SEPARATING THE POWER OF MONOTONE SPAN PROGRAMS OVER DIFFERENT FIELDS*

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Abstract. Monotone span programs are a linear-algebraic model of computation. They are equivalent to linear secret sharing schemes and have various applications in cryptography and complexity. A fundamental question is how the choice of the field in which the algebraic operations are performed effects the power of the span program. In this paper we prove that the power of monotone span programs over finite fields of different characteristics is incomparable; we show a super-polynomial separation between any two fields with different characteristics, answering an open problem of Pudlák and Sgall 1998. Using this result we prove a super-polynomial lower bound for monotone span programs for a function in uniform $-\mathcal{NC}^2$ (and therefore in $\mathcal{P}$), answering an open problem of Babai, Wigderson, and Gál 1999. (All previous super-polynomial lower bounds for monotone span programs were for functions not known to be in $\mathcal{P}$.) Finally, we show that quasi-linear secret sharing schemes, a generalization of linear secret sharing schemes introduced in Beimel and Ishai 2001, are stronger than linear secret sharing schemes. In particular, this proves, without any assumptions, that non-linear secret sharing schemes are more efficient than linear secret sharing schemes.

Key words. Monotone span programs, Algebraic models of computation, Lower bounds, Secret sharing

AMS subject classifications. 68Q05, 68Q17, 68R05, 68Q70

1. Introduction. The relation between computational complexity and linear algebra is an important research direction with two main avenues. On one hand, algebraic techniques were used to prove lower bounds in combinatorics [1, 15, 17] and complexity, e.g., [27, 20, 24]. On the other hand, algebraic computational models, which capture the essence of linear algebra, were defined. Such models include, for example, arithmetic circuits, Boolean circuits with MOD_p gates, and the Blum-Shub-Smale model of computation [9].

In this paper we discuss the algebraic computational model of span programs, introduced by Karchmer and Wigderson [18]. Intuitively, span programs capture the power of basic linear algebraic operations—the rank and dependency of a set of vectors. More specifically, a monotone span program is presented as a matrix over some field, with rows labelled by variables. The span program accepts an input if the rows whose variables are satisfied by the input span a fixed nonzero vector. The size of a span program is its number of rows. A detailed definition is given in §2.

This paper deals with the role of the field in algebraic models of computation. Part of the specification of algebraic models of computation, in particular span programs, is the field in which the arithmetic operations are performed. A fundamental question is how the choice of the field, and especially its characteristic, effects the power of the model. As different fields may differ substantially in their structure, especially when the characteristics of the fields are different, it would be natural to expect computational models defined over different fields to differ significantly in their power. A major result separating the power of algebraic models of computation over different

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fields was the seminal paper by Smolensky for bounded depth circuits with MOD_p gates [27]. Lower bounds related to the characteristic of the field are also known for polynomial calculus proofs [6]. However, the power of the field in algebraic models of computation is yet to be fully understood.

Our Results. The main contribution of this paper is showing that the power of monotone span programs over finite fields of different characteristic is incomparable. Prior to this work, the best separation known for monotone span programs, was a logarithmic separation for the threshold function [18].[1] In this paper we show a super-polynomial separation between any two fields with different characteristics, answering an open problem of [23]. That is, for every fixed prime number p we describe a function which has a small monotone span program over the field with p elements, but requires a monotone span program of size $n^{\Omega(\sqrt{\log n})}$ over any field whose characteristic is not p (including fields with characteristic 0).

Our second contribution concerns the functions for which lower bounds for monotone span programs have been proved. The best known lower bound for monotone span programs, proved by Gál [13], is $n^{\Omega(\log n)}$ (improving previous results of [3, 2]). However, all the known super-polynomial lower bounds [2, 13, 14] were for functions in $\mathcal{NP}$, not known to be in $\mathcal{P}$. We show a lower bound of $n^{\Omega(\sqrt{\log n})}$ for a function in uniform $-\mathcal{NC}^2$ (and therefore in $\mathcal{P}$), thus answering an open problem of [2].[2]

Our third contribution concerns secret sharing schemes, which are an important tool in cryptography, introduced by Blakley [8], Shamir [25], and Ito, Saito, and Nishizeki [16]. A *secret sharing scheme* enables a dealer to share a secret among a set of parties, such that only some pre-defined authorized subsets will be able to reconstruct the secret from their shares. The authorized sets correspond to a monotone Boolean function $f : \{0,1\}^n \to \{0,1\}$, where n is the number of parties and the authorized subsets are the subsets with their characteristic vectors in $f^{-1}(1)$. The efficiency of a secret sharing scheme is the overall size of the shares given to the parties. Monotone span programs are equivalent to a subclass of secret sharing schemes called *"linear secret sharing schemes."* Monotone span programs were also used in other cryptographic applications, e.g., [22, 11]. Beimel and Ishai [4] showed functions that, under plausible assumptions, have no efficient linear secret sharing scheme but yet have an efficient non-linear secret sharing scheme. Furthermore, they introduced the class of quasi-linear secret sharing schemes. In this paper we show that quasi-linear secret sharing schemes are stronger than linear schemes. In particular, this proves, without any assumptions, that non-linear schemes are more efficient than linear schemes.

Highlights of the Techniques. Proving a separation between the power of two models of computation requires a function with both a lower bound for one model, and an upper bound for the other. To get the lower bound for monotone span program over a certain field, we use the method of [13], which is based on [24]. In the center of Gál's method is a matrix whose rank over the field is much larger than its combinatorial cover number. To get the upper bound for the same function for monotone span programs over another field, we require that the cover has an additional property which is related to the characteristic of the field. As an example, for GF(2) we

[1] It was known that span programs over finite fields with the same characteristic basically have the same power.

[2] We note that every function which has a polynomial size monotone $\mathcal{NC}^1$ circuit has a polynomial size monotone span program, and every function which has a polynomial size span program over a small field has a polynomial size $\mathcal{NC}^2$ circuit.

require that each entry of the matrix is covered by an odd number of rectangles. Our use of combinatorial covers and their properties is borrowed from communication complexity (see [19] for background on communication complexity). In particular, we use ideas similar to [12], where they considered the model of counting communication complexity.

The main technical contribution of this paper is constructing such a matrix and proving that it satisfies the desired properties. In particular, the matrix we construct checks whether two linear subspaces over GF(p) have non-trivial intersection. Not surprisingly, the matrix reflects linear algebraic computations over GF(p), which are difficult to simulate over fields with characteristics different than p.

Organization. In §2 we supply some preliminaries. In §3 we give a general method for proving a separation between the power of monotone span programs over fields with different characteristics. Next, in §4 we apply this general method to achieve a separation of $n^{\Omega(\sqrt{\log n})}$ for an explicit function. Finally, in §5, we use this separation to exhibit a monotone function in uniform $-\mathcal{NC}^2$ that has no polynomial size monotone span program, and to prove that there exist secret sharing schemes stronger than the linear secret sharing schemes.

2. Preliminaries. We start with the definition of our main computational model — span programs.

DEFINITION 2.1 (Span Program [18]). *A* span program *over a field F is a triplet $\widehat{M} = \langle M, \rho, \vec{v}\rangle$, where M is a matrix over F, $\vec{v}$ is a nonzero row vector called the target vector (it has the same number of coordinates as the number of columns in M), and ρ is a labelling of the rows of M by literals from $\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$ (every row is labelled by one literal, and the same literal can label many rows).*

A span program accepts or rejects an input by the following criterion. For every input $u \in \{0,1\}^n$ define the sub-matrix M_u of M consisting of those rows whose labels are satisfied by the assignment u. The span program $\widehat{M}$ accepts *u if and only if $\vec{v} \in$ span(M_u), i.e., some linear combination of the rows of M_u gives the target vector $\vec{v}$. A span program* computes *a Boolean function f if it accepts exactly those inputs u where $f(u) = 1$. The* size *of $\widehat{M}$ is the number of rows in M.*[3]

A span program is called monotone *if the labels of the rows are only positive literals $\{x_1, \ldots, x_n\}$. Monotone span programs compute only monotone functions, and every monotone Boolean function can be computed by a monotone span program. The size of the smallest monotone span program over F that computes f is denoted by* $\mathrm{mSP}_F(f)$.

