# $\text{Li}^{(\text{p})}$-SERVICE? AN ALGORITHM FOR COMPUTING p-ADIC POLYLOGARITHMS

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ABSTRACT. We describe an algorithm for computing Coleman's $p$-adic polylogarithms up to a given precision.

## 1. INTRODUCTION

It is well known that, for a number field $k$ with ring of integers $\mathcal{O}_k$, there is a relation between the regulator of the group of units of $\mathcal{O}_k$, $\mathcal{O}_k^*$, and the residue of $\zeta_k(s)$ at $s = 1$. In terms of $K$-theory, $\mathcal{O}_k^* \cong K_1(\mathcal{O}_k^*)$, and Borel in [Bor77] showed that this relation generalizes, for $n = 2, 3, \ldots$, to a similar relation between a suitably defined regulator of the higher $K$-group $K_{2n-1}(\mathcal{O}_k)$ and the value of $\zeta_k(s)$ at $s = n$.

However, it is far more difficult to find explicit non-trivial elements in those higher $K$-groups than in $\mathcal{O}_k^*$. But $K_{2n-1}(\mathcal{O}_k) \cong K_{2n-1}(k)$ for $n \geq 2$ and Zagier in [Zag91] gave a conjectural description of the latter groups tensored with $\mathbb{Q}$. His construction always gives a $\mathbb{Q}$-subspace (see, e.g., [dJ95]), and gives the whole of $K_3(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $K_5(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all number fields $k$, as well as of $K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \geq 2$ if $k$ is cyclotomic (see loc. cit. where this is deduced from results by Suslin, Goncharov and Zagier [Zag91] respectively).

The regulator for $\mathcal{O}_k^*$ is defined as the determinant of a matrix with as entries the logarithm of the absolute value of certain elements of $\mathcal{O}_k^*$ embedded into $\mathbb{C}$. In Zagier's conjecture the Borel regulator for $K_{2n-1}(k) \cong K_{2n-1}(\mathcal{O}_k)$ is obtained as the determinant of a matrix with as entries suitable $\mathbb{Z}$ -linear combinations of the values at certain elements of $k$ embedded into $\mathbb{C}$ of the $n$-th (real) polylogarithm.

This $n$-th real polylogarithm is obtained from the complex polylogarithm $\text{Li}_n(z)$, which is defined by the power series

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad (n \geq 1) \tag{1.1}$$

on the open unit disc in $\mathbb{C}$. It can be continued analytically to a multi-valued function on $\mathbb{C} \setminus \{0, 1\}$. The real-valued modification is easily computed from this. For the numerical verification of the conjectures in the context of Zagier's conjecture, it is important to have an efficient implementation of the complex polylogarithm as in PARI-GP.

There is a conjectural $p$-adic analogue of Borel's theorem which fits into the general context of relations between regulators of $K$-groups and special values of $L$-functions as conjectured by Beilinson (see [Bei85] or [Sch88]). Its predictions are rather similar, but somewhat more involved, using the syntomic regulator [Bes00] rather than the Beilinson regulator. In the case of totally real number fields it follows from [BdJ03] that the $p$-adic regulator on the part of the $K$-groups given by Zagier's conjecture[1] is given by a determinant as before, but with the complex polylogarithm replaced with the *p-adic polylogarithm.*

The $p$-adic polylogarithm is an analogue of its complex cousin. It was defined by Coleman [Col82] using the technique now referred to as *Coleman integration.* On the open unit disc in $\mathbb{C}_p$, the so called field of $p$-adic complex numbers, it is again defined by the power series (1.1). To extend it to a function defined on $\mathbb{C}_p \setminus \{1\}$ Coleman uses a technique he called "analytic continuation along Frobenius", which is rather involved. As a consequence, it is not so easy to compute the functions defined in this way. To our knowledge there has been only one attempt to compute Coleman integrals in the literature [CT94], with very limited precision.

Recently we embarked on a project of testing numerically the $p$-adic analogue of Borel's theorem (see [BBdJR]). This requires the computation of $p$-adic L-functions and $p$-adic polylogarithms up to a given precision. The present work concerns an algorithm for the latter. In the process of describing the set-up for the algorithm we obtain bounds for $|\mathrm{Li}_n(\zeta)|$ for a root of unity $\zeta$ of order not a power of $p$ that may be of independent interest (see the end of Section 4).

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## 2. The $p$-adic polylogarithm

A precise definition of Coleman integration is beyond the scope of the present paper. Fortunately, in the case of $p$-adic polylogarithms there is a certain simplification that is sufficient for the computations that we give here.

Recall that the field of $p$-adic complex numbers, $\mathbb{C}_p$, is the completion of the algebraic closure of the field of $p$-adic numbers $\mathbb{Q}_p$. We let $|\cdot|$ be the absolute value on $\mathbb{C}_p$, normalized such that $|p| = p^{-1}$. It corresponds to the valuation $v_p$ normalized by $v_p(p) = 1$. The residue field of $\mathbb{C}_p$ is the algebraic closure of the field with $p$ elements, $\overline{\mathbb{F}}_p$.

We consider the projective line $X = \mathbb{P}^1(\mathbb{C}_p)$. A standard open disc in $X$ with centre $a$ in $\mathbb{C}_p$ and radius $r$ is the subset $D(a,r) = \{z \in X \,|\, |z-a| < r\}$. A standard annulus with centre $a$ in $\mathbb{C}_p$ is a subset of the form $A(a,s,r) = \{z \in X \,|\, s < |z-a| < r\}$. Rigid analytic functions on $D(a,r)$ are power series $\sum_{i=0}^{\infty} b_i(z-a)^i$ that converge on $D(a,r)$, and rigid analytic functions on $A(a,s,r)$ are Laurent series $\sum_{i=-\infty}^{\infty} b_i(z-a)^i$ that converge on $A(a,s,r)$. For $a = \infty$ we take $D(\infty,r) = \{z \in \mathbb{C}_p \,|\, |z| > 1/r\} \cup \{\infty\}$ and rigid analytic functions on it are power series in $1/z$ that converge on $D(\infty,r)$. Similarly, $A(\infty,s,r) = \{z \in \mathbb{C}_p \,|\, 1/s > |z| > 1/r\}$ and rigid analytic functions on it are Laurent series in $1/z$ that converge on $A(\infty,s,r)$.

**Definition 2.1.** A branch of the $p$-adic logarithm is a group homomorphism

$$\log : \mathbb{C}_p^* \to \mathbb{C}_p$$

given by the usual power series $\log(1+z) = z - z^2/2 + z^3/3 - \cdots$ when $|z| < 1$.

[1] There is a technical condition, fulfilled for almost all primes, that we ignore here for the sake of exposition. See the introduction of [BdJ03], in particular Theorem 1.10.

**Remark 2.2.** A branch of the $p$-adic logarithm is determined by specifying $\log(p)$ in $\mathbb{C}_p$, as follows. If $v_p(z) = 0$ then $z$ reduces to an element of $\overline{\mathbb{F}}_p^*$ and therefore $z^n$ reduces to 1 for some positive $n$. Then $\log(z^n)$ is independent of the branch as it is determined by the power series given above, and $\log(z) = \log(z^n)/n$, independent of $n$. In general, if $bv_p(z) = a$ for integers $a$ and $b$ with $b$ positive, then $v_p(z^b/p^a) = 0$ and $\log(z) = a\log(p)/b + \log(z^b/p^a)/b$, independent of the choice of $a$ and $b$.

We now once and for all fix a branch of the $p$-adic logarithm. All constructions that follow do depend on this choice in principle. (For the precise dependence of the functions $\mathrm{Li}_n(z)$ that we are about to describe on the choice of the branch of the logarithm we refer to Remark 7.4 or [BdJ03, Proposition 2.6].)

We define log functions on $A(a, s, r)$ to be polynomials $\sum f_i \cdot (\log(z-a))^i$ if $a \neq \infty$ and polynomials $\sum f_i \cdot (\log z)^i$ if $a = \infty$, with the $f_i$ rigid analytic functions on $A(a, s, r)$. We let $\mathcal{O}_{\log}(U)$ denote the space of rigid analytic functions on $U$ if $U = D(a, r)$, and the space of log functions on $U$ if $U = A(a, s, r)$. We differentiate functions formally, with the rule that the derivative of $\log(z-a)$ is $1/(z-a)$. It is a basic fact, and rather easy to prove, that differentiation gives a surjective map from $\mathcal{O}_{\log}(U)$ to itself, with kernel consisting of the constants.

Consider now the system of differential equations

$$\begin{aligned} \mathrm{dLi}_1(z) &= \frac{\mathrm{d}z}{1-z} \\ \mathrm{dLi}_{n+1}(z) &= \mathrm{Li}_n(z)\frac{\mathrm{d}z}{z} \quad (n \geq 1) \end{aligned} \tag{2.3}$$

or, equivalently

$$\begin{aligned} \mathrm{Li}_0(z) &= \frac{z}{1-z} \\ \mathrm{dLi}_{n+1}(z) &= \mathrm{Li}_n(z)\frac{\mathrm{d}z}{z} \quad (n \geq 0)\,. \end{aligned} \tag{2.4}$$

The complex polylogarithms are defined by the same system. It has singularities at 0, 1 and $\infty$. It follows from the properties of the rings $\mathcal{O}_{\log}(U)$ discussed before that on each disc or annulus $U$ not containing 0, 1 or $\infty$ (2.3) or (2.4) can be solved with $\mathrm{Li}_n(z)$ in $\mathcal{O}_{\log}(U)$. In fact, such solutions on $U$ are unique up to adding

$$c_{n-1} + \frac{c_{n-2}}{1!}\log z + \cdots + \frac{c_0}{(n-1)!}\log^{n-1} z$$

with $c_j$ in $\mathbb{C}_p$ corresponding to the constant of integration in (2.4) for $n = j$.

In particular, all such solutions are rigid analytic for all but three "residue discs" $U_a$, consisting of those points in $\mathbb{P}^1(\mathbb{C}_p)$ reducing to the same point as $a$. The three residue discs for which this does not hold a priori are $U_0$, $U_1$ and $U_\infty$. On $U_0$ it is well known (and immediate) that the series in (1.1) for $z$ in $\mathbb{C}_p$ with $|z| < 1$ and $n \geq 0$ satisfy the systems (2.3) and (2.4), and we shall in fact assume that the $\mathrm{Li}_n(z)$ on $U_0$ are given by those series. But on $U_a$ with $a = 1$ or $\infty$ the $\mathrm{Li}_n(z)$ only belong to $\mathcal{O}_{\log}(U)$ for $U = U_a \setminus \{a\}$.

Clearly, what has been said so far does not suffice to determine $\mathrm{Li}_n(z)$ uniquely. The real magic of Coleman's theory is that there is a canonical way of choosing solutions to differential equations such as in (2.3) (in general, unipotent differential equations) using a principle known as *Frobenius equivariance*. As we mentioned before, a general discussion of Coleman's theory is beyond the scope of the present work but we explain what it means in the present context.

