}_d(\mathcal{U})$, i.e., $\mathbf{R} = (R_1, \dots, R_d) \in \mathcal{CAL}(\mathcal{K})^d \Rightarrow F(\mathbf{R}) \in \mathcal{AL}(\mathcal{U} \otimes \mathcal{K})$
2. F has an Agler decomposition: $F(z) + F(w)^* = \sum_{k=1}^d (z_k + \overline{w_k}) K_k(z,w)$ for positive kernels K_k on $(\mathbb{C}_+)^d$
3. $F(z) = C\&D \begin{bmatrix} (Z(z) - A|_{\mathcal{X}})^{-1} B \\ I \end{bmatrix}$ where $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is a CT-impedance-conservative operator node

CT-impedance-conservative operator node

$\mathbf{U} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is a CT-impedance-conservative operator node means $\mathbf{U}$ has domain $\mathcal{D}$ dense in $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ whose graph $\mathcal{G}(\mathbf{U}) := \begin{bmatrix} A\&B \\ C\&D \\ I \;\; 0 \\ 0 \;\; I \end{bmatrix} \mathcal{D}$ is $\mathcal{J}$-Lagrangian in $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ where now $\mathcal{J} = \begin{bmatrix} 0 & 0 & I_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & -I_{\mathcal{U}} \\ I_{\mathcal{X}} & 0 & 0 & 0 \\ 0 & -I_{\mathcal{U}} & 0 & 0 \end{bmatrix}$

$\Rightarrow$ $\mathbf{U}$ is a system node

Bounded case

Then $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ with $A = -A^*$, $D = -D^*$

Bessmertnyi form

Then $F(z) = D + B^*(Z(z) - A)^{-1}B = D(z) - C(z)A(z)^{-1}B(z)$
where
$\mathbf{U}(z) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} -A & B \\ -B^* & D \end{bmatrix} + \begin{bmatrix} Z(z) & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{U}_0 + \sum_{k=1}^d z_k \mathbf{U}_k$
with $\mathbf{U}_0 = \begin{bmatrix} -A & B \\ -B^* & D \end{bmatrix} = -\mathbf{U}_0^*$, $\mathbf{U}_k = \begin{bmatrix} P_k & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ for $k = 1, \dots, d$

$\Rightarrow F \in \mathcal{B}_{d,enlarged}(\mathcal{U})$
(but with A, B, B^*, D all unbounded in general (H2) case)

Proof of realization for $\mathcal{HA}_d(\mathcal{U})$-class with no $z\beta$ term

Method 1

Simultaneous diagonal & Cayley transform to Schur-Agler class over $\mathbb{D}^d$. Use (H2) to show that -1 not an eigenvalue of A in DT-$\mathcal{SA}_d(\mathcal{U})$ realization of $F\left(\frac{Z(z)-\mathbf{1}}{Z(z)+\mathbf{1}}\right) \Rightarrow$ can apply diagonal transform to get CT-$\mathcal{HA}_d(\mathcal{U})$ realization (Staffans MTNS 2002)

Method 2

Lurking isotropic subspace: Apply (H2) to show that $\mathcal{J}$-Lagrangian subspace is a graph space

Method 3: $d = 1$

Use the integral representation with $\beta = 0$: nice exercise: suggested in Staffans MTNS 2002 but with details left to the reader!

Agler-McCarthy-Young realization

Theorem 1

Given a generic $F \in \mathcal{HA}_d(\mathcal{U})$, there exist a state space $\mathcal{X}$, a spectral resolution $(P_1, \dots, P_d)$ on $\mathcal{X}$ with $Z(z) = \sum_{k=1}^d z_k P_k$, a possibly unbounded operator $X = -X^*$ on $\mathcal{X}$, a bounded operator $B \colon \mathcal{U} \to \mathcal{X}$, a bounded operator $D = -D^*$ on $\mathcal{X}$, so that

$$F(z) = D + B^* Z(z) B - B^* (Z(z) + Z(\overline{z_0}))(X + Z(z))^{-1}(Z(z) - Z(z_0))B$$

where z_0 is a fixed point in $(\mathbb{C}_+)^d$

Proof

For generic F, 1 is not an eigenvalue of unitary colligation matrix $\mathbf{U}$ of unitary GR-realization for $\mathcal{C} \circ F\left(\frac{Z(z)-\mathbf{1}}{Z(z)+\mathbf{1}}\right)$ $\Rightarrow$ can apply Cayley transform of the whole colligation matrix!

Theorem 2 (Agler-McCarthy-Young)

Assume F generic in $\mathcal{HA}_c(\mathcal{U})$. Then T.F.A.E.:

1. d-variable (H1): $\lim_{x\to\infty} x|\langle F(x\mathbf{1})u, u\rangle| < \infty$ for all $u \in \mathcal{U}$
2. There exist state space $\mathcal{X}$, spectral resolution $(P_1, \ldots, P_d)$ on $\mathcal{X}$ with $Z(z) = \sum_{k=1}^{d} z_k P_k$, possibly unbounded $X = -X^*$ on $\mathcal{X}$ and bounded $B\colon \mathcal{U} \to \mathcal{X}$ so that $F(z) = B^*(Z(z) + X)^{-1}B$

$d = 1$: $=$ Nevanlinna's result

Bessmertnyi form of Agler-McCarthy-Young realization

$F(z) =$
$D + B^* Z(z) B - B^*(Z(z) + Z(\overline{z_0}))(X + Z(z))^{-1}(Z(z) - Z(z_0))B$
$= D(z) - C(z) + A(z)^{-1} B(z)$ where
$\begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} X & -Z(z_0)B \\ B^* Z(\overline{z_0}) & D \end{bmatrix} + \begin{bmatrix} Z(z) & Z(z)B \\ B^* Z(z) & B^* Z(z) B \end{bmatrix} =$
$\mathbf{U}_0 + \sum_{k=1}^d z_k \mathbf{U}_k$ where
$\mathbf{U}_0 = \begin{bmatrix} X & -Z(z_0)B \\ B^* Z(\overline{z_0}) & D \end{bmatrix} = -\mathbf{U}_0^*,$
$\mathbf{U}_k = \begin{bmatrix} P_k & P_k B \\ B^* P_k & B^* P_k B \end{bmatrix} = \begin{bmatrix} I \\ B^* \end{bmatrix} P_k \begin{bmatrix} I & B \end{bmatrix} \geq 0$
$\Rightarrow$ generic $F \in \mathcal{HA}_d(\mathcal{U})$ is in $\mathcal{B}_{d,enlarged}(\mathcal{U})$
(but with a single unbounded operator X)

versus Staffans realization for general (H2) case:
$\mathbf{U}_0 = \begin{bmatrix} X & -B \\ B^* & D \end{bmatrix} = -\mathbf{U}_0^*$ where X, B, B^*, D all unbounded in general