EXAMPLE 2.2. Consider the following monotone span program over GF(2):

x_2	1	1	0	0	0
x_2	0	1	1	1	0
x_1	0	1	1	1	0
x_3	0	1	0	1	1
x_4	0	0	1	0	1

In this example, the target vector is $\vec{v} = \langle 1, 0, 0, 1, 1\rangle$. There are 4 different variables labelling the rows of the matrix and the inputs are of size 4. Consider the input

[3]The choice of the fixed nonzero vector $\vec{v}$ does not effect the size of the span program. It is always possible to replace $\vec{v}$ by another nonzero vector $\vec{v}'$ via a change of basis without changing the function computed and the size of the span program. Most often $\vec{v}$ is chosen to be the $\vec{1}$ vector (with all entries equal 1).

$\langle 0,1,0,1\rangle$. Since x_2 and x_4 are satisfied by the input, we consider the submatrix consisting of rows labelled by these variables:

x_2	1	1	0	0	0
x_2	0	1	1	1	0
x_4	0	0	1	0	1

The question is whether the rows of this submatrix span the target vector $\langle 1,0,0,1,1\rangle$. Since $\langle 1,0,0,1,1\rangle$ is the sum, over GF(2), of the rows of the submatrix, the input is accepted by the program.

Now consider the input $\langle 1,1,0,0\rangle$. Again, we focus on the submatrix of rows labelled by x_1 and x_2, the variables satisfied by the input assignment:

x_2	1	1	0	0	0
x_2	0	1	1	1	0
x_1	0	1	1	1	0

Looking at the rightmost coordinate, we see the all the submatrix entries in this column are 0, while in the target vector $\langle 1,0,0,1,1\rangle$ it is 1. Hence, no linear combination of the rows of submatrix gives the target vector. Therefore, the input is rejected by the program.

2.1. Combinatorial Rectangles and Covers. Combinatorial rectangles and covers are a useful tool in communication complexity, and are used in this work in a similar way. Let X and Y be arbitrary finite sets. A combinatorial rectangle is a set $X_0 \times Y_0$, where $X_0 \subseteq X$ and $Y_0 \subseteq Y$. A *cover* of $X \times Y$ is a set $\mathcal{R}$ of rectangles such that every pair $\langle x,y\rangle \in X \times Y$ belongs to at least one rectangle in $\mathcal{R}$.

Let M be a Boolean $|X| \times |Y|$ matrix such that the rows of M are indexed by the elements of X, and the columns of M are indexed by the elements of Y. We say that a rectangle $R_0 = X_0 \times Y_0$, where $X_0 \subseteq X$ and $Y_0 \subseteq Y$, is a *monochromatic* rectangle if there exists a $b \in \{0,1\}$ such that for every $x \in X_0$ and $y \in Y_0$ it holds that $M[x,y] = b$. If $b = 1$ we call R_0 a 1-*rectangle*, and if $b = 0$ we call R_0 a 0-*rectangle*. We say that a cover $\mathcal{R}$ is a *monochromatic* cover of M if every rectangle $R \in \mathcal{R}$ is a monochromatic rectangle. If $\mathcal{R}$ is a set of 1-rectangles that cover all the 1-entries of M, then $\mathcal{R}$ is called a 1-*cover* of M. If $\mathcal{R}$ is a set of 0-rectangles that cover all the 0-entries of M, we call $\mathcal{R}$ a 0-*cover* of M.

2.2. Linear Subspaces. We use basic linear algebra to find a function that is easy for span programs over one field and hard for span programs over another field. For a prime number p, we denote by GF(p) the unique finite field with p elements.

Let k be a positive integer, and let p be a prime. Denote by $V_k^{2k}(p)$ the set of all k-dimensional subspaces of $\mathrm{GF}(p)^{2k}$, and denote by $v_k^{2k}(p)$ the number of such subspaces, that is, $v_k^{2k}(p) = \left|V_k^{2k}(p)\right|$. To prove our result, we count the number of subspaces satisfying a certain property. Towards this aim, we will use the following easy algebraic claim. We say that two linear spaces U and W are different if there exists a vector $\vec{v}$ such that $\vec{v} \in U$ and $\vec{v} \notin W$ or vice versa.

Claim 2.3. *Let k be a positive integer, F be a field, and M be a matrix with k rows such that* $\mathrm{rank}_F(M) = k$. *Let T_1, T_2 be matrices with k rows each, where $T_1 \neq T_2$. Define M_1 (respectively, M_2) to be the matrix resulting from concatenating the matrix T_1 (respectively, T_2) to M, that is $M_i = (M|T_i)$ for $i \in \{1,2\}$. Then, the linear spaces spanned by the rows of M_1 and M_2 are different.*

Proof. Since $T_1 \neq T_2$, there exists an index $j \in \{1, \ldots, k\}$, such that the rows $T_1[j]$ and $T_2[j]$ are different. Let $\vec{r} = M_1[j]$, that is, $\vec{r}$ is the jth row of M_1. We show that $\vec{r}$ is not spanned by the rows of M_2. Assume there exist a combination of the rows of M_2 that spans $\vec{r}$. That is, $\vec{r} = \sum_{i=1}^{k} \alpha_i M_2[i]$ for some $\alpha_1, \ldots, \alpha_k \in F$. Let m be the number of columns in M, and consider the restriction of the above sum to the first m coordinates. It holds that $M[j] = \sum_{i=1}^{k} \alpha_i M[i]$. Since M has k rows and $\mathrm{rank}_F(M) = k$, we get that $\alpha_j = 1$ and $\alpha_i = 0$ for every $i \neq j$. Thus, $\vec{r} = M_2[j]$, that is, $M_1[j] = M_2[j]$, contradicting the fact that $T_1[j] \neq T_2[j]$. □

One application of Claim 2.3 is the following known corollary, which gives a lower bound on $v_k^{2k}(p)$.[4]

COROLLARY 2.4. *Let k be a positive integer, and let p be a prime. Then $v_k^{2k}(p) \geq p^{k^2}$.*

Proof. Let I_k be the $k \times k$ unit matrix, T be an arbitrary $k \times k$ matrix over $\mathrm{GF}(p)$, and M_1 be the $k \times 2k$ matrix that is a concatenation of I_k and T. There are p^{k^2} different choices of T, and therefore p^{k^2} different ways to construct M_1. By Claim 2.3, each such M_1 represents a different element of $V_k^{2k}(p)$, and thus $v_k^{2k}(p) \geq p^{k^2}$. □

It is easy to see that $v_k^{2k}(p) < p^{2k^2}$, since this is the number of ways to choose *any* k vectors from $\mathrm{GF}(p)^{2k}$, and thus, we have $p^{k^2} \leq v_k^{2k}(p) < p^{2k^2}$.[5]

We denote by $\vec{e}_j$ the jth unit vector, that is, the vector that is 1 in the jth coordinate, and 0 in all the others. We say that a nonzero vector has a *leading* 1, if the first non zero coordinate in the vector is 1. Let p be a prime, ℓ be a positive integer, and U be a subspace of dimension ℓ over $\mathrm{GF}(p)$. Then, the number of vectors with a leading 1 in U is $\frac{p^{\ell}-1}{p-1}$. We denote by $\mathrm{char}(F)$ the characteristic of the field F. Finally, we denote by $[n]$ the set $\{1, \ldots, n\}$.