For this we also need the functions

$$\mathrm{Li}_n^{(p)}(z) := \mathrm{Li}_n(z) - \frac{1}{p^n}\mathrm{Li}_n(z^p)\,, \tag{2.5}$$

a priori defined for $z$ in $\mathbb{C}_p$ with $z^p \neq 1$. They satisfy conditions similar to (2.4), namely

$$\begin{aligned} \mathrm{Li}_0^{(p)}(z) &= \frac{z}{1-z} - \frac{z^p}{1-z^p} \\ \mathrm{d}\mathrm{Li}_{n+1}^{(p)}(z) &= \mathrm{Li}_n^{(p)}(z)\frac{\mathrm{d}z}{z} \quad (n \geq 0)\,. \end{aligned} \tag{2.6}$$

**Theorem 2.7. (Coleman)** *For any branch of the $p$-adic logarithm there exists a unique sequence of functions*

$$\mathrm{Li}_n : \mathbb{P}^1(\mathbb{C}_p) \setminus \{1, \infty\} \to \mathbb{C}_p \qquad (n \geq 0)$$

*with the properties:*

1. *the restrictions of the $\mathrm{Li}_n$ to every residue disc $U = U_a$ other than $U_1$ and $U_\infty$, and to the annuli $U = U_1 \setminus \{1\}$ and $U = U_\infty \setminus \{\infty\}$, belong to $\mathcal{O}_{\log}(U)$ and satisfy* (2.4)*;*
2. *the restrictions of the $\mathrm{Li}_n$ to $U_0$ is given by the series* (1.1)*;*
3. *for each $n \geq 0$ the function $\mathrm{Li}_n^{(p)}(z)$ with $\mathrm{Li}_n^{(p)}(\infty) = 0$ extends to a function that on the set*

$$\mathbb{P}^1(\mathbb{C}_p) \setminus \{z \text{ in } \mathbb{C}_p \text{ with } |z-1| \leq p^{-1/(p-1)}\}$$

*is given by a convergent power series in $1/(1-z)$.*

*Proof.* We use [Col82]. The $\mathrm{Li}_n(z)$ are defined there in section VI (page 195) exactly to satisfy (2.4) (the definition of $\mathrm{Li}_0(z)$ in loc. cit. is incorrect) as well as $\lim_{z\to 0}\mathrm{Li}_n(z) = 0$. The fact that the polylogarithms belong to $\mathcal{O}_{\log}(U)$ for all residue discs but $U_0$ is part of the properties of Coleman integration. Using induction on $n$ it follows directly from the definition that $\mathrm{Li}_n(z)$ on $U_0$ is given by (1.1), hence lies in $\mathcal{O}_{\log}(U_0)$. The power series expansion of $\mathrm{Li}_n^{(p)}(z)$ in (3) is Proposition 6.2.

As for uniqueness, we first notice that the power series expansion of $\mathrm{Li}_n^{(p)}(z)$ is uniquely determined by (2.6) and its value at $\infty$ (cf. Proposition 4.3 below). Assuming that, on each residue disc $U$, $\mathrm{Li}_{n-1}(z)$ in $\mathcal{O}_{\log}(U)$ has already been determined, the differential equation in (2.4) determines $\mathrm{Li}_n(z)$ up to a constant. Therefore $\mathrm{Li}_n(z)$ is determined up to adding a function $C(z)$ that is constant on each $U$. Since the domain of convergence of the power series expansion of $\mathrm{Li}_n^{(p)}(z)$ touches every $U$ we have that $C(z) - C(z^p)/p^n = 0$. Because $z$ and $z^{p^f}$ lie in the same residue disc for some $f > 0$ this implies that $C$ is the zero function. □

**Remark 2.8.** (1) In fact, in Coleman's theory a much weaker condition than the condition in part (3) is required. This condition is iteratively defined and hard to explain. Fortunately, the stronger condition in part (3) is satisfied by [Col82, Proposition 6.2] and implies the required condition.

(2) The part of $\mathbb{P}^1(\mathbb{C}_p)$ that has to be removed in part (3) of Theorem 2.7 is the disc around 1 that contains all the singularities of the differential equation satisfied by $\mathrm{Li}_n^{(p)}(z)$ except for 0 and $\infty$, i.e., the $p$-th roots of unity. The convergence of the power series in $1/(1-z)$ on the indicated domain implies a growth condition

on its coefficients. We will in fact deduce, by explicit computation, a more precise growth condition on these coefficients (see Proposition 6.1).

We will need some further results about $\mathrm{Li}_n(z)$.

**Proposition 2.9.** (1) *For $m \geq 1$ and $z$ in $\mathbb{C}_p$ with $z^m \neq 1$,*

$$\mathrm{Li}_n(z^m) = m^{n-1} \sum_{\zeta^m=1} \mathrm{Li}_n(\zeta z);$$

(2) $\mathrm{Li}_n(z) + (-1)^n \mathrm{Li}_n(z^{-1}) = -\frac{1}{n!} \log^n(z)$.

*Proof.* Those are (the correct version of) Proposition 6.1 and Proposition 6.4(i) of [Col82]. □

## 3. Method of computation on $U_0$ and $U_\infty$

On $U_0$ we can use the standard expansion in (1.1),

$$\mathrm{Li}_n(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^n}, \tag{3.1}$$

which we shall denote by $F_{n,0}(t)$.

**Remark 3.2.** As an immediate consequence of the power series expansion of $\mathrm{Li}_n(z)$ on $U_0$ and the definition of $\mathrm{Li}_n^{(p)}(z)$ in (2.5) we see that on $U_0$

$$\mathrm{Li}_n^{(p)}(z) = {\sum_{k\geq 1}}' \frac{z^k}{k^n}, \tag{3.3}$$

where the prime indicates that we only sum over those $k$ for which $p \nmid k$. We can collect terms $z^k/k^n$ in $\mathrm{Li}_n(z) = \sum_{k\geq 1} \frac{z^k}{k^n}$ for which $v_p(k) = m$ and find that

$$\mathrm{Li}_n(z) = \sum_{m\geq 0} \frac{\mathrm{Li}_n^{(p)}(z^{p^m})}{p^{mn}} \tag{3.4}$$

as well.

For the disc $U_\infty$ we use that

$$\mathrm{Li}_n(z) + (-1)^n \mathrm{Li}_n(1/z) = -\frac{1}{n!} \log^n(z)$$

as in part (2) of Proposition 2.9. This reduces the calculation to one for $1/z$, which is in $U_0$, and the calculation of $\log(z)$.

## 4. Method of computation in the generic case

In this section we explain how to compute $\mathrm{Li}_n(z)$ on all residue discs except $U_0$, $U_1$ and $U_\infty$. The residue discs $U_0$ and $U_\infty$ were discussed in the previous section and $U_1$ will be dealt with in Section 5.

We begin with a well known observation.

**Proposition 4.1.** *Every residue disc other than $U_0$ and $U_\infty$ is $U_\zeta$ for a unique root of unity $\zeta$ of order dividing $p^f - 1$ for some $f > 0$, known as the* Teichmüller representative *of this residue disc.*

*Proof.* For the residue disc $U_b$ the reduction $\bar{b}$ satisfies $\bar{b}^{p^f-1} = 1$ for some $f > 0$ because $U_b \neq U_0$ or $U_\infty$. Since $\mathbb{C}_p$ is complete we can apply Hensel's lemma to lift $\bar{b}$ to a unique solution of $x^{p^f-1} = 1$ in $U_b$, which is $\zeta$. □

The key observation for the computation of $\mathrm{Li}_n(z)$ is the following.

**Proposition 4.2.** *Suppose that $\zeta \neq 1$ is a $(p^f - 1)$-th root of unity. Then*

$$\mathrm{Li}_n(\zeta) = \left(p^{nf} - 1\right)^{-1} \left(\mathrm{Li}_n^{(p)}(\zeta)p^{nf} + \mathrm{Li}_n^{(p)}(\zeta^p)p^{n(f-1)} + \cdots + \mathrm{Li}_n^{(p)}(\zeta^{p^{f-1}})p^n\right).$$

*Proof.* This formula is derived in [Bes02] as part of the proof of Corollary 2.2 there. For completeness, we recall the easy proof. For any $z$ in $\mathbb{C}_p$ with $z^{p^k} \neq 1$ we have

$$\sum_{r=0}^{k-1} \mathrm{Li}_n^{(p)}(z^{p^r})p^{-rn} = \sum_{r=0}^{k-1}(\mathrm{Li}_n(z^{p^r}) - p^{-n}\mathrm{Li}_n(z^{p^{r+1}}))p^{-rn} = \mathrm{Li}_n(z) - p^{-nk}\mathrm{Li}_n(z^{p^k})$$

since the second sum is telescopic. Setting $z = \zeta$ and $k = f$ we have $\zeta^{p^f} = \zeta$ hence

$$\sum_{r=0}^{f-1} \mathrm{Li}_n^{(p)}(\zeta^{p^r})p^{-rn} = (1 - p^{-nf})\mathrm{Li}_n(\zeta) = p^{-nf}(p^{nf} - 1)\mathrm{Li}_n(\zeta)$$

so

$$\mathrm{Li}_n(\zeta) = (p^{nf} - 1)^{-1} \sum_{r=0}^{f-1} \mathrm{Li}_n^{(p)}(\zeta^{p^r})p^{n(f-r)}$$

as required. □

Proposition 4.2 implies that if we are able to compute $\mathrm{Li}_n^{(p)}(z)$, then we can find $\mathrm{Li}_n(z)$ at least for $z = \zeta$. For this calculation we use a power series expansion of $\mathrm{Li}_n^{(p)}(z)$ around $\infty$ as in the next result (but see also Remark 8.4). Then Proposition 4.4 below will show how to use the value of $\mathrm{Li}_n(\zeta)$ in order to compute $\mathrm{Li}_n(z)$ for $z$ in $U_\zeta$.

**Proposition 4.3.** *We have $\mathrm{Li}_n^{(p)}(z) = g_n(1/(1-z))$ for a power series $g_n(v)$ in $\mathbb{Q}[[v]]$, convergent for $v$ in $\mathbb{C}_p$ with $|v| < p^{1/(p-1)}$. It is determined inductively by*

$$g_0(v) = v - 1 - \frac{(v-1)^p}{v^p - (v-1)^p}$$

$$g'_{n+1}(v) = -\frac{g_n(v)}{v - v^2} = -\frac{g_n(v)}{v}(1 + v + v^2 + \cdots) \qquad (n \geq 0)\,,$$

*and*

$$g_n(0) = 0 \qquad (n \geq 1)\,.$$

*Proof.* By Theorem 2.7 we can write $\mathrm{Li}_n^{(p)}(z) = g_n(1/(1-z))$ with $g_n(v)$ a power series that converges for $|v| < p^{1/(p-1)}$. To determine the relations satisfied by the $g_n(v)$ we first write $u = 1 - z$ and let $f_n(u) = \mathrm{Li}_n^{(p)}(1-u)$ for $n \geq 0$. Using (2.5) the equations in (2.6) become

$$f_0(u) = \frac{1-u}{u} - \frac{(1-u)^p}{1-(1-u)^p} \quad \text{and} \quad f'_{n+1}(u) = -\frac{f_n(u)}{1-u}\,.$$

Next we set $v = 1/u$ and let $g_n(v) = f_n(1/v)$ to find

$$g_0(v) = v - 1 - \frac{(v-1)^p}{v^p - (v-1)^p} \quad \text{and} \quad g'_{n+1}(v) = -\frac{g_n(v)}{v - v^2} = -\frac{g_n(v)}{v}(1 + v + v^2 + \cdots)$$

as required. We then have $g_n(0) = \mathrm{Li}_n^{(p)}(\infty) = 0$ for all $n \geq 0$ by Theorem 2.7(3). Clearly, these relations determine $g_n(v)$ inductively by integration. Because the denominator $v^p - (v-1)^p$ of $g_0(v)$ is a unit in $\mathbb{Z}[[v]]$ we see that $g_0(v)$ is in $\mathbb{Z}[[v]]$ and hence that $g_n(v)$ is in $\mathbb{Q}[[v]]$ for $n \geq 1$. Finally, we observe that $v = 1/(1-z)$. □