3. The General Method for Separation.

We want to construct a function that is hard for monotone span programs over fields with characteristic different than p, and easy for monotone span programs over $\mathrm{GF}(p)$, where p is a prime. We use the method of [13] to get the lower bound for monotone span programs over fields with characteristic different than p. In the center of this method is a matrix with a large gap between its rank and the size of its monochromatic cover. To get a small upper bound for monotone span programs over $\mathrm{GF}(p)$, we shall require the cover to have an additional property which we call 1-mod-p, that is, for every entry of the matrix, the number of rectangles covering it is equivalent to 1 modulo p. Generally speaking, the number of variables in f, the function we prove the separation for, is equal to the number of rectangles in a cover. A detailed description is given below.

3.1. The Lower Bound.

Let M be a matrix and $\mathcal{R}$ be a monochromatic cover of M. Recall that $\mathcal{R}$ is a set of rectangles. Denote $n = |\mathcal{R}|$, and $\mathcal{R} = \{R_1, \ldots, R_n\}$, where $R_i = X_i \times Y_i$. A vector in $\{0,1\}^n$ can be viewed as a characteristic vector of a subset of $\mathcal{R}$. Throughout the paper, we identify each such vector with its corresponding subset. We define two subsets of $\{0,1\}^n$, Acc and Rej. These are exactly the same sets defined by Razborov in [24], constructing a monotone function associated with any cover. We will focus on functions that accept every $x \in \mathrm{Acc}$ and reject every $y \in \mathrm{Rej}$.

[4] Corollary 2.4 can be proved by directly counting the elements of V_k^{2k}. However, since we need Claim 2.3 for other purposes, we use it to prove Corollary 2.4 as well.

[5] Actually, $v_k^{2k}(p) = O(p^{k^2})$.

We first define a set Acc. For every row x of M we define a vector $\vec{z}_x \in \{0,1\}^n$. The ith coordinate in $\vec{z}_x$ indicates if the rectangle R_i covers the row x of M. That is, $\vec{z}_x[i] = 1$ if $x \in X_i$, and $\vec{z}_x[i] = 0$ otherwise. The set Acc contains the vectors $\vec{z}_x$ for every row x of the matrix M. That is, $\mathrm{Acc} = \{\vec{z}_x : x \in X\}$. An example for Acc is described in Fig. 3.1. For example, the set in Acc corresponding to x in the figure is $\{R_4, R_5, R_6\}$, the rectangles that cover the row x, and $\vec{z}_x = \langle 0,0,0,1,1,1\rangle$.

We now define a set Rej. For every column y of M we define a vector $\vec{w}_y \in \{0,1\}^n$. The ith coordinate of $\vec{w}_y$ indicates if the rectangle R_i *does not* cover the column y of the matrix M. That is, $\vec{w}_y[i] = 1$ if $y \notin Y_i$, and $\vec{w}_y[i] = 0$ otherwise. The set Rej contains the vectors $\vec{w}_y$ for every column y of the matrix M. That is, $\mathrm{Rej} = \{\vec{w}_y : y \in Y\}$. For example, the set in Rej corresponding to y described in Fig. 3.1 is $\{R_1, R_2, R_4\}$, the rectangles that *do not* cover the column y, and $\vec{w}_y = \langle 1,1,0,1,0,0\rangle$.

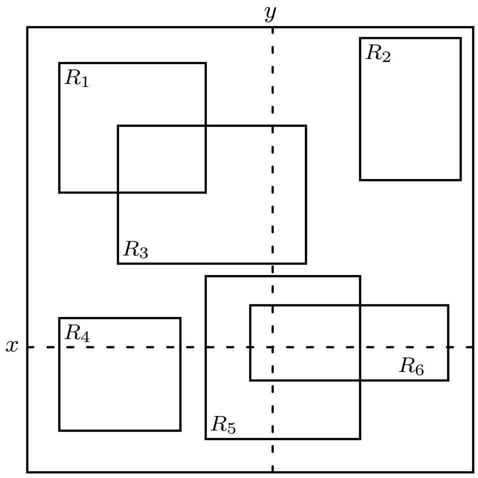

FIG. 3.1. *An illustration of elements in the sets* Acc *and* Rej*. Note that the rectangles in the figure* do not *form a cover.*

The lower bound on the size of monotone span programs is achieved using the following theorem, which is a restatement of Theorem 4.1 of [13].

THEOREM 3.1 ([13]). *Let M be a Boolean matrix, $\mathcal{R}$ be a monochromatic cover of M of size n, and* Acc *and* Rej *as defined above. If $f : \{0,1\}^n \to \{0,1\}$ is a monotone function such that $f(x) = 1$ for every $x \in \mathrm{Acc}$ and $f(y) = 0$ for every $y \in \mathrm{Rej}$, then $\mathrm{mSP}_F(f) \geq \mathrm{rank}_F(M)$, for every field F.*

That is, we get the lower bound for every function f accepting Acc, and rejecting Rej. Note that there are no requirements concerning inputs $t \notin (\mathrm{Acc} \cup \mathrm{Rej})$ (except for monotonicity). One can observe that such a function exists.

3.2. The Upper Bound. To prove a gap between the power of monotone span programs over the different fields, we need a function that has a small monotone span program over $\mathrm{GF}(p)$. Towards this aim, we require the cover $\mathcal{R}$ to be a *monochromatic* 1-mod-p *cover*, according to the following definition:

DEFINITION 3.2. *Let M be a Boolean matrix. A set $\mathcal{R}$ of combinatorial rectangles is called a* monochromatic 1-mod-p cover *of M, if $\mathcal{R}$ is a monochromatic cover of M, and, for each entry of M, the number of rectangles covering it is equivalent to* 1 *modulo p.*

Given a small monochromatic 1-mod-p cover of M, we construct a monotone span program over GF(p) that accepts Acc and rejects Rej. The gap will hold for the function computed by this span program.

Consider the following monotone span program $\widehat{P}$ over GF(p). The program $\widehat{P}$ associates a row with each rectangle of $\mathcal{R}$, and a column with each column of the matrix M. Therefore, the rectangle $R_j \in \mathcal{R}$ is represented by the variable x_j. The row associated with the rectangle $R_i = X_i \times Y_i$ is 1 in the column labeled by y if $y \in Y_i$, that is, if the rectangle R_i covers the column y in M. Otherwise, this entry in $\widehat{P}$ is 0. The target is $\vec{1}$. Note that $size(\widehat{P}) = n$, that is, there is exactly one row for each variable.

The following lemma is a simple special case of the upper bound part of Theorem 3.4 in [13]. In fact, the program $\widehat{P}$ considered here is exactly the program that one obtains by applying the construction from the proof of the upper bound in [13] to this simple special case.

LEMMA 3.3. *The program $\widehat{P}$ accepts every $\vec{z}_x \in$ Acc and rejects every $\vec{w}_y \in$ Rej.*

Proof. We first prove that $\widehat{P}$ accepts every $\vec{z}_x \in$ Acc. Specifically, we will show that since $\mathcal{R}$ is a 1-mod-p cover, the sum of the rows labeled by the rectangles of $\vec{z}_x$ is the vector $\vec{1}$, and thus $\vec{z}_x$ is accepted by $\widehat{P}$. That is, we show that for every column of $\widehat{P}$, the rows labeled by variables satisfied by $\vec{z}_x$ sum to 1 in this column. Towards this goal, fix a column y. Since $\vec{z}_x \in$ Acc, it is the characteristic vector of the set of rectangles covering the row x of M. According to the definition of $\widehat{P}$, for every rectangle R_j such that $\vec{z}_x[j] = 1$ the entry $\langle R_j, y\rangle$ of $\widehat{P}$ is 1 if and only if R_j covers the column y, that is $y \in Y_j$. On the other hand, $\vec{z}_x[j] = 1$ if and only if R_j covers the row x. Thus, the sum over the rows of $\widehat{P}$ associated with $\vec{z}_x$ in the column y is exactly the number of rectangles covering both y and x, that is, the number of rectangles covering the entry $\langle x, y\rangle$ in M. Since $\mathcal{R}$ is a 1-mod-p cover, this number is 1 modulo p. To conclude, the sum of the rows labeled by variables that are satisfied by $\vec{z}_x$ is the vector $\vec{1}$, and $\vec{z}_x$ is accepted by $\widehat{P}$.

Let $\vec{w}_y \in$ Rej. We show that there is no linear combination of the rows labeled by the rectangles of y that give the vector $\vec{1}$. Since $\vec{w}_y \in$ Rej, it is the characteristic vector of the subset of rectangles from $\mathcal{R}$ that *do not* cover the column y of M. Hence, all the rows of $\widehat{P}$ corresponding to variables satisfied by $\vec{w}_y$ are 0 in the column associated with y. Therefore, every combination of the rows labeled by variables satisfied by $\vec{w}_y$ is 0 in this column. Thus, the vector $\vec{1}$ is not a linear combination of these rows, and $\vec{w}_y$ is rejected by $\widehat{P}$. ∎

Combining Theorem 3.1 and Lemma 3.3, we get the separation theorem.

THEOREM 3.4 (Separation Theorem). *Let M be a Boolean matrix, and $\mathcal{R}$ be a monochromatic* 1-mod-p *cover of M of size n. Then there exists a monotone function f, with n variables, such that* $\mathrm{mSP}_{\mathrm{GF}(p)}(f) = n$ *and* $\mathrm{mSP}_F(f) \geq \mathrm{rank}_F(M)$ *for every field F.*

Proof. Denote by $f_{\widehat{P}}$ the function computed by $\widehat{P}$. By Lemma 3.3, $f_{\widehat{P}}$ accepts Acc and rejects Rej, and thus by Theorem 3.1 $\mathrm{mSP}_F(f_{\widehat{P}}) \geq \mathrm{rank}_F(M)$. On the other hand, $\mathrm{size}(\widehat{P}) = n$ and thus $\mathrm{mSP}_{\mathrm{GF}(p)}(f) = n$. The function $f_{\widehat{P}}$ is monotone as it is computed by a monotone span program. ∎

4. The Linear Subspaces Zero Intersection Function. In this section we show an explicit matrix with a high rank over fields with characteristic different than p, and a small monochromatic 1-mod-p cover. Thus, by Theorem 3.4 we get a function f with a super-polynomial gap between $\mathrm{mSP}_{\mathrm{GF}(p)}(f)$ and $\mathrm{mSP}_F(f)$ where F is any field such that $\mathrm{char}(F) \neq p$. We define the desired matrix in two steps: in the first step we define the matrix M_{ZI}, and prove it has full rank over fields with $\mathrm{char}(F) \neq p$. In the second step we use M_{ZI} to define another matrix, M_{LZI}, which has both a high rank over fields with char $\neq p$, and a small monochromatic 1-mod-p cover.

Let k be a positive integer and p be a prime.[6] The *Zero Intersection* (ZI) function determines whether the intersection of two k-dimensional linear subspaces of $\mathrm{GF}(p)^{2k}$ is the subspace $\{\vec{0}\}$. More formally, define $\mathrm{ZI}_k^p : V_k^{2k}(p) \times V_k^{2k}(p) \to \{0,1\}$ as follows: $\mathrm{ZI}_k^p(U,W) = 1$, where U and W are subspaces in $V_k^{2k}(p)$, if and only if $\dim(U \cap W) = 0$. Recall that the intersection of any two linear subspaces is a linear subspace.

We represent ZI_k^p by a $v_k^{2k}(p) \times v_k^{2k}(p)$ matrix denoted $M_{\mathrm{ZI}_k^p}$. Each row and each column of $M_{\mathrm{ZI}_k^p}$ is labeled by a subspace $U \in V_k^{2k}(p)$, and each entry $M_{\mathrm{ZI}_k^p}[U,W]$ is equal to $\mathrm{ZI}_k^p(U,W)$. Denote by r_U the row in $M_{\mathrm{ZI}_k^p}$ associated with the subspace $U \in V_k^{2k}(p)$. We will use ZI instead of ZI_k^p, and M_{ZI} instead of $M_{\mathrm{ZI}_k^p}$, when k and p are clear from the context.

4.1. Analyzing the Rank of M_{ZI}. The next theorem shows that M_{ZI} has full rank over any field with char $\neq p$.

THEOREM 4.1. *Let k be a positive integer, p be a prime, and F be a field such that* $\mathrm{char}(F) \neq p$. *Then, $M_{\mathrm{ZI}_k^p}$ has full rank over F.*

Proof. To prove that the matrix has full rank, it is sufficient to show that any unit vector is spanned by the rows of the matrix. Recall that the columns of the matrix are labeled by subspaces from $V_k^{2k}(p)$. For every $U \in V_k^{2k}(p)$ we consider the unit vector $\vec{e}_U \in \mathrm{GF}(p)^{v_k^{2k}(p)}$ and show that it is spanned by the rows of M_{ZI}. Specifically, we show a combination of the rows of the matrix spanning $\vec{e}_U$ having a special structure: The coefficient of $\vec{r}_Z$, the row labeled by $Z \in V_k^{2k}(p)$, depends only on the dimension of the subspace $U \cap Z$. More precisely, we show there are constants $\alpha_0, \ldots, \alpha_k \in F$, such that

$$\vec{e}_U = \sum_{d=0}^{k} \alpha_d \sum_{\substack{Z \in V_k^{2k}(p) \\ \dim(Z \cap U) = d}} \vec{r}_Z. \tag{4.1}$$

Fix $W \in V_k^{2k}(p)$, and consider $\vec{c}_W$, the column of M_{ZI} associated with W. We have to show that with the appropriate constants $\alpha_0, \ldots, \alpha_k \in F$, the above expression is 0 in this column if $W \neq U$, and is 1 if $W = U$. When computing the sum in the column $\vec{c}_W$, we add α_d for every subspace Z such that $\mathrm{ZI}(Z,W) = 1$ (i.e., $\dim(Z \cap W) = 0$) and $\dim(Z \cap U) = d$. This motivates the following definition:

DEFINITION 4.2. *Let $U, W \in V_k^{2k}(p)$ be subspaces, and let ℓ be an integer such that* $\dim(U \cap W) = \ell$. *Define $H_k^p(\ell, d)$ to be the* number *of subspaces $Z \in V_k^{2k}(p)$ such that* $\dim(U \cap Z) = d$ *and* $\dim(W \cap Z) = 0$. From symmetry arguments, the number $H_k^p(\ell, d)$ is independent of the choice of U and W. We will write $H_k(\ell, d)$ instead of $H_k^p(\ell, d)$, when p is clear from the context.

To summarize, we need to show that there are constants $\alpha_0, \ldots, \alpha_k \in F$ such that:

1. For each $0 \leq \ell \leq k-1$, it holds that $\sum_{d=0}^{k} \alpha_d \cdot H_k(\ell, d) = 0$. That is, the sum over any column labeled with $W \neq U$ equals 0, where for a subspace $W \in V_k^{2k}(p)$ such that $\dim(U \cap W) = \ell$, the relevant equation is the ℓ-th equation.

[6] Through this section the reader should think of k as small. That is, we construct a function with n variables and $k \approx \sqrt{\log n}$.

2. $\sum_{d=0}^{k} \alpha_d \cdot H_k(k,d) = 1$. That is, the sum over the column associated with U is 1.