**Proposition 4.4.** *Let $\zeta \neq 1$ be a $(p^f - 1)$-th root of unity. For $z$ in $U_\zeta$ we have that $\mathrm{Li}_n(z) = F_{n,\zeta}(z - \zeta)$ for a power series $F_{n,\zeta}(t)$ with coefficients in $\mathbb{Q}_p(\zeta)$. It converges for $|t| < 1$ and can be found inductively by the formulae*

$$F_{0,\zeta}(t) = \frac{\zeta + t}{(1-\zeta) - t} = (1-\zeta)^{-1}\left(\zeta + \frac{t}{1-\zeta} + \frac{t^2}{(1-\zeta)^2} + \cdots\right) \tag{4.5}$$

*and, for $n \geq 0$,*

$$F'_{n+1,\zeta}(t) = \frac{F_{n,\zeta}(t)}{\zeta + t} = \zeta^{-1} F_{n,\zeta}(t)(1 - \zeta^{-1}t + \zeta^{-2}t^2 - \cdots) \tag{4.6}$$

*as well as*

$$F_{n+1,\zeta}(0) = \mathrm{Li}_{n+1}(\zeta)\,. \tag{4.7}$$

*Proof.* The fact that $\mathrm{Li}_n(z)$ is rigid analytic on $U_\zeta$ and therefore has a power series expansion as above was stated in Theorem 2.7(1). The first two formulae are immediate consequences of (2.4). Integration then determines $F_{n+1,\zeta}(t)$ for $n \geq 0$, except for its constant term, which is given by the last equation. Since by Proposition 4.3 $\mathrm{Li}_n^{(p)}(\zeta)$ is in $\mathbb{Q}_p(\zeta)$ the same holds for $\mathrm{Li}_n(\zeta)$ by Proposition 4.2. The claim about the coefficients is then clear from the inductive formulae. □

For use in some of the estimates in the following sections we also prove a few results about the absolute values of $\mathrm{Li}_n^{(p)}(z)$ and $\mathrm{Li}_n(z)$ (cf. [Bes02]). Note that if $|z| < 1$ then $|\mathrm{Li}_n^{(p)}(z)| = |z|$ by (3.3).

**Proposition 4.8.** *If $z$ in $\mathbb{C}_p$ satisfies $|z - 1| = 1$ then $|\mathrm{Li}_n^{(p)}(z)| \leq 1$.*

*Proof.* This is a slight generalization of [Bes02, Proposition 2.1], with the same proof. We use a formula found by Coleman [Col82, Lemma 7.2],

$$\mathrm{Li}_n^{(p)}(z) = \int_{\mathbb{Z}_p^*} x^{-n} d\mu_z(x)\,,$$

where $\mu_z$ is the measure such that $\mu_z(a + p^k\mathbb{Z}_p) = \frac{z^a}{1 - z^{p^k}}$ for $a = 1, \ldots, p^k - 1$ not divisible by $p$. Since for the specified values of $z$ this measure takes values with absolute value at most 1 the same holds for $\mathrm{Li}_n^{(p)}(z)$. □

**Corollary 4.9** ([Bes02, Corollary 2.2])**.** *If $\zeta \neq 1$ is a root of unity of order prime to $p$ then $\mathrm{Li}_n(\zeta)$ is in $p^n\mathbb{Z}_p(\zeta)$.*

*Proof.* We have that $\mathrm{Li}_n^{(p)}(\zeta)$ is in $\mathbb{Q}_p(\zeta)$ by Proposition 4.3 and by Proposition 4.8 we have $|\mathrm{Li}_n^{(p)}(\zeta)| \leq 1$ so that $\mathrm{Li}_n^{(p)}(\zeta)$ is in $\mathbb{Z}_p(\zeta)$. The result is now immediate from Proposition 4.2. □

**Corollary 4.10.** *If $\zeta$ is a root of unity of order $p^k m$ with $m > 1$ not divisible by $p$ then $|\mathrm{Li}_n(\zeta)| \le p^{(k-1)n}$.*

*Proof.* For $k = 0$ this is part of Corollary 4.9. For $k > 0$ it then follows by induction since $\mathrm{Li}_n(\zeta) = p^{-n}\mathrm{Li}_n(\zeta^p) + \mathrm{Li}_n^{(p)}(\zeta)$ and $|\mathrm{Li}_n^{(p)}(\zeta)| \le 1$ by Proposition 4.8. □

## 5. Method of computation on $U_1$

To compute $\mathrm{Li}_n(z)$ for $z \ne 1$ in $U_1$ we use the following result.

**Proposition 5.1** ([Col82, Proposition 7.1])**.** *For $n \ge 2$ the function*

$$E_n(z) = \mathrm{Li}_n(z) - \frac{1}{n-1}\log(z)\mathrm{Li}_{n-1}(z) \tag{5.2}$$

*extends to a rigid analytic function on $U_1$.*

It is then clear that $E_n(z)$ is defined for $z \ne 0$. Moreover, Proposition 2.9(1) together with the identities $\log(z^m) = m\log(z)$ and $\log(\zeta z) = \log(z)$ for a root of unity $\zeta$ implies that we have a distribution relation

$$E_n(z^m) = m^{n-1}\sum_{\zeta^m=1} E_n(\zeta z) \tag{5.3}$$

for $m \ge 1$ and $z$ in $\mathbb{C}_p^*$. In particular

$$E_n(z^2) = 2^{n-1}(E_n(z) + E_n(-z))\,. \tag{5.4}$$

**Proposition 5.5.** *For $n \ge 2$ set $G_n(t) = E_n(t+1)$. Then $G_n(t)$ for $|t| < 1$ is given by a power series in $\mathbb{Q}_p[[t]]$ that satisfies*

$$G_2'(t) = 1 - t/2 + t^2/3 - \cdots$$

*and*

$$G_n'(t) = \frac{n-2}{n-1}\frac{G_{n-1}(t)}{1+t} \qquad \text{for } n \ge 3\,.$$

*Proof.* By Proposition 5.1 we know that$G_n(t)$ is given by a power series that converges for $|t| < 1$. From (2.3) we have that $E_2'(z) = -\log(z)/(1-z)$ so that

$$G_2'(t) = E_2'(t+1) = \frac{\log(1+t)}{t}$$

which gives the first formula. Similarly, for $n \ge 3$, $E_n'(z) = \frac{n-2}{n-1}\frac{E_{n-1}(z)}{z}$, giving the second.

That $G_n(0)$ is in $\mathbb{Q}_p$ for $p \ne 2$ will be proved in Proposition 5.6, so that, by induction on $n$, $G_n(t)$ is in $\mathbb{Q}_p[[t]]$ for all $n \ge 2$ in this case. For $p = 2$ it will follow by induction on $n$ in conjunction with Proposition 5.7 that $G_n(0)$, hence all the coefficients of all $G_n(t)$, are in $\mathbb{Q}_2$. □

The proposition implies that we may find $G_n(t)$ from $G_{n-1}(t)$ except for its constant term $G_n(0) = E_n(1)$. For this we use Proposition 5.6 if $p \ne 2$ and Proposition 5.7 if $p = 2$. Note that it follows from the definition of $E_n(z)$ and part (2) of Proposition 2.9 that $E_n(z) + (-1)^n E_n(1/z) = \log^n(z)/(n!(n-1))$ so that $G_n(0) = E_n(1) = 0$ for $n \ge 2$ even.

**Proposition 5.6.** *We have*

$$G_n(0) = 2^{n-1}\mathrm{Li}_n(-1)/(1-2^{n-1})\,.$$

*Proof.* Putting $z = 1$ in (5.4) gives $(1 - 2^{n-1})E_n(1) = 2^{n-1}E_n(-1) = 2^{n-1}\mathrm{Li}_n(-1)$ from which the formula follows immediately. ☐

Proposition 5.6 suffices to compute $G_n(0)$ if $p \neq 2$ since then $-1$ is not in $U_1$ and $\mathrm{Li}_n(-1)$ can be computed using the methods of Section 4. In particular, $-1$ is a Teichmüller representative for such $p$ so that $\mathrm{Li}_n(-1)$, hence $G_n(0)$, is in $\mathbb{Q}_p$ by Proposition 4.4.

If $p = 2$ then $-1$ is in $U_1$ and we use the next result to compute $G_n(0)$.

**Proposition 5.7.** *Suppose $p = 2$ and let $G_n(t) = G_n(0) + \widetilde{G}_n(t)$. Then, in $\mathbb{Q}_2$,*

$$G_n(0) = \frac{2^{n-1}}{1 - 2^n}\widetilde{G}_n(-2)\,.$$

*Proof.* Substituting $z = 1$ in (5.4) gives

$$G_n(0) = 2^{n-1}\left(G_n(0) + G_n(-2)\right) = 2^{n-1}\left(2G_n(0) + \widetilde{G}_n(-2)\right)$$

and the result follows. That $\widetilde{G}_n(-2)$ is in $\mathbb{Q}_2$ is clear since $\widetilde{G}_n(t)$ lies in $\mathbb{Q}_2[[t]]$ by induction in conjunction with Proposition 5.5. ☐

It is now clear that we may compute $\mathrm{Li}_n(z)$ for $z$ in $U_1$ inductively using the relation (5.2). We can even get an explicit formula as follows.

**Proposition 5.8.** *We have, for $z \neq 1$ in $U_1$ and $n \geq 2$,*

$$\mathrm{Li}_n(z) = \sum_{j=0}^{n-2} \frac{(n-j-1)!}{(n-1)!}\log^j(z)E_{n-j}(z) + \frac{1}{(n-1)!}\log^{n-1}(z)\mathrm{Li}_1(z)\,.$$

*Proof.* For $n \geq 2$ we get from (5.2) that

$$(n-1)!\,E_n(z) = (n-1)!\,\mathrm{Li}_n(z) - (n-2)!\log(z)\mathrm{Li}_{n-1}(z)\,.$$

Setting $E_1(z) = \mathrm{Li}_1(z)$ we have an equality of generating power series in $T$,

$$\sum_{n\geq 1}(n-1)!\,E_n(z)T^n = (1 - \log(z)T)\left(\sum_{n\geq 1}(n-1)!\,\mathrm{Li}_n(z)T^n\right),$$

so that

$$\sum_{n\geq 1}(n-1)!\,\mathrm{Li}_n(z)T^n = \left(\sum_{n\geq 1}(n-1)!\,E_n(z)T^n\right)(1 + \log(z)T + \log^2(z)T^2 + \cdots)\,,$$

from which the result follows easily. ☐

**Remark 5.9.** The formula in Proposition 5.8 makes clear that, around $z = 1$, $\mathrm{Li}_n(z)$ is the sum of a rigid analytic function and

$$\frac{1}{(n-1)!}\log^{n-1}(z)\mathrm{Li}_1(z) = -\frac{1}{(n-1)!}\log^{n-1}(z)\log(1-z)\,.$$

Around $z = 1$ $\log(z)$ is rigid analytic while $\log(1-z)$ is not. Consequently $\mathrm{Li}_n(z)$ is not rigid analytic around $z = 1$.

## 6. Estimates

In this section we provide estimates of the coefficients of the power series $g_n(v)$ of Proposition 4.3, $F_{n,\varsigma}(t)$ of Proposition 4.4 and $G_n(t)$ of Proposition 5.5. We shall use those in Section 7 to know how many terms of those power series we have to calculate in order to compute $\mathrm{Li}_n(z)$ up to a specified precision for a given $z \neq 1$ in $\mathbb{C}_p$.

For the coefficients of $g_n(v)$ we have the following result.