Putting things differently, we view the numbers $H_k(\ell, d)$ for $\ell, d \in \{0, \ldots, k\}$ as a $(k+1) \times (k+1)$ matrix over F.[7] According to the above conditions we have to prove there are $\alpha_0, \ldots, \alpha_k \in F$ such that $H_k \langle \alpha_0, \alpha_1, \ldots, \alpha_k \rangle^T = \langle 0, \ldots, 0, 1 \rangle^T$.

In the next two claims, we show that H_k is upper-left triangular, where the numbers on the secondary diagonal are nonzero in F, thus H_k has full rank over F. The structure of H_k is illustrated in Fig. 4.1. In Claim 4.3 we show that the numbers below the secondary diagonal are all 0. In Claim 4.4 we show that the numbers on the secondary diagonal are all powers of p which are nonzero since $\mathrm{char}(F) \neq p$. The numbers above the secondary diagonal may take any value from F.

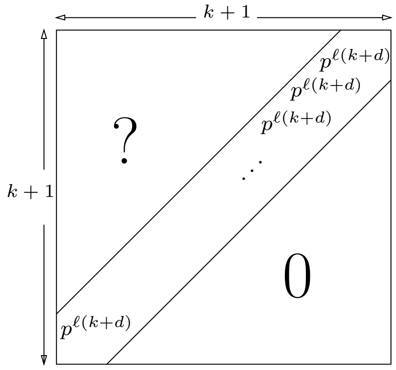

FIG. 4.1. *The structure of the matrix H_k.*

CLAIM 4.3. *Let k be a positive integer, ℓ and d be non-negative integers, p be a prime, and H_k be as above. If $\ell + d > k$ then $H_k^p(\ell, d) = 0$.*

Proof. Let $U, W \in V_k^{2k}(p)$, where $\dim(U \cap W) = \ell$. We have to show that since $\ell + d > k$ there is no subspace $Z \in V_k^{2k}(p)$, such that $\dim(Z \cap U) = d$ and $\dim(Z \cap W) = 0$. Assume toward contradiction that there exists such Z. Let $B_{U \cap W} = \langle \vec{w}_1, \ldots, \vec{w}_\ell \rangle$ be a basis of the subspace $U \cap W$. Let $B_{U \cap Z} = \langle \vec{z}_1, \ldots, \vec{z}_d \rangle$ be a basis for $U \cap Z$. Consider the set of vectors $X = B_{U \cap W} \cup B_{U \cap Z}$. First note that $X \subseteq U$, that is, all the vectors in X are in the subspace U. Since $\dim(U) = k$ and $|X| = \ell + d > k$, the set X must be linearly dependent. Thus, there must be a nontrivial combination of the vectors of X, giving the vector $\vec{0}$, that is, $\sum_{i=1}^{\ell} \lambda_i \vec{w}_i + \sum_{i=1}^{d} \delta_i \vec{z}_i = \vec{0}$. Since both $B_{U \cap W}$ and $B_{U \cap Z}$ are linearly independent, the nonzero vector $\vec{v} = \sum_{i=1}^{\ell} \lambda_i \vec{w}_i$ is spanned by both $B_{U \cap W}$ and $B_{U \cap Z}$. Since $U \cap W \subseteq W$ and $U \cap Z \subseteq Z$, we get that $\vec{v} \in W \cap Z$ and thus, $\dim(W \cap Z) > 0$, contradicting the assumption that $\dim(W \cap Z) = 0$. □(Claim 4.3)

We shell need the following notation for the next claim: Let $B = \langle \vec{v}_1, \ldots, \vec{v}_{2k} \rangle$ be a basis of $\mathrm{GF}(p)^{2k}$. Let $Z \in V_k^{2k}(p)$ and $B_Z = \langle \vec{z}_1, \ldots, \vec{z}_k \rangle$ be a basis for Z. Thus there must be unique constants such that for every $i \in [k]$ we have $z_i = \sum_{j=1}^{2k} \beta_{i,j} \vec{v}_j$. Then we call the $k \times 2k$ matrix $(\beta_{i,j})$ the *representation matrix* of B_Z *according to* B. Notice that for every basis B of Z we get a different representation.

[7]Since $H_k(\ell, d)$ may be a number not in F, we will replace it by $H_k(\ell, d) \bmod c$, where c is the characteristic of F. If the characteristic of F is 0, then $H_k(\ell, d)$ will always be in F.

Claim 4.4. *Let k be a positive integer, ℓ and d be non-negative integers, and p be a prime. If $\ell + d = k$ then $H_k^p(\ell, d) = p^{\ell(k+d)}$.*

Proof. Let $U, W \in V_k^{2k}(p)$ be any subspaces such that $\dim(U \cap W) = \ell$. We must show that the number of subspaces Z such that $\dim(Z \cap U) = d$ and $\dim(Z \cap W) = 0$ is $p^{\ell(k+d)}$. We will first define the term *canonical representation* of a subspace in $V_k^{2k}(p)$. Next, we will show that each subspace Z such that $\dim(Z \cap U) = d$ and $\dim(Z \cap W) = 0$ has a canonical representation. Then we will show that every canonical representation is associated with a unique subspace Z such that $\dim(Z \cap U) = d$ and $\dim(Z \cap W) = 0$. Thus, the number of such subspaces is equal to the number of different canonical representations. To complete the proof, we will show that the number of such canonical representations is $p^{\ell(k+d)}$. The canonical representation is defined according to a specific basis of $\mathrm{GF}(p)^{2k}$. Consider a basis $B_{U,W}$ of $\mathrm{GF}(p)^{2k}$ defined as follows:

$$B_{U,W} = \langle \vec{v}_1, \ldots, \vec{v}_\ell, \vec{u}_1, \ldots, \vec{u}_d, \vec{w}_1, \ldots, \vec{w}_d, \vec{x}_1, \ldots, \vec{x}_\ell \rangle$$

where:

(i) $\langle \vec{v}_1, \ldots, \vec{v}_\ell \rangle$ is a basis of $U \cap W$. Recall that $\dim(U \cap W) = \ell$.

(ii) $\langle \vec{u}_1, \ldots, \vec{u}_d \rangle$ is an expansion of $\langle \vec{v}_1, \ldots, \vec{v}_\ell \rangle$ to a basis of U. Recall that $\dim(U) = k$ and $d + \ell = k$.

(iii) $\langle \vec{w}_1, \ldots, \vec{w}_d \rangle$ is an expansion of $\langle \vec{v}_1, \ldots, \vec{v}_\ell \rangle$ to a basis of W. Recall that $\dim(W) = k$ as well.

(iv) $\langle \vec{x}_1, \ldots, \vec{x}_\ell \rangle$ is an expansion of $\langle \vec{v}_1, \ldots, \vec{v}_\ell, \vec{u}_1, \ldots, \vec{u}_{k-\ell}, \vec{w}_1, \ldots, \vec{w}_{k-\ell} \rangle$ to a basis of $\mathrm{GF}(2)^{2k}$. Here there are ℓ vectors since $2k - (\ell + d + d) = \ell$.

We say that a subspace $Z \in V_k^{2k}(p)$ has a *canonical representation* according to $B_{U,W}$ if it has a basis whose representation matrix according to $B_{U,W}$ is as described in Fig. 4.2. The matrix in Fig. 4.2 is a $k \times 2k$ matrix. Each entry in zones (b), (g), and (h) must be 0. The entries in zones (d) and (f) must form the unit matrices I_ℓ and I_d respectively. Each entry in zones (a), (c), and (e) can take any value from $\mathrm{GF}(p)$.