**Proposition 6.1.** *For $n \geq 1$ let*

$$c(n,p) = \frac{p}{p-1} - \frac{n-1}{\log(p)} + (n-1)\log_p\left(\frac{n(p-1)}{\log(p)}\right) + \log_p\left(\frac{p(p-1)n}{\log(p)}\right) + \log_p(2)\,.$$

*If*

$$g_n(v) = a_{n,1}v + a_{n,2}v^2 + \cdots$$

*then*

$$v_p(a_{n,k}) \geq \max\left(0, \frac{k}{p-1} - \log_p(k) - c(n,p)\right)\,.$$

*Proof.* We first show that $v_p(a_{n,k}) \geq 0$. For this we recall that for a power series $f(z) = \sum_i b_i z^i$ converging on the closed unit disc, we have that $\max|b_i| = \max_{|z|=1}|f(z)|$, where $z$ must be considered in the algebraic closure of the field of coefficients (see [FvdP04, Example 3.3.2]), which will be $\mathbb{Q}_p$ in our case. It thus suffices to show that $g_n(z) = \mathrm{Li}_n^{(p)}(1-z^{-1})$ takes on integral values when $|z| = 1$. But then $|(1-z^{-1})-1| = 1$ so that we can apply Proposition 4.8.

Next we estimate $v_p(a_{0,k})$. Recall from Proposition 4.3 that

$$g_0(v) = v - 1 - \frac{(v-1)^p}{v^p-(v-1)^p} = v - 1 + (v-1)^p\sum_{i=0}^{\infty}(pf(v))^i\,.$$

where $v^p - (v-1)^p = 1 - pf(v)$ for some polynomial $f(v)$ in $\mathbb{Z}[v]$ of degree $p-1$. Then any term contributing to $v^k$ in $g_0(v)$ for $k > 1$ will come from a product containing $(pf(v))^i$ with $i \geq \lceil (k-p)/(p-1)\rceil$ where, $\lceil x\rceil$ is the smallest integer greater than or equal to $x$. Therefore, also for $k = 1$,

$$v_p(a_{0,k}) \geq \left\lceil\frac{k-p}{p-1}\right\rceil\,, \tag{6.2}$$

proving the result for $n = 0$ since $c(n,p) > 0$.

We now prove the estimate for $n \geq 1$. By Proposition 4.3 we have

$$\begin{aligned} g'_{n+1}(v) &= -\frac{g_n(v)}{v}(1+v+v^2+\cdots) \\ &= -a_{n,1} - (a_{n,1}+a_{n,2})v - (a_{n,1}+a_{n,2}+a_{n,3})v^2 - \cdots \end{aligned}$$

and consequently $ka_{n+1,k} = -(a_{n,1}+\cdots+a_{n,k})$ for $k \geq 1$. Substituting $v = 1$, which is in the range of convergence for $g_n$, we find

$$\sum_{i=1}^{\infty} a_{n,i} = g_n(1) = \mathrm{Li}_n^{(p)}(0) = 0\,.$$

It follows that $ka_{n+1,k} = a_{n,k+1} + a_{n,k+2} + \cdots$. Therefore

$$v_p(a_{n+1,k}) \geq \min_{j\geq k+1}\{v_p(a_{n,j}) - v_p(k)\}\,.$$

Iterating this and using (6.2) we obtain

$$v_p(a_{n,k}) \geq \min_{k=j_0<j_1<j_2<\cdots<j_n} \left\{ \left\lceil \frac{j_n - p}{p-1} \right\rceil - \sum_{i=0}^{n-1} v_p(j_i) \right\}$$
$$\geq \min_{k=j_0<j_1<j_2<\cdots<j_n} \left\{ \frac{j_n - p}{p-1} - \sum_{i=0}^{n-1} v_p(j_i) \right\}.$$

We shall bound the last expression from below. We do this by first considering possible values of $j_n$. Suppose that $k + p^l \leq j_n < k + p^{l+1}$ for some integer $l \geq 0$. We then have the lower bound

$$\frac{j_n - p}{p-1} \geq \frac{k + p^l - p}{p-1}. \tag{6.3}$$

We now bound from above the sum $\sum_{i=0}^{n-1} v_p(j_i)$ for all possible sequences of $j$'s. Clearly, in the range from $k$ to $k + p^{l+1} - 1$ there is only one integer divisible by $p^{l+1}$, and its valuation is bounded by $\log_p(k + p^{l+1})$. The valuation of the other $n-1$ possible $j$'s is bounded by $l$. Thus, we have

$$\sum_{i=0}^{n-1} v_p(j_i) \leq (n-1)l + \log_p(k + p^{l+1}). \tag{6.4}$$

Combining the estimates (6.3) and (6.4) and taking the minimum over all possible $l$'s we finally arrive at the estimate

$$\begin{aligned} v_p(a_{n,k}) &\geq \min_{0\leq l\in\mathbb{Z}} \left\{ \frac{k + p^l - p}{p-1} - (n-1)l - \log_p(k + p^{l+1}) \right\} \\ &\geq \min_{0\leq l\in\mathbb{R}} \left\{ \frac{k + p^l - p}{p-1} - (n-1)l - \log_p(k + p^{l+1}) \right\}. \end{aligned} \tag{6.5}$$

Computing this last minimum is a standard problem. We have

$$\frac{d}{dl}\left( \frac{k + p^l - p}{p-1} - (n-1)l - \log_p(k + p^{l+1}) \right) = \frac{\log(p) \cdot p^l}{p-1} - (n-1) - \frac{p^{l+1}}{k + p^{l+1}}.$$

This derivative is clearly positive for large $l$ and is negative for $l = 0$ when $n \geq 2$. Consequently, for $n \geq 2$ it must vanish at the value of $l$ where the right-hand side of (6.5) attains its minimal value, so that we get

$$\frac{\log(p) \cdot p^l}{p-1} = n - 1 + \frac{p^{l+1}}{k + p^{l+1}}.$$

Since the last summand is always between 0 and 1 we obtain the inequalities

$$n > \frac{\log(p) \cdot p^l}{p-1} > n - 1,$$

which implies that $l < \log_p(n(p-1)/\log(p))$, $p^l/(p-1) > (n-1)/\log(p)$ and $p^{l+1} < p(p-1)n/\log(p)$.

We observe that those inequalities also hold if $n = 1$ and $l = 0$ so that they hold where the right-hand side of (6.5) attains its minimum. Using those inequalities as well as $\log(x+y) \le \log(2\max(x,y)) \le \log(2)+\log(x)+\log(y)$ for $x$, $y \ge 1$ we find that $v_p(a_{n,k})$ is bigger than

$$\frac{k-p}{p-1} + \frac{n-1}{\log(p)} - (n-1)\log_p\left(\frac{n(p-1)}{\log(p)}\right) - \log_p(2) - \log_p(k) - \log_p\left(\frac{p(p-1)n}{\log(p)}\right)$$
$$= \frac{k}{p-1} - \log_p(k) - c(n,p)$$

as required. □

**Remark 6.6.** Proposition 6.1 implies that $g_n(v)$ converges for $|v| < p^{1/(p-1)}$, as stated in Theorem 2.7(3) and Propisition 4.3. The bound seems to have the right behaviour and only the constant $c(n,p)$ may possibly be improved.

We now move on to estimates concerning the $F_{n,\zeta}(t)$'s that were introduced in Proposition 4.4. It is clear from (3.1) that the coefficient of $t^k$ in $F_{n,0}(t)$ has valuation at least $-n\log_p(k)$ for all $k \ge 1$. For the corresponding statement for the $F_{n,\zeta}(t)$'s (with $\zeta \ne 1$ a root of unity of order relatively prime to $p$) we have to work a little more.

**Proposition 6.7.** *Let $\zeta \ne 1$ be a $(p^f-1)$-th root of unity and write*

$$F_{n,\zeta}(t) = a_{n,0} + a_{n,1}t + a_{n,2}t^2 + \cdots$$

*in $\mathbb{Q}_p(\zeta)[[t]]$. Then $a_{n,0}$ is in $p^n\mathbb{Z}_p(\zeta)$ and for all $k \ge 1$ we have $v_p(a_{n,k}) \ge -n\log_p(k)$.*

*Proof.* By Corollary 4.9 $a_{n,0} = \mathrm{Li}_n(\zeta)$ lies in $p^n\mathbb{Z}_p(\zeta)$. We proceed to prove the other statement by induction on $n$. For $n=0$ we have that $\mathrm{Li}_0(z) = \frac{z}{1-z}$ and so $F_{0,\zeta}(t) = \frac{\zeta+t}{(1-\zeta)-t}$. Because $\zeta$ does not reduce to 1 by assumption $(1-\zeta)-t$ is a unit in $\mathbb{Z}_p(\zeta)[[t]]$, so $F_{0,\zeta}(t)$ is in $\mathbb{Z}_p(\zeta)[[t]]$. For $n \ge 1$ we see from (4.6) that

$$ka_{n+1,k} = -\sum_{j=0}^{k-1}(-\zeta)^{j-k}a_{n,j}\,. \tag{6.8}$$

Therefore $k\,a_{n+1,k}$ is a sum of elements with valuations $v_p(a_{n,j})$, $0 \le j \le k-1$, and so for $k \ge 1$, $v_p(a_{n+1,k}) \ge \min_{j=0,\dots,k-1} v_p(a_{n,j}) - v_p(k) \ge -n\log_p(k) - \log_p(k)$. □

**Remark 6.9.** The proof of Proposition 6.7 actually shows that we have the slightly better estimate $v_p(a_k) \ge -\log_p\left(\frac{k!}{\max(0,k-n)!}\right)$ for all $k \ge 0$.

Finally, we consider the series $G_n(t)$ for $n \ge 2$ that were introduced in Proposition 5.5.

**Proposition 6.10.** *Denote $(n-1)G_n(t)$ by $H_n(t)$ and write $H_n(t) = \sum_{k\ge0} b_{n,k}t^k$. With $\epsilon_p = 2$ for $p=2$ and $\epsilon_p = 1$ otherwise we have that $v_p(b_{n,0}) \ge n - \epsilon_p$, and $v_p(b_{n,k}) \ge -n\log_p(k)$ if $k > 0$.*

*Proof.* We begin with the statement for $k=0$. It follows easily from the distribution relation (5.3) and Corollary 4.9 that

$$v_p(G_n(0)) = v_p(E_n(1)) \ge n - \min_{p\nmid m>1} v_p(m^{n-1}-1).$$

Now $m^{n-1}-1$ will be divisible by $p^s$ for all $m$ relatively prime to $p$ precisely when the exponent of $(\mathbb{Z}/p^s)^*$ divides $n-1$. Therefore any such $s$ satisfies $s \le v_p(n-1)+\epsilon_p$ so that $v_p(G_n(0)) \ge n - v_p(n-1) - \epsilon_p$ and $v_p(H_n(0)) \ge n - \epsilon_p$.

For the statement for $k > 0$ we note that by Proposition 5.5 and what we noticed just before Proposition 5.6 (namely that $G_n(0) = 0$ if $n \geq 2$ is even)

$$H_2(t) = G_2(t) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k^2} t^k \tag{6.11}$$

so that $v_p(b_{2,k}) \geq -2\log_p(k)$. Since by the same proposition

$$H_n'(t) = \frac{H_{n-1}(t)}{1+t} = H_{n-1}(t)(1 - t + t^2 - \cdots) \tag{6.12}$$

for $n > 2$, we see as in the proof of Proposition 6.7 that

$$v_p(kb_{n,k}) \geq \min_{0\leq j<k} v_p(b_{n-1,k}) \,.$$

The result follows from this by induction since $(n-1) - \epsilon_p \geq 0$. □

## 7. The algorithm

In this section we use the material from the previous sections in order to give an algorithm for computing $\mathrm{Li}_n(z)$ for $n \geq 2$ and $z \neq 1$ up to a given precision and analyze its efficiency.