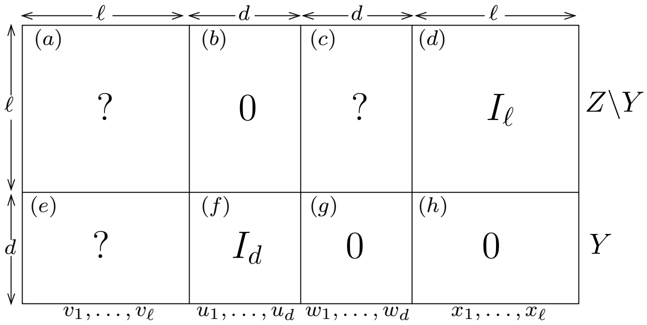

Fig. 4.2. *A canonical representation of a subspace $Z \in V_k^{2k}(p)$ with $\dim(U \cap Z) = d$ and $\dim(W \cap Z) = 0$.*

First we show that every subspace $Z \in V_k^{2k}(p)$ such that $\dim(Z \cap U) = d$ and $\dim(Z \cap W) = 0$ has a canonical representation according to $B_{U,W}$. Let $Y = Z \cap U$. Note that $\dim(Y) = d$. Let $B_Y = \langle \vec{y}_1, \ldots, \vec{y}_d \rangle$ be a basis of Y, and let $B_Z = \langle \vec{y}_1, \ldots, \vec{y}_d, \vec{z}_1, \ldots, \vec{z}_\ell \rangle$ be an expansion of B_Y to a basis of Z. Consider M_Z, the representation matrix of B_Z according to $B_{U,W}$. Since $Y \subseteq U$, all the entries in the zones (g) and (h) are 0 as required. We claim that we can perform elementary operations on the lower part of M_Z so that we get the matrix I_d in zone (f). Otherwise, we would get a row $\vec{r}$ that is $\vec{0}$ in zone (f), but this would leave all the nonzero entries of $\vec{r}$ in zone (e). Since zone (e) represents the basis vectors from $U \cap W$, this would mean $\dim(Z \cap W) > 0$, contradicting the properties of Z. It is left to set zone (d) to I_ℓ and all the entries in zone (b) to 0. Setting all the entries in zone (b) to 0 can be done by elementary operations on the upper part of M_Z using the rows from the lower part, which now form the unit matrix I_d in zone (f). (This would change the entries in zone (a), but we have no constraints on this zone.) We claim that we can set zone (d) to be I_ℓ by elementary operations on the upper part of M_Z. Otherwise we would get a row $\vec{r}$ that is all zero in zone (d). Thus $\vec{r}$ has nonzero entries only in zones (a) and (c), but then it again implies that $\vec{r}$ represents a vector from W, contradicting the fact that $\dim(Z \cap W) = 0$.

Next we prove that *every* subspace $Z \in V_k^{2k}(p)$ which can be represented in the above canonical form, satisfies $\dim(Z \cap W) = 0$ and $\dim(Z \cap U) = d$. Let M_Z be a canonical representation of Z according to $B_{U,W}$. Since M_Z has I_ℓ and I_d as sub-matrices, we have $\mathrm{rank}_{\mathrm{GF}(p)}(M_Z) = k$ and thus $Z \in V_k^{2k}(p)$. Now suppose $\dim(Z \cap W) > 0$. Then we can span a vector $w \in W$ by the rows of M_Z. This vector has to be zero in the coordinates labeled by $\vec{u}_1, \ldots, \vec{u}_d$, and by $\vec{x}_1, \ldots, \vec{x}_\ell$, but this cannot be done by a non-trivial combination of the rows of M_Z. Thus, $\dim(W \cap Z) = 0$. The lower part of M_Z is nonzero only in coordinates labeled by vectors from U, and since it has I_d as a sub-matrix, we get that $\dim(Z \cap U) \geq d$. Now suppose that $\dim(Z \cap U) = d' > d$. Then we have $\dim(Z \cap U) = d'$, $\dim(Z \cap W) = 0$, and $\dim(U \cap W) = \ell$, where $\ell + d' > \ell + d = k$, which is impossible by Claim 4.3. Therefore, $\dim(U \cap Z) = d$.

To complete the proof, we show that any two subspace who have different canonical representations over $B_{U,W}$ are different. To see that, note that the matrix

$$S = \begin{pmatrix} 0 & I_\ell \\ I_d & 0 \end{pmatrix}$$

is a sub-matrix of any canonical representation. The matrix S is clearly of rank k, and thus, by Claim 2.3 any two subspaces with different canonical representation are different.

Therefore, when constructing a subspace Z, with $\dim(Z \cap U) = d$ and $\dim(Z \cap W) = 0$, the freedom in exactly in the entries marked with '?' in Fig. 4.2. Since there are p possibilities for every such entry, and the number of such entries is $(k \cdot \ell) + (\ell \cdot d) = \ell(k + d)$, we conclude that $H_k(\ell, d) = p^{\ell(k+d)}$. $\square$(Claim 4.4)

Since the characteristic of F is different than p, every power of p is non zero over F. Therefore, as argued above, we proved that H_k has full rank over F, and the theorem follows. $\square$(Theorem 4.1)

In Corollary 2.4 we proved that $v_k^{2k}(p) \geq p^{k^2}$. Since M_{ZI_k} is a $v_k^{2k}(p) \times v_k^{2k}(p)$ matrix, $\mathrm{rank}_F(M_{\mathrm{ZI}_k}) \geq p^{k^2}$.

4.2. A Small 1-mod-p Cover for the Zeros of M_{ZI}. To apply Theorem 3.4 to an explicit matrix, we need this matrix to have a small monochromatic 1-mod-p cover. We next show that there is a small 1-mod-p cover for the 0's of M_{ZI}. We do not know if there exists a small 1-mod-p cover for the 1's of M_{ZI}. Thus, we are not able to use M_{ZI} directly, and we use it in §4.3 to build the matrix M_{LZI}, which has a small 1-mod-p cover for both the 1's and the 0's.

To give some intuition on the cover of M_{LZI} we show a 1-mod-p cover for the 0's of M_{ZI} of size less than p^{2k}. This should be compared to the number of rows in M_{ZI} which is $p^{\Theta(k^2)}$. Define the cover $\mathcal{R}$ as follows: Let $\vec{v} \in GF(p)^{2k}$ be a vector with a leading 1, that is, the first nonzero coordinate of $\vec{v}$ is 1. We add the rectangle $R_{\vec{v}} = X_{\vec{v}} \times Y_{\vec{v}}$ to the cover $\mathcal{R}$, where:

$$X_{\vec{v}} = \left\{U \in V_k^{2k}(p) : \vec{v} \in U\right\} \text{ and } Y_{\vec{v}} = \left\{W \in V_k^{2k}(p) : \vec{v} \in W\right\}.$$

That is, $R_{\vec{v}}$ contains the rows and the columns of M_{ZI} labeled by subspaces that contain the vector $\vec{v}$. The rectangle $R_{\vec{v}}$ is a 0-rectangle, since for each $U \in X_{\vec{v}}$ and $W \in Y_{\vec{v}}$ it holds that $\vec{v} \in U \cap W$, hence $\dim(U \cap W) \neq 0$, and thus $\mathrm{ZI}(U,W) = 0$. We claim $\mathcal{R}$ is a 1-mod-p cover of the 0's of M_{ZI}. Let $\langle U, W\rangle$ be an entry of M_{ZI}, such that $\mathrm{ZI}(U,W) = 0$. Then $\dim(U \cap W) > 0$. Therefore, the entry $\langle U, W\rangle$ is covered by any rectangle $R_{\vec{v}}$ such that $\vec{v} \in U \cap W$, and $\vec{v}$ has a leading one. Since $U \cap W$ is a linear subspace of $GF(p)^{2k}$, it has $\frac{p^\ell - 1}{p-1}$ vectors with a leading 1, where $\ell = \dim(U \cap W) \geq 1$. Since $\frac{p^\ell-1}{p-1} \equiv \frac{-1}{-1} \equiv 1 \pmod p$, the number of rectangles covering the entry $\langle U, W\rangle$ is equivalent to 1 modulo p. Since there are $\frac{p^{2k}-1}{p-1}$ different vectors with a leading 1 in $\mathrm{GF}(p)^{2k}$, the size of the 0-cover is $\frac{p^{2k}-1}{p-1}$.

4.3. The List Version of the Zero Intersection Function. To get a matrix with a high rank over fields with characteristic different than p, and a small monochromatic 1-mod-p cover, we define the function LZI, the list version of the Zero Intersection function. The idea of using the list version of functions has been used in communication complexity [20] (see, e.g., [19]). Define $\mathrm{LZI}_k^p : (V_k^{2k}(p))^k \times (V_k^{2k}(p))^k \to \{0,1\}$ as follows:

$$\mathrm{LZI}_k^p(\langle A_1,\dots,A_k\rangle, \langle B_1,\dots,B_k\rangle) = 1 \iff \exists i \in \{1\dots k\} \text{ such that } \mathrm{ZI}_k^p(A_i,B_i) = 1.$$

That is, LZI_k^p gets k instances of ZI_k^p, and outputs the value 1 iff ZI_k^p outputs 1 on at least one of the given instances. The matrix M_{LZI}, representing LZI, is defined in a similar way to M_{ZI}. The next two lemmas show that M_{LZI} has a small 1-mod-p cover.

LEMMA 4.5. *There is a monochromatic* 1-mod-p *cover of the* 0*'s of* M_{LZI} *of size smaller than* p^{2k^2}.

Proof. We build the 0-cover $\mathcal{R}_0$ of the 0's of M_{LZI} in a similar way to the 0-cover for M_{ZI} built in §4.2. Let $\langle \vec{v}_1,\dots,\vec{v}_k\rangle \in (\mathrm{GF}(p)^{2k})^k$ be a tuple of k vectors from $\mathrm{GF}(p)^{2k}$, each with a leading 1. The rectangle in $\mathcal{R}_0$ corresponding to $\langle \vec{v}_1,\dots,\vec{v}_k\rangle$ is $R = X \times Y$ where:

$$X = \{\langle A_1,\dots,A_k\rangle \in (V_k^{2k}(p))^k : \vec{v}_i \in A_i \text{ for each } i \in [k]\}$$

and

$$Y = \{\langle B_1,\dots,B_k\rangle \in (V_k^{2k}(p))^k : \vec{v}_i \in B_i \text{ for each } i \in [k]\}.$$

First we show R is a 0-rectangle. If $\langle A_1,\dots,A_k\rangle \in X$ and $\langle B_1,\dots,B_k\rangle \in Y$, then $\vec{v}_i \in A_i \cap B_i$ for every $i \in [k]$, and thus $\mathrm{ZI}(A_i,B_i) = 0$ for every $i \in [k]$. Therefore, $\mathrm{LZI}(\langle A_1,\dots,A_k\rangle, \langle B_1,\dots,B_k\rangle) = 0$.

Next we show that for every 0-entry of M_{LZI}, the number of rectangles covering it is equivalent to 1 modulo p. Let $\langle\langle A_1,\dots,A_k\rangle, \langle B_1,\dots,B_k\rangle\rangle \in (V_k^{2k}(p))^k \times (V_k^{2k}(p))^k$ such that $\mathrm{LZI}(\langle A_1,\dots,A_k\rangle, \langle B_1,\dots,B_k\rangle) = 0$. The entry $\langle\langle A_1,\dots,A_k\rangle, \langle B_1,\dots,B_k\rangle\rangle$ is covered by any rectangle associated with a tuple of k nonzero vectors $\langle \vec{v}_1,\dots,\vec{v}_k\rangle$, such that $\vec{v}_i \in A_i \cap B_i$, for every $i \in [k]$, and has a leading 1. Since $A_i \cap B_i$ is a linear subspace, the number of vectors with a leading 1 in $A_i \cap B_i$ is $\frac{p^{\ell_i}-1}{p-1}$ where $\ell_i = \dim(A_i \cap B_i) \geq 1$. Thus, the number of rectangles covering $\langle\langle A_1,\dots,A_k\rangle, \langle B_1,\dots,B_k\rangle\rangle$ is a product of numbers that are equivalent to 1 modulo p, and therefore is equivalent to 1 modulo p itself.

The number of 0-rectangles in $\mathcal{R}_0$ is the number of tuples of k vectors with a leading 1 from $\mathrm{GF}(p)^{2k}$, that is, $(\frac{p^{2k}-1}{p-1})^k < p^{2k^2}$. (This is much smaller than the number of rows in M_{LZI}, which is $p^{\Omega(k^3)}$.) ∎

Now we show a cover $\mathcal{R}_1$ for the 1's of M_{LZI}. The natural way to do it would be to associate a rectangle $R = X \times Y$ with each pair $\langle i, U\rangle$, such that $i \in [k]$, and $U \in V_k^{2k}(p)$, where:

$$X = \left\{\langle A_1, \ldots, A_k\rangle \in (V_k^{2k}(p))^k : A_i = U\right\}$$

and

$$Y = \left\{\langle B_1, \ldots, B_k\rangle \in (V_k^{2k}(p))^k : \dim(U \cap B_i) = 0\right\}.$$

That is, any input pair having $ZI(A_i, B_i) = 1$ in the ith instance, will be covered by the rectangle associated with i and A_i.

The problem with this choice of $\mathcal{R}_1$ is that it is not a 1-mod-p cover. For example, if $\langle A_1, \ldots, A_k\rangle$ and $\langle B_1, \ldots, B_k\rangle$ have exactly p instances $\langle A_i, B_i\rangle$ such that $\mathrm{ZI}(A_i, B_i) = 1$, then the number of rectangles covering the entry

$$\langle\langle A_1, \ldots, A_k\rangle, \langle B_1, \ldots, B_k\rangle\rangle$$

will be equivalent to 0 modulo p. To solve this problem, we require that i is the index of the *first* instance of ZI, such that $\mathrm{ZI}(A_i, B_i) = 1$.

LEMMA 4.6. *There is a monochromatic* 1-mod-p *cover for the* 1*'s of* M_{LZI} *of size smaller than* p^{4k^2}.[8]

Proof. Associate a rectangle $R = X \times Y$ with any pair $\langle\langle \vec{v}_1, \ldots, \vec{v}_{i-1}\rangle, U\rangle$, where $\langle \vec{v}_1, \ldots, \vec{v}_{i-1}\rangle$ is a tuple of $i-1$ vectors with a leading 1 from $\mathrm{GF}(p)^{2k}$ where $1 \le i \le k$, and $U \in V_k^{2k}(p)$ is a subspace. The sets X and Y are defined as follows:

$$X = \{\langle A_1, \ldots, A_k\rangle \in (V_k^{2k}(p))^k : \vec{v}_j \in A_j \text{ for each } j \in [i-1] \text{ and } A_i = U\}, \text{ and}$$
$$Y = \{\langle B_1, \ldots, B_k\rangle \in (V_k^{2k}(p))^k : \vec{v}_j \in B_j \text{ for each } j \in [i-1] \text{ and } \dim(B_i \cap U) = 0\}.$$

To see that R is a 1-rectangle take $\langle A_1, \ldots, A_k\rangle \in X$ and $\langle B_1, \ldots, B_k\rangle \in Y$. Then, $\dim(A_i \cap B_i) = \dim(U \cap B_i) = 0$, and thus $\mathrm{ZI}(A_i, B_i) = 1$. Therefore, $\mathrm{LZI}(\langle A_1, \ldots, A_k\rangle, \langle B_1, \ldots, B_k\rangle) = 1$.