First of all we formalize the notion of "up to a given precision".

**Definition 7.1.** (1) For any number $\alpha$ in $\mathbb{C}_p$ we say that we know $\alpha$ up to precision $N$ if we have $\beta$ in $\mathbb{C}_p$ such that $v_p(\alpha - \beta) > N$.

(2) We say that we know $\alpha \neq 0$ in $\mathbb{C}_p$ up to relative precision $N$ if we have $\beta$ in $\mathbb{C}_p$ such that $v_p(\beta/\alpha - 1) > N$.

**Remark 7.2.** (1) Note that those notions are absolute; even if $\alpha$ is in a finite extension of $\mathbb{Q}_p$ they do not take the ramification of this extension into account.

(2) For $\alpha$ and $\beta$ as in the first part of the definition we shall refer to $\beta$ as an approximation of $\alpha$ up to precision $N$.

(3) If we know $\alpha \neq 0$ up to precision $N$ then we know $\alpha$ up to relative precision $N - v_p(\alpha)$, and conversely.

(4) If $z \neq 0$ is known up to relative precision $N > 0$ then so is $1/z$. In particular, if we know $z \neq 0$ up to precision $N > v_p(z)$ then we know $z$ and $1/z$ up to relative precision $N - v_p(z)$, and $1/z$ up to precision $N - 2v_p(z)$.

We assume that we want to compute $\mathrm{Li}_n(z)$ up to precision $N > 0$ for $z \neq 1$ in a complete subfield $F$ of $\mathbb{C}_p$. If $z$ in $F$ does not lie in $U_0$, $U_1$ or $U_\infty$ then it lies in $U_\zeta$ for some Teichmüller representative $\zeta \neq 1$. Since $F$ is complete one sees as in the proof of Proposition 4.1 that $\zeta$ lies in $F$.

We shall also assume that we know $z$ up to precision $N' > v_p(z)$ so that we can at least decide in which residue disc $z$ lies and, in fact, we know $v_p(z)$. In Algorithm 7.6 we will also give a value of $N'$ that suffices for the computation of $\mathrm{Li}_n(z)$ up to precison $N$.

**Remark 7.3.** For the algorithm it is not necessary to assume that $F$ is a finite extension of $\mathbb{Q}_p$, but with that assumption it is possible to give universal estimates (see Remark 8.1) and to quantify its efficiency (see Proposition 8.2). If we want to know $\mathrm{Li}_n(z)$ up to precision $N$ for arbitrary $z$ in $\mathbb{C}_p$ then from the algorithm we can determine $N'$ such that if $\tilde{z}$ is an approximation of $z$ up to precision $N'$ then $\mathrm{Li}_n(\tilde{z})$ is an approximation of $\mathrm{Li}_n(z)$ up to precision $N$. By taking such $\tilde{z}$ in $\overline{\mathbb{Q}}_p$ we reduce to calculations in $\mathbb{Q}_p(\tilde{z})$, a finite extension of $\mathbb{Q}_p$ and a complete field.

**Remark 7.4.** We shall now discuss that $\mathrm{Li}_n(z)$ is in $F$ because $F$ is complete. We shall also clarify how $\mathrm{Li}_n(z)$ depends on the branch of the logarithm (see Definition 2.1 and Remark 2.2), which was also made explicit in [BdJ03, Proposition 2.6] by a different method.

It is clear from (3.1) that $F_{n,0}(t)$ lies in $\mathbb{Q}[[t]]$, so that $\mathrm{Li}_n(z)$ for $z$ in $U_0$ lies in $F$ and is independent of the branch of the logarithm.

For $z \neq \infty$ in $U_\infty$ it follows from Proposition 2.9(2) that $\mathrm{Li}_n(z)$ is in $F$ provided we use a branch of the logarithm for which $\log(p)$ is in $F$. The dependence on this branch is also clear from this.

If $z$ in $U_\zeta$, where $\zeta \neq 1$ is a $(p^f-1)$-th root of unity for some $f > 0$, then by Proposition 4.4 the coefficients of $F_{n,\zeta}(t)$ are in $\mathbb{Q}_p(\zeta) \subseteq F$ so that $\mathrm{Li}_n(z)$ lies in $F$. Since the statements of Propositions 4.2, 4.3 and 4.4 do not depend on the branch of the logarithm, neither does $\mathrm{Li}_n(z)$ for such $z$.

Finally, for $z \neq 1$ in $U_1$ it follows from Proposition 5.5, together with Proposition 5.6 for $p \neq 2$ and Proposition 5.7 for $p = 2$, that $E_n(z)$ for $z$ in $U_1$ is independent of the branch of the logarithm and is in $F$. The dependence of $\mathrm{Li}_n(z)$ for such $z$ on the branch of the logarithm is clear from Proposition 5.8 since the only part that depends on it is $\mathrm{Li}_1(z) = -\log(1-z)$. This formula also makes clear that $\mathrm{Li}_n(z)$ is in $F$ provided that $\log(p)$ is in $F$.

**Remark 7.5.** Note that we cannot know 0 up to precision $N' > v_p(0)$ so that we shall assume that $z \neq 0$ in Algorithm 7.6 below. However, clearly $\mathrm{Li}_n(0) = 0$, and if we know that $|z| < p^{-N'} < 1$ then $|\mathrm{Li}_n(z)| < \max_{m\geq 0}\{p^{mn-N'p^m}\}$ by (3.4) because $|\mathrm{Li}_n^{(p)}(z)| = |z|$ when $|z| < 1$. If $n \leq N'\log(p)$ then this maximum is attained for $m = 0$ and equals $p^{-N'}$ but if $n > N'\log(p)$ then it may be much bigger.

We now give the algorithm for computing $\mathrm{Li}_n(z)$ up to precision $N$ for $z \neq 0$, while also giving a sufficient precision for $z$ for this. The various steps will be justified afterwards. We assume that the fixed branch of the logarithm, $\log(z)$, is readily computable.

**Algorithm 7.6.** In order to compute $\mathrm{Li}_n(z)$ for $z \neq 0, 1$ in $F$ and $n \geq 2$ up to precision $N > 0$ we first determine in which residue disc $z$ lies and then do the following.

(1) If $z$ is in $U_0$ then we find $M \geq 0$ such that $p^m v_p(z) - mn > N$ for all $m > M$. For each $m = 0, \ldots, M$ we find $\mathsf{tsl}_m$ such that $kp^m v_p(z) - mn > N$ for $k \geq \mathsf{tsl}_m$. Working in $F$ up to precision $N + nM$ we then calculate

$$\sum_{m=0}^{M} p^{-mn} {\sum_{k=1}^{\mathsf{tsl}_m-1}}' b_k \tilde{z}^{kp^m} ,$$

where the prime indicates that we sum only over $k$ that are not divisible by $p$, $\tilde{z}$ is an approximation of $z$ up to precision $N+nM$ and $b_k$ is an approximation of $1/k^n$ up to the same precision.

(2) If $z$ is in $U_\infty$ then we calculate $(-1)^{n-1}\mathrm{Li}_n(1/z) - \log^n(z)/n!$. Here $\mathrm{Li}_n(1/z)$ is computed using (1) and it can be calculated up to precision $N$ if we know $z$ up to precision $N + nM + 2v_p(z)$ where $M$ is such that $-p^m v_p(z) - mn > N$ for all $m > M$. We can calculate $\log^n(z)/n!$ up to precision $N$ by first finding $V$ with $|\log(z)| \leq p^{-V}$ and knowing $z$ up to precision $N' > v_p(z)$ satisfying

$$\max\{v_p(V), N + v_p(n!) - (n-1)v_p(V)\} \leq \min_{m\geq 0}\{(N' - v_p(z))p^m - m\} .$$

(3) If $z$ lies in $U_\zeta$ with $\zeta \neq 1$ in $F$ a root of unity of order dividing $p^f - 1$ then we proceed in several steps.

(a) We find $\mathsf{tsl} \geq 0$ such that $kv_p(z-\zeta) - n\log_p(k) > N$ for all $k \geq \mathsf{tsl}$. We find an approximation $\tilde{\zeta}$ of $\zeta$ such that $v_p(\tilde{\zeta}-\zeta) > N + n\log_p(\mathsf{tsl}-1)$.

(b) We find $\mathsf{gsl}$ such that $\frac{k}{p-1} - \log_p(k) - c(m,p) > N - m + (n-m)\log_p(\mathsf{tsl}-1)$ for $m = 1, \ldots, n$ and all $k \geq \mathsf{gsl}$, where $c(m,p)$ is as in Proposition 6.1.

(c) We calculate the classes of $g_m(v)$ in $\mathbb{Q}_p[[v]]/(v^{\mathsf{gsl}})$ for $m = 1, \ldots, n$ inductively using Proposition 4.3, starting with the coefficients of $g_0(v)$ up to precision $N + n\log_p(\mathsf{tsl}-1) + n\lfloor\log_p(\mathsf{gsl}-1)\rfloor$.

(d) We compute $\mathrm{Li}_m^{(p)}(\zeta^{p^j})$ for $j = 1, \ldots, f$ and $m = 1, \ldots, n$ up to precision $N - m + (n-m)\log_p(\mathsf{tsl}-1)$ by evaluating the terms of degree smaller than $\mathsf{gsl}$ in $g_m(v)$ on $1/(1-\tilde{\zeta})$, which according to Remark 7.2(4) is known up to the same precision as $\zeta$, namely $N + n\log_p(\mathsf{tsl}-1)$.

(e) We calculate $\mathrm{Li}_m(\zeta)$ up to precision $N + (n-m)\log_p(\mathsf{tsl}-1)$ for $m = 1, \ldots, n$ by using Proposition 4.2, with $\zeta$ replaced by $\tilde{\zeta}$.

(f) Working in $\mathbb{Q}_p(\zeta)[[t]]/(t^{\mathsf{tsl}})$ with coefficients up to precision $N + n\log_p(\mathsf{tsl}-1)$ we use (4.5), (4.6) and (4.7), but with $\zeta$ replaced by $\tilde{\zeta}$ and the $\mathrm{Li}_m(\zeta)$ replaced by the approximations obtained in (e), in order to compute approximations to the terms of degree less than $\mathsf{tsl}$ in $F_{n,\zeta}(t)$.

(g) We then evaluate the terms of degree less than $\mathsf{tsl}$ in the result on $\tilde{z} - \tilde{\zeta}$ where $\tilde{z}$ is an approximation of $z$ of precision $N + n\log_p(\mathsf{tsl}-1)$, and we work in $F$ up to precision $N$.

(4) If $z \neq 1$ lies in $U_1$ then we calculate $\mathrm{Li}_n(z)$ up to precision $N$ by calculating all summands in

$$\mathrm{Li}_n(z) = \sum_{j=0}^{n-2} \frac{(n-j-1)!}{(n-1)!} \log^j(z) E_{n-j}(z) + \frac{1}{(n-1)!} \log^{n-1}(z)\mathrm{Li}_1(z) \tag{7.7}$$

up to precision $N$, as follows.

(a) We find $V$ and $V_1$ with $V \leq v_p(\log(z))$ and $V_1 \leq v_p(\mathrm{Li}_1(z)) = v_p(\log(1-z))$.

(b) We take $W \leq 0$ such that $W \leq \min_{k\geq 1}\{kv_p(z-1) - n\log_p(k)\}$.

(c) If necessary we increase $N$ so that

$$N + v_p((n-1)!) - v_p((n-j-2)!) - W - jV \geq 0$$

for $j = 0, \ldots, n-2$, as well as

$$N + v_p((n-1)!) - V_1 - (n-1)V \geq 0\,.$$

We let $N_E = \max_{j=0,\ldots,n-2}\{N + v_p((n-1)!) - v_p((n-j-2)!) - jV\}$.

(d) We compute $\log^{n-1}(z)$ up to precision $N + v_p((n-1)!) - V_1$ as well as $\mathrm{Li}_1(z) = -\log(1-z)$ up to precision $N + v_p((n-1)!) - (n-1)V$. Their product divided by $(n-1)!$ is the last term in (7.7) up to precision $N$. The calculation of $\log^{n-1}(z)$ up to the required precision can be done if we know $z$ up to precision $N'_{n-1} > 0$ satisfying

$$\max\{V, N + v_p((n-1)!) - V_1 - (n-2)V\} \leq \min_{m\geq 0}\{N'_{n-1}p^m - m\}\,.$$

The calculation of $\mathrm{Li}_1(z) = -\log(1-z)$ can be done up to the required precision if we know $z$ up to precision $N' > v_p(1-z)$ satisfying

$$N + v_p((n-1)!) - (n-1)V \le \min_{m\ge 0}\{(N' - v_p(1-z))p^m - m\}\,.$$

(e) We compute $\log^j(z)$ up to precision $N + v_p((n-1)!) - v_p((n-j-2)!) - W$ for $j = 1, \ldots, n-2$, putting $\log^0(z) = 1$. For $j > 0$ this can be done if we know $z$ up to precision $N'_j > 0$ satisfying

$$\max\{V, N + v_p((n-1)!) - v_p((n-j-2)!) - W - (j-1)V\} \le \min_{m\ge 0}\{N'_j p^m - m\}\,.$$

(f) We find $\mathsf{hsl} > 0$ such that $kv_p(z-1) > N_E + n\log_p(k)$ for all $k \ge \mathsf{hsl}$; if $p = 2$ then we increase $\mathsf{hsl}$ if necessary to ensure that $k > N_E - (m-1) + (n-m)\log_p(\mathsf{hsl}-1)$ for all $k \ge \mathsf{hsl}$ and all $m = 2, \ldots, n$.

(g) If $p \ne 2$ then we compute $H_m(0) = (m-1)2^{m-1}\mathrm{Li}_m(-1)/(1-2^{m-1})$ for $m = 2, \ldots, n$ up to precision $N_E + (n-m)\log_p(\mathsf{hsl}-1)$ using (3)(a)-(e) with $\zeta = 1$; we compute the terms of degree less than $\mathsf{hsl}$ in $H_n(t)$ by iterated integration using (6.12) starting from (6.11), but with the $H_m(0)$ replaced by the approximations just obtained and working in $\mathbb{Q}_p[[t]]/(t^{\mathsf{hsl}})$, up to precision $N_E + (n-2)\log_p(\mathsf{tsl}-1)$ for the coefficients.
If $p = 2$ then we compute the terms of degree less than $\mathsf{hsl}$ in $H_m(t)$ for $m = 2, \ldots, n$ using (6.12) and (6.11), working in $\mathbb{Q}_p[[t]]/(t^{\mathsf{hsl}})$ up to precision $N_E + (n-2)\log_p(\mathsf{tsl}-1)$ for the coefficients, but at every stage determining an approximation of $b_{m,0}$ as $2^{m-1}(1-2^m)^{-1}\sum_{k=1}^{\mathsf{hsl}-1}\tilde b_{m,k}(-2)^k$ where the $\tilde b_{m,k}$ are the approximate coefficients of $H_m(t)$ we have already calculated.

(h) We then evaluate the terms of degree less than $\mathsf{hsl}$ in the resulting approximation of $H_m(t)$ ($m = 2, \ldots, n$) on $\tilde z - 1$ where $\tilde z$ is an approximation of $z$ of precision $N_E + n\log_p(\mathsf{hsl}-1)$.

(i) For $j = 0, \ldots, n-2$ we multiply the outcome of (h) for $m = n-j$, corresponding to $(n-j-1)E_{n-j}(z)$ up to precision $N_E$, with the approximation of $\log^j(z)$ from (e) and $(n-j-2)!/(n-1)!$, sum the results and add the product that we obtained in (d).

**Remark 7.8.** The conditions on $\mathsf{tsl}_m$, $\mathsf{tsl}$, $\mathsf{gsl}$ and $\mathsf{hsl}$ in (1), (3)(a), (3)(b) and (4)(f) of the algorithm are of the form $c_1k - c_2\log(k) > c_3$ for all $k \ge \mathsf{tsl}_m$, etc., with positive $c_i$. Since the left-hand side is increasing for $k > c_2/c_1$ it is very easy to find the minimum values for $\mathsf{tsl}_m$, etc.

We now justify the various steps in Algorithm 7.6. For (1) we use the next proposition.

**Proposition 7.9.** *Let $N > 0$ and let $z \ne 0$ be in $U_0$. If $M \ge 0$ is such that $p^m v_p(z) - mn > N$ for all $m > M$, $\tilde z$ is an approximation of $z$ up to precision $N + nM$, and for $m = 0, \ldots, M$ we choose $\mathsf{tsl}_m > 0$ such that $kp^m v_p(z) - mn > N$ for $k \ge \mathsf{tsl}_m$, then*

$$\sum_{m=0}^{M} p^{-mn} \sideset{}{'}\sum_{k=1}^{\mathsf{tsl}_m-1} \frac{\tilde z^{kp^m}}{k^n}\,,$$

*where the prime indicates that we sum only over $k$ that are not divisible by $p$, is an approximation of $\mathrm{Li}_n(z)$ up to precision $N$. Moreover, we can replace each $1/k^n$ by an approximation up to precision $N + nM$.*

*Proof.* It is clear from (3.3) that $|\mathrm{Li}_n^{(p)}(z)| = |z|$ when $|z| < 1$. Therefore, from (3.4) we see that in order to compute $\mathrm{Li}_n(z)$ up to precision $N$ we only have to compute $\sum_{m=0}^{M} p^{-mn}\mathrm{Li}_n^{(p)}(z^{p^m})$ if $p^m v_p(z) - mn > N$ for all $m > M$. So we reduce to the calculation of $p^{-mn}\mathrm{Li}_n^{(p)}(z^{p^m})$ up to precision $N$ for $m = 0, \ldots, M$. In the corresponding power series $p^{-mn}\sum_{k=1}^{\prime\infty} z^{kp^m}/k^n$ we can ignore terms with valuation bigger than $N$, i.e., where $kp^m v_p(z) - mn > N$.

Finally, for each term $p^{-mn} z^{kp^m}/k^n$ that we compute we can replace $z$ with an approximation $\tilde{z}$ satisfying $v_p(z-\tilde{z}) > N+mn$ and $1/k^n$ with an approximation $b_k$ satisfying $v_p(1/k^n - b_k) > N + mn$, and obtain that term up to precision $N$. This follows from the identity $z^{kp^m}/k^n - b_k\tilde{z}^{kp^m} = (z^{kp^m} - \tilde{z}^{kp^m})/k^n + (1/k^n - b_k)\tilde{z}^{kp^m}$ since $v_p(k^n) = 0$, and $v_p(z) > 0$ implies that $v_p(z^{kp^m} - \tilde{z}^{kp^m}) \geq v_p(z - \tilde{z})$ and $v_p(\tilde{z}^{kp^m}) > 0$. □

For (2) we use that $\mathrm{Li}_n(z) = (-1)^{n-1}\mathrm{Li}_n(1/z) - \log^n(z)/n!$ by Proposition 2.9(2). For the term $\mathrm{Li}_n(1/z)$ we use (1), and the corresponding precision of $z$ also follows from this part together with Remark 7.2(4) because we are assuming that we know $z$ up to positive relative precision. The term $-\log^n(z)/n!$ can be readily calculated using standard methods, so we only give estimates for the precision of $z$ that enables us to calculate it up to precision $N$.

**Lemma 7.10.** *If $y$ in $\mathbb{C}_p$ satisfies $|y| \leq p^{-V}$ and is known up to precision $N' \geq V$ then $y^n$ for $n \geq 1$ is known up to precision $N' + (n-1)V$.*

*Proof.* By assumption we know $y + \varepsilon$ for some $\varepsilon$ in $\mathbb{C}_p$ with $|\varepsilon| < p^{-N'}$. Then $|(y+\varepsilon)^n - y^n| = |\varepsilon| \cdot |y^{n-1} + \varepsilon y^{n-2} + \cdots + \varepsilon^{n-2}y + \varepsilon^{n-1}| < p^{-N'}p^{-(n-1)V}$ because $N' \geq V$. □

As for the logarithm we have the following result.

**Lemma 7.11.** *If we know $z$ in $\mathbb{C}_p^*$ up to precision $N' > v_p(z)$ then we can calculate $\log(z)$ up to precision $\min_{m\geq 0}\{(N' - v_p(z))p^m - m\}$. If $1 \leq (N' - v_p(z))\log(p)$ then we can calculate $\log(z)$ up to precision $N' - v_p(z)$.*

*Proof.* Since we know $z$ up to precision $N'$ we know $\tilde{z} = z + \varepsilon$ with $|\varepsilon| < p^{-N'}$. Then $\log(\tilde{z}) - \log(z) = \log(1 + \varepsilon/z) = -\mathrm{Li}_1(-\varepsilon/z)$. Since $v_p(\varepsilon/z) > N' - v_p(z) > 0$ the estimates in Remark 7.5 apply. □

We can now give a sufficient precision for $z$ in order to compute $\log^n(z)/n!$ up to precision $N$. Note that Lemma 7.11 in practice allows us to find a lower bound for $v_p(\log(z))$, as is needed in (2) and the next proposition. (But see also Remark 8.1.)

**Proposition 7.12.** *If, for $z$ in $\mathbb{C}_p^*$, $|\log(z)| \leq p^{-V}$, and we know $z$ up to precision $N' > v_p(z)$ satisfying*

$$\max\{V, N + v_p(n!) - (n-1)V\} \leq \min_{m\geq 0}\{(N' - v_p(z))p^m - m\},$$

*then we can compute $\log^n(z)/n!$ up to precision $N$.*

*Proof.* It suffices to calculate $\log^n(z)$ up to precision $N+v_p(n!)$. By Lemma 7.10 we can do this if we know $\log(z)$ up to precision $N''=\max\{V,N+v_p(n!)-(n-1)V\}$. In order to compute $\log(z)$ up to precision $N''$ by Lemma 7.11 it suffices to know $z$ up to precision $N'>v_p(z)$ satisfying $\min_{m\geq 0}\{(N'-v_p(z))p^m-m\}\geq N''$. □

Step (3) of the algorithm follows from Propositions 6.7 and 7.13, Remark 7.14, and Proposition 7.15 together with its proof.

**Proposition 7.13.** *Let $g_n(v)$ be as in Proposition 4.3 and $c(n,k)$ as in Proposition 6.1. To compute $g_n(\alpha)$ with $v_p(\alpha)=0$ with precision $N>0$, it suffices to know $\alpha$ up to precision $N$ and evaluate the sum of the terms in $g_n(v)$ of degree less than $\mathsf{gsl}$ at the approximation of $\alpha$, where $\mathsf{gsl}$ is such that $\frac{k}{p-1}-\log_p(k)-c(n,p)>N$ for all $k\geq\mathsf{gsl}$. In fact, it suffices to use approximations up to precision $N$ for the coefficients of the terms of degree less than $\mathsf{gsl}$.*

*Similarly, in order to know $\mathrm{Li}_n^{(p)}(\zeta)$ up to precision $N>0$ for a root of unity $\zeta\neq 1$ of order not divisible by $p$, it suffices to know $\zeta$ and the coefficients of the terms of $g_n(v)$ of degree less than $\mathsf{gsl}$ up to precision $N$, where $\mathsf{gsl}$ is as before.*

*Proof.* The first statement is clear from the integrality of the coefficients of $g_n$ as given in Proposition 6.1. The second statement is clear since $\mathrm{Li}_n^{(p)}(\zeta)=g_n(1/(1-\zeta))$ and $1/(1-\zeta)$ is known to the same precision as $\zeta$ by Remark 7.2(4) because $v_p(1/(1-\zeta))=0$. □

**Remark 7.14.** Computing the coefficients of $g_n(v)$ as rational numbers is very inefficient so, instead, we use coefficients in $\mathbb{Q}_p$ with finite precision. If $g_n(v)=a_{n,1}v+a_{n,2}v^2+\cdots$ then $ka_{n+1,k}=-(a_{n,1}+a_{n,2}+\cdots+a_{n,k})$ by Proposition 4.3. So the error in an approximation of the $a_{n,k}$ for $k\leq\mathsf{gsl}-1$ can grow by at most $\lfloor\log_p(\mathsf{gsl}-1)\rfloor$ to an error in the $a_{n+1,k}$. Therefore, using the method of Proposition 4.3, we can compute the $a_{n,k}$ for $k=1,\ldots,\mathsf{gsl}-1$ up to precision $N$ if we know the $a_{0,k}$ for $k=1,\ldots,\mathsf{gsl}-1$ up to precision $N+n\lfloor\log_p(\mathsf{gsl}-1)\rfloor$.

For the power series $F_{n,\zeta}(t)$ instead of $g_n(t)$, where $\zeta\neq 1$ is a $(p^f-1)$-th root of unity, the corresponding statements in the next proposition are more involved.

**Proposition 7.15.** *Assume that $z$ lies in the residue disc $U_\zeta$ for $\zeta\neq 1$ a $(p^f-1)$-th root of unity and let*

$$F_{n,\zeta}(t)=a_{n,0}+a_{n,1}t+a_{n,2}t^2+\cdots \tag{7.16}$$

*be the Taylor expansion of $\mathrm{Li}_n(z)$ around $\zeta$ as in Proposition 4.4. Let $\mathsf{tsl}>0$ be such that $v_p(a_{n,k}(z-\zeta)^k)>N$ for all $k\geq\mathsf{tsl}$ and assume that the $\mathrm{Li}_m(\zeta)$ for $m=1,\ldots,n$ are known up to precision $N+(n-m)\log_p(\mathsf{tsl}-1)$. If $\zeta$ and $z$ are known up to precision $N+n\log_p(\mathsf{tsl}-1)$ then we can compute $\mathrm{Li}_n(z)$ up to precision $N$ by computing the terms of degree smaller than $\mathsf{tsl}$ in (7.16) using the methods of Proposition 4.4 but with the approximate values for $\zeta$ and $\mathrm{Li}_m(\zeta)$, and evaluating those on the approximation of $z-\zeta$.*

*Proof.* Note that $\mathsf{tsl}$ as in the statement of the proposition exists by Proposition 6.7 or Remark 6.9. We can therefore compute $\mathrm{Li}_n(z)$ up to precision $N$ by computing $a_{n,0}+a_{n,1}(z-\zeta)+\cdots+a_{n,\mathsf{tsl}-1}(z-\zeta)^{\mathsf{tsl}-1}$. Using an approximation $\tilde{\zeta}$ of $\zeta$ in (4.5) we have an approximation

$$\widetilde{F}_{m,\zeta}(t)=\tilde{a}_{m,0}+\tilde{a}_{m,1}t+\cdots$$

of $F_{m,\zeta}(t)$ for $m=0$. We use $\widetilde{F}_{m,\zeta}(t)$ instead of $F_{m,\zeta}(t)$ and $\tilde{\zeta}$ instead of $\zeta$ in (4.6) in order to inductively obtain an approximation

$$\widetilde{F}_{m+1,\zeta}(t) = \tilde{a}_{m+1,0} + \tilde{a}_{m+1,1}t + \cdots$$

of $F_{m+1,\zeta}(t)$. Here $k\tilde{a}_{m+1,k} = -\sum_{j=0}^{k-1}(-\tilde{\zeta})^{j-k}\tilde{a}_{m,j}$ for $k=1,2,\ldots,\mathsf{tsl}-1$, as in (6.8). Inductively we may assume that $v_p(\tilde{a}_{m,j}-a_{m,j}) > N+(n-m)\log_p(\mathsf{tsl}-1)$ for $j=0,\ldots,\mathsf{tsl}-1$ since this holds for $m=0$ by our assumption on $v_p(\tilde{\zeta}-\zeta)$, Remark 7.2(4) and (4.5). Then for $k=1,\ldots,\mathsf{tsl}-1$ we have, as in the proof of Proposition 6.7,

$$\begin{aligned} k(a_{m+1,k}-\tilde{a}_{m+1,k}) &= \sum_{j=0}^{k-1}\Big((-\tilde{\zeta})^{j-k}\tilde{a}_{m,j} - (-\zeta)^{j-k}a_{m,j}\Big) \\ &= \sum_{j=0}^{k-1}(-1)^{j-k}\left[\tilde{\zeta}^{j-k}(\tilde{a}_{m,j}-a_{m,j}) + (\tilde{\zeta}^{j-k}-\zeta^{j-k})a_{m,j}\right]. \end{aligned}$$

Since $v_p(\tilde{\zeta}^{j-k}) \geq 0$ and $v_p(\tilde{\zeta}^{j-k}-\zeta^{j-k}) > N + n\log_p(\mathsf{tsl}-1)$ by our assumptions, by Proposition 6.7 it follows as in the proof of that proposition that

$$v_p(\tilde{a}_{m+1,k}-a_{m+1,k}) > N + (n-(m+1))\log_p(\mathsf{tsl}-1)$$

for $k=1,\ldots,\mathsf{tsl}-1$. Finally, $v_p(\tilde{a}_{m+1,0}-a_{m+1,0}) > N+(n-(m+1))\log_p(\mathsf{tsl}-1)$ since we assume that we have such an approximation $\tilde{a}_{m+1,0}$ for $a_{m+1,0}=\mathrm{Li}_{m+1}(\zeta)$.

Now we consider each term $a_{n,j}(z-\zeta)^j$ in (7.16) that we have to evaluate for $j<\mathsf{tsl}$. If $j=0$ then $v_p(a_{n,0}-\tilde{a}_{n,0}) > N$ by our assumptions. If $j=1,\ldots,\mathsf{tsl}-1$ then, with $\tilde{z}$ an approximation of $z$ of precision $N+n\log_p(\mathsf{tsl}-1)$, the corresponding approximation is $\tilde{a}_{n,j}(\tilde{z}-\tilde{\zeta})^j$. But

$$a_{n,j}(z-\zeta)^j - \tilde{a}_{n,j}(\tilde{z}-\tilde{\zeta})^j = (a_{n,j}-\tilde{a}_{n,j})(z-\zeta)^j + \tilde{a}_{n,j}((z-\zeta)^j - (\tilde{z}-\tilde{\zeta})^j)$$

and we have just seen that $v_p(a_{n,j}-\tilde{a}_{n,j}) > N$. That $z$ lies in $U_\zeta$ means that $v_p(z-\zeta)>0$ so that the first term on the right has valuation bigger than $N$. Since also $v_p(z-\tilde{z})>0$ and $v_p(\tilde{\zeta}-\zeta)>0$ we see that the second term has valuation at least $v_p(\tilde{a}_{n,j}) + v_p((z-\zeta)-(\tilde{z}-\tilde{\zeta}))$. Because $v_p(\tilde{a}_{n,j}) \geq \min\{v_p(a_{n,j}-\tilde{a}_{n,j}), v_p(a_{n,j})\} \geq -n\log_p(\mathsf{tsl}-1)$ by Proposition 6.7 this valuation is at least $N$ by our assumptions on $v_p(z-\tilde{z})$ and $v_p(\zeta-\tilde{\zeta})$. □

Finally we justify part (4) of Algorithm 7.6. In order to calculate $\alpha\beta$ up to precision $N$ if the valuations of $\alpha$ and $\beta$ are bounded below as $V_\alpha \leq v_p(\alpha)$ and $V_\beta \leq v_p(\beta)$, it suffices to compute $\alpha$ up to precision $N-V_\beta$ and $\beta$ up to precision $N-V_\alpha$, at least if $N-V_\alpha-V_\beta \geq 0$. By increasing $N$ if necessary we can always assume this.

We want to apply this to each of the terms in the formula in Proposition 5.8 for $z\neq 1$ in $U_1$. Using Lemma 7.11 we can find $V$ and $V_1$ with $V \leq v_p(\log(z))$ and $V_1 \leq v_p(\mathrm{Li}_1(z)) = v_p(\log(1-z))$. Also, if we take $W\leq 0$ such that

$$W \leq \min_{k\geq 1}\{kv_p(z-1) - n\log_p(k)\}$$

then $W \leq v_p((n-j-1)E_{n-j}(z))$ for $j=2,\ldots,n$ by Proposition 6.10. If necessary we now increase $N$ so that

$$N + v_p((n-1)!) - v_p((n-j-2)!) - W - jV \geq 0$$

for $j = 0, \dots, n-2$, as well as

$$N + v_p((n-1)!) - V_1 - (n-1)V \geq 0\,,$$

and compute:

(i) $\log^j(z)$ up to precision $N + v_p((n-1)!) - v_p((n-j-2)!) - W$ for $j = 1, \dots, n-2$, putting $\log^0(z) = 1$;
(ii) $(n-j-1)E_{n-j}(z)$ up to precision $N + v_p((n-1)!) - v_p((n-j-2)!) - jV$ for $j = 0, \dots, n-2$;
(iii) $\log^{n-1}(z)$ up to precision $N + v_p((n-1)!) - V_1$;
(iv) $\mathrm{Li}_1(z)$ up to precision $N + v_p((n-1)!) - (n-1)V$.

Since $\mathrm{Li}_1(z) = -\log(1-z)$ we can deal with (i), (iii) and (iv) by applying Proposition 7.12 and Lemma 7.11, taking into account that $v_p(z) = 0$. This gives (4)(d) and (4)(e) of Algorithm 7.6 For (ii) we use the following result, but in the algorithm we use it, and, if $p = 2$, also Proposition 7.19, for the uniform precision

$$N_E = \max_{j=0,\dots,n-2} \{N + v_p((n-1)!) - v_p((n-j-2)!) - jV\}$$

rather than $N$.

**Proposition 7.17.** *Assume $z$ is in the residue disc $U_1$ and, for $n \geq 2$, let*

$$H_n(t) = b_{n,0} + b_{n,1}t + b_{n,2}t^2 + \cdots \tag{7.18}$$

*be the Taylor expansion of $(n-1)E_n(z)$ around 1 as in Proposition 6.10. Let $\mathsf{hsl} > 0$ be such that $v_p(b_{n,k}(z-1)^k) > N$ for all $k \geq \mathsf{hsl}$ and assume that $(m-1)E_m(1)$ for $m = 2, \dots, n$ is known up to precision $N + (n-m)\log_p(\mathsf{hsl}-1)$. If $z$ is known up to precision $N + n\log_p(\mathsf{hsl}-1)$ then we can compute $(n-1)E_n(z)$ up to precision $N$ by computing the terms of degree smaller than $\mathsf{hsl}$ in (7.18) by integration of (6.12) but with the approximate value for $H_n(0) = (n-1)E_n(1)$, and evaluating those on the approximation of $z-1$.*

*Proof.* This is the exact analogue of Proposition 7.15 but for $H_n(t)$, and with the simplification that the role of $\zeta$ is now played by 1 so that we can take $\tilde{\zeta} = 1$. The proof is the same (or somewhat simpler) since the estimates for $v_p(b_{m,k})$ for $k > 0$ (see Proposition 6.10) are the same as for $v_p(a_{m,k})$ and $v_p(b_{m,0}) \geq 0$ (see Proposition 6.7). In this case they allow us to show, starting from the expression for $H_2(t)$ as in (6.11) but with coefficients up to precision $N + (n-2)\log(\mathsf{hsl}-1)$, that for the approximations $\tilde{b}_{m,k}$ of $b_{m,k}$ we obtain iteratively by using (6.12) as well as the approximations of $H_m(0) = (m-1)E_m(1)$, we have $v_p(\tilde{b}_{m,k} - b_{m,k}) > N + (n-m)\log_p(\mathsf{hsl}-1)$ for $k = 0, \dots, \mathsf{hsl}-1$ and $m = 2, \dots, n$. □

By Proposition 7.17 we reduce to calculating $(m-1)E_m(1)$ up to precision $N+(n-m)\log_p(\mathsf{hsl}-1)$ for $m = 2, \dots, n$. For $p \neq 2$ we can already do this since, by Proposition 5.6, $(m-1)E_m(1) = (m-1)G_n(0) = (m-1)2^{m-1}\mathrm{Li}_m(-1)/(1-2^{m-1})$ so we only have to calculate $\mathrm{Li}_m(-1)$ up to precision

$$N + (n-m)\log_p(\mathsf{hsl}-1) + v_p(1-2^{m-1}) - v_p(m-1)$$

for $m = 2, \dots, n$ and we can do using step (3) of Algorithm 7.6.

But for $p = 2$ we use Proposition 5.7. For this we formulate a supplement to Proposition 7.17. Note that we can always attain the condition on $\mathsf{hsl}$ below by increasing it if necessary.

**Proposition 7.19.** *Let $p = 2$ and assume that* $\mathsf{hsl}$ *in Proposition 7.17 also satisfies*

$$k + v_p(b_{m,k}) > N - (m-1) + (n-m)\log_p(\mathsf{hsl} - 1)$$

*for all $k \geq \mathsf{hsl}$ and all $m = 2, \ldots, n$. Then we can, in the inductive process in the proof of Proposition 7.17, calculate $(m-1)E_m(1)$ as $2^{m-1}H_m(-2)/(1-2^m)$ using the non-constant terms in the approximation of $H_m(t)$ that we obtain, for $m = 2, \ldots, n$, up to precision $N + (n-m)\log_p(\mathsf{hsl} - 1)$.*

*Proof.* By multiplying the expression in Proposition 5.7 for $n = m$ by $m-1$ we see that for $m \geq 2$ we can compute $(m-1)E_m(1)$ as

$$(m-1)E_m(1) = H_m(0) = 2^{m-1}\frac{H_m(-2) - H_m(0)}{1 - 2^m}.$$

The extra condition on $\mathsf{hsl}$ means that the terms of degree less than $\mathsf{hsl}$ in $H_m(t)$ suffice to compute this expression up to precision $N + (n-m)\log_p(\mathsf{hsl} - 1)$. Since the estimate $v_p(\tilde{b}_{m,k} - b_{m,k}) > N + (n-m)\log_p(\mathsf{hsl} - 1)$ for $k = 1, \ldots, \mathsf{hsl} - 1$ still applies at each stage in the process we see that we can use the approximate coefficients for the calculation of $H_m(-2) - H_m(0)$ and still compute $H_m(-2)$ up to precision $N + (n-m)\log_p(\mathsf{hsl} - 1)$. Using those values in Proposition 7.17 we see that we can find the coefficients of the terms of degree less than $\mathsf{hsl}$ in $H_n(t)$ up to precision $N$. □

This concludes the justification of Algorithm 7.6.

## 8. Concluding remarks

In this section we describe how to make the estimates in it uniform for all elements in a fixed finite extension of $\mathbb{Q}_p$, analyze the corresponding asymptotic time and make a remark about an alternative approach for computing the constant term of the $\mathbb{F}_{n,\zeta}(t)$.

**Remark 8.1.** In case one wants to compute $\mathrm{Li}_n(z)$ for several $z$ in a field $F$ with finite ramification degree $e$ over $\mathbb{Q}_p$ it is probably more efficient to compute the (approximations of the truncated) power series in Algorithm 7.6 as they are needed using universal estimates and remember them.

Namely, if $z$ lies in the residue disc $U_a$ with $a \neq \infty$ then $v_p(z-a) \geq 1/e$. For $\log(z)$ with $z$ in $F^*$ we observe that $v_p(\log(z)) \geq v_p(\log(p)) + v_p(\log(y)) - 2v_p(e)$ for some $y$ in $U_1$ because we can take $b = e$ in Remark 2.2 and $\log(\eta) = 0$ for any root of unity $\eta$. Then $y = 1 + x$ with $v_p(x) \geq 1/e$ and, for $m \geq 0$, $y^{p^m} = (1+x)^{p^m} = 1 + x'$ with

$$v_p(x') \geq \min\{m + 1/e, (m-1) + p/e, \ldots, 1 + p^{m-1}/e, p^m/e\}$$

as one sees easily by induction on $m$. If we choose $m \geq 0$ such that this minimum is at least $1/(p-1)$, i.e., such that $p^m \geq e/(p-1)$, then $v_p(p^m \log(1+x)) = v_p(\log(1+x')) \geq v_p(x') \geq 1/(p-1)$ by [Was97, Lemma 5.5]. Therefore $v_p(\log(z)) \geq v_p(\log(p)) + 1/(p-1) - 2v_p(e) - m$ for all $z$ in $F^*$.

Using those bounds one can obtain, in each of the four cases in the algorithm, universal estimates for the lengths of the power series involved, etc., or the precision required for $z$.

We conclude by analyzing the efficiency of Algorithm 7.6.

**Proposition 8.2.** *Suppose that $z$ belongs to a finite extension $F$ of $\mathbb{Q}_p$ with ramification index $e$ and residue extension degree $f$. Then Algorithm 7.6 computes $\mathrm{Li}_n(z)$ to precision $N$ in asymptotic time $O(fN^2(nfp+ne^{1+\epsilon}+e^{2+\epsilon})(Nfp)^\epsilon)$ with respect to $N$.*

*Proof.* We note that we assume that fast arithmetic is used, and that we have used the epsilon notation to hide all kinds of log factors. We only count multiplications. We assume that $F$ is given explicitely as a purely ramified extension of degree $e$ of an unramified extension $F^{\mathrm{unr}}$ of $\mathbb{Q}_p$ of degree $f$. Let us follow the steps of the algorithm in this case. We have to do three types of multiplications, whose time estimates for precision $x$ are as follows:

- multiplications in $\mathbb{Q}_p$, which take time
$$O((x\log(p))^{1+\epsilon}) = O(x^{1+\epsilon}p^\epsilon)\,;$$
- multiplications in $F^{\mathrm{unr}}$, which take time
$$O((x\log(p^f))^{1+\epsilon}) = O((xf)^{1+\epsilon}p^\epsilon)\,;$$
- multiplications in $F$, which take time
$$O((xe\log(p^f))^{1+\epsilon}) = O((xfe)^{1+\epsilon}p^\epsilon)\,.$$

Now let us consider each of the steps in Algorithm 7.6(3).

(a) By its definition and Remark 8.1 we clearly have $\mathsf{tsl} = O(Ne)$ and we find $\tilde\zeta$ in $F^{\mathrm{unr}}$ by using Newton's method to solve the equation $x^{p^f-1} = 1$. As the required precision is about $N$ the algorithm will be about $\log(N) = N^\epsilon$ steps, each consisting of about $\log(p^f) = fp^\epsilon$ multiplications which are carried out again to precision $N$ in $F^{\mathrm{unr}}$. The total complexity is
$$O(N^\epsilon fp^\epsilon(Nf)^{1+\epsilon}p^\epsilon) = O(N^{1+\epsilon}f^{2+\epsilon}p^\epsilon)\,.$$

(b) Here we have $\mathsf{gsl} = O(pN)$.

(c) Now we have to compute the expansion of the $g_m(v)$'s to $\mathsf{gsl}$ places. There are $n$ steps, each consisting of a polynomial multiplication of length $O(pN)$ with the coefficients up to precision $N$. Note that this computation is done in $\mathbb{Q}_p$. It is done in time
$$O(n(pN)^{1+\epsilon}N^{1+\epsilon}p^\epsilon) = O(np^{1+\epsilon}N^{2+\epsilon})\,.$$

(d) Each of our $g_m(v)$ have to be evaluated at $f$ elements derived from powers of the $\tilde\zeta$. This is done in $F^{\mathrm{unr}}$. As the precision is $O(N)$ and the polynomials are of length $O(pN)$ the complexity is
$$O(nfNp(Nf)^{1+\epsilon}p^\epsilon) = O(np^{1+\epsilon}(Nf)^{2+\epsilon})\,.$$

(e) The calculation of $\mathrm{Li}_m(\zeta)$ is now done without any further multiplications, other than with powers of $p$.

(f) The computation of $F_{n,\zeta}(t)$ involves $n$ times multiplication of polynomials of degree $O(Ne)$ with coefficients in $F^{\mathrm{unr}}$ hence takes time
$$O(n(Ne)^{1+\epsilon}(Nf)^{1+\epsilon}p^\epsilon) = O(np^\epsilon(fe)^{1+\epsilon}N^{2+\epsilon})\,.$$

(g) This step consists in evaluating a polynomial of degree $O(Ne)$ at elements of $F$, so takes time
$$O((Ne)^{1+\epsilon}(Nfe)^{1+\epsilon}p^\epsilon) = O(p^\epsilon f^{1+\epsilon}(Ne)^{2+\epsilon})\,.$$

Looking at the time complexities of all the steps we see that those of (d), (f) and (g) dominate, and their sum gives our time estimate. □

**Remark 8.3.** (1) The estimates in Proposition 8.2 are based on computing the value at $z$ from scratch. As mentioned in Remark 8.1, if we want to compute $\mathrm{Li}_n(z)$ for several $z$ in the same residue disc $U_\zeta$ then it is more efficient to compute the approximation of the truncation of $F_{n,\zeta}(t)$ and remember it since then only (g) will have to be performed again.

(2) The time estimates in Proposition 8.2 are worst case estimates. If $z$ is closer to the root of unity $\zeta$ the complexities of (f) and (g) are reduced since one needs fewer terms in the power series expansion around $\zeta$ to achieve the required complexity.

**Remark 8.4.** In Proposition 5.7 (where $p = 2$) we calculate the constant term of $G_n(t)$ without ever using $g_n(v)$. We can do this also for $F_{n,\zeta}(t)$ if $p$ is any prime and $\zeta$ is any root of unity of order dividing $p^f - 1$, at the cost of possibly having to adjoin the $p^f$-th roots of unity to the field. Namely, by Proposition 2.9(1) we have that

$$\mathrm{Li}_n(\zeta^{p^f}) = p^{f(n-1)} \sum_{\eta^{p^f}=1} \mathrm{Li}_n(\eta\zeta)$$

and since $\zeta^{p^f} = \zeta$ this determines the constant of integration in (4.6) just as in Proposition 5.7 because all $\eta\zeta$ lie in $U_\zeta$. However, comparing the estimates for steps (a)-(e) in the proof of Proposition 8.2 with those of performing (g) for all $\eta\zeta$ we see that this method in general is less efficient than calculating the power series $g_n(v)$ and evaluating it at $1/(1-\zeta^{p^j})$ for $j = 1, \ldots, f$, as is necessary for applying Proposition 4.2.

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