We next show that for every 1-entry of M_{LZI}, the number of rectangles covering it is equivalent to 1 modulo p. Let $\langle\langle A_1, \ldots, A_k\rangle, \langle B_1, \ldots, B_k\rangle\rangle \in (V_k^{2k}(p))^k \times (V_k^{2k}(p))^k$ such that $\mathrm{LZI}(\langle A_1, \ldots, A_k\rangle, \langle B_1, \ldots, B_k\rangle) = 1$. Let i be the smallest index such that $\dim(A_i \cap B_i) = 0$. Then the entry $\langle\langle A_1, \ldots, A_k\rangle, \langle B_1, \ldots, B_k\rangle\rangle$ is covered by a rectangle if and only if it is associated with a pair $\langle\langle \vec{v}_1, \ldots, \vec{v}_{i-1}\rangle, A_i\rangle$, such that $\vec{v}_j \in A_j \cap B_j$ for every $j \in \{1, \ldots, i-1\}$. Since the number of vectors with a leading 1 in $A_j \cap B_j$ for every $j \in [i]$ is equivalent to 1 modulo p, the number of such rectangles is equivalent to 1 modulo p as well.

The size of $\mathcal{R}_1$ is smaller than the number of ways to choose k vectors with a leading 1 from $\mathrm{GF}(p)^{2k}$, and a subspace from $V_k^{2k}(p)$, and thus is smaller than $p^{2k^2} \cdot v_k^{2k}(p) < p^{4k^2}$. ∎

By taking the union of the 0-cover from Lemma 4.5 and the 1-cover from Lemma 4.6 we get the following corollary.

COROLLARY 4.7. M_{LZI} *has a monochromatic* 1-mod-p *cover of size smaller than* p^{5k^2}.

[8] It may seem that this number is too big, but this should be compared to the dimensions of M_{LZI} which is $p^{\Omega(k^3)}$.

We proved in Theorem 4.1 that $\mathrm{rank}_F(M_{\mathrm{ZI}_k}) \geq p^{k^2}$. We use this fact to analyze the rank of M_{LZI_k} over F. Let A be an $m \times n$ matrix, and B be an $r \times s$ matrix. Then the *Kronecker product* of A and B, denoted $A \otimes B$, is an $mr \times ns$ matrix, formed by multiplying each element of A by the entire matrix B and putting it in the place of the element of A. For any field F, for every two matrices A and B, it holds that $\mathrm{rank}_F(A \otimes B) = \mathrm{rank}_F(A)\,\mathrm{rank}_F(A)$. This property of the Kronecker product, together with De Morgan laws, imply the following Lemma.

LEMMA 4.8. *Let k be a positive integer and let p be a prime. Then*

$$\mathrm{rank}_F(M_{\mathrm{LZI}_k^p}) = p^{\Omega(k^3)}.$$

We are ready to prove our main result:

THEOREM 4.9 (Main Result). *Let p be a fixed prime. Then there exist a family of functions $\{f_n\}_{n\in\mathcal{N}}$, such that $\mathrm{mSP}_{\mathrm{GF}(p)}(f_n) = n$ and for every field F with characteristic different than p, it holds that $\mathrm{mSP}_F(f_n) = n^{\Omega(\sqrt{\log n})}$.*

Proof. For a positive number k, denote by n_k the size of the monochromatic 1-mod-p cover for M_{LZI} given by Corollary 4.7. We first show f_n for each n of the form $n = n_k$ for some positive k. According to Corollary 4.7, the matrix M_{LZI_k} has a monochromatic 1-mod-p cover of size n, which is smaller than p^{5k^2}. According to Lemma 4.8, we have that $\mathrm{rank}_F(M_{\mathrm{LZI}_k}) = p^{\Omega(k^3)}$. In terms of n, we have

$$n^{\sqrt{\log_p n}} \leq (p^{5k^2})^{\sqrt{\log_p(p^{5k^2})}} = (p^{5k^2})^{\sqrt{5k^2}}.$$

By Theorem 3.4, there is a function f_n in n variables, such that $\mathrm{mSP}_{\mathrm{GF}(p)}(f) = n$ and $\mathrm{mSP}_F(f) \geq p^{\Omega(k^3)} = n^{\Omega(\sqrt{\log n})}$. The last equality holds since p is a constant. By padding arguments, the result holds for every value of n. □

5. A Super-polynomial Lower Bound for a Function in uniform $-\mathcal{NC}^2$**.** In this section we show a monotone function that is computable by uniform-$\mathcal{NC}^2$ circuits, and does not have a polynomial size monotone span program over any field.[9] For comparison, all the previous super-polynomial lower bounds are for function not known to be in $\mathcal{P}$.

Denote by $f^2 = \{f_n^2\}_{n\in\mathcal{N}}$ and $f^3 = \{f_n^3\}_{n\in\mathcal{N}}$ the families of functions given by Theorem 4.9 for $p = 2$ and $p = 3$ respectively. Define the family of functions $f = \{f_{2n}\}_{n\in\mathcal{N}}$ to be $f_{2n}(x_1,\ldots,x_n,y_1,\ldots,y_n) = f_n^2(x_1,\ldots,x_n) \wedge f_n^3(y_1,\ldots,y_n)$.

We show a uniform $-\mathcal{NC}^2$ family of circuits for f. Let $\widehat{P_2}$ be the monotone span program over GF(2) that computes f^2. Recall that $\mathrm{size}(\widehat{P_2}) = n$. As mentioned in §2, we can assume w.l.o.g. that the number of columns in $\widehat{P_2}$ is not larger than the number of rows, which is n. Therefore, since linear algebra over fixed finite fields is in log-space uniform-$\mathcal{NC}^2$ [7, 21, 10, 18], there exists an $\mathcal{NC}^2$ family of circuits $\{C^2\}_{n\in\mathcal{N}}$ that computes f^2. Similarly, there exists an $\mathcal{NC}^2$ family of circuits $\{C_n^3\}_{n\in\mathcal{N}}$ that computes f^3. Thus, the $\mathcal{NC}^2$ family of circuits $\{C_{2n}\}_{n\in\mathcal{N}}$, where $C_{2n} = C_n^2 \wedge C_n^3$, computes f.

The problem with the family of circuits $\{C\}_{n\in\mathcal{N}}$, as described, is that it is not uniform. The mere *existence* of a monotone span program with a small number of columns does not yield a uniform-$\mathcal{NC}^2$ circuit. To get uniform circuits we have to

[9] In this paper uniform means log-space uniform.

show an *explicit* monotone span program with a small number of columns that can be generated in space $O(\log n)$. We do this in §5.1.

We next show that f has no small monotone span program over any field. Assume there is a polynomial size monotone span program $\widehat{Q}$ that computes f over some field F. Let c be the characteristic of F. If $c \neq 2$ then the restriction of $\widehat{Q}$ to inputs of the form $x_1, \ldots, x_n \cdot 1^n$, gives a new monotone span program $\widehat{Q}_2$ of polynomial size over F that computes f^2 (as any restriction of a function with a small monotone span program has a small monotone span program [18]), contradicting the fact that f^2 has no polynomial size monotone span program over fields with characteristic different than 2. If $c = 2$ then $c \neq 3$ and we get the contradiction for f^3 in a similar way. Thus,

THEOREM 5.1. *There exist a family of monotone functions* $\{f_n\}_{n\in\mathcal{N}}$ *that is computable by a* $\mathrm{uniform}-\mathcal{NC}^2$ *family of circuits having* $\mathrm{mSP}_F(f_n) = n^{\Omega(\sqrt{\log n})}$ *for every field* F*.*

5.1. Reducing the Number of Columns. In Theorem 4.9 we introduced a function $f_{\widehat{P}}$ such that $\mathrm{mSP}_{\mathrm{GF}(p)}(f_n) = n$ and $\mathrm{mSP}_F(f_n) = n^{\Omega(\sqrt{\log n})}$. In this section we want to construct a family of uniform-$\mathcal{NC}^2$ circuits for a function that accepts Acc and rejects Rej.

It is known that any function that has a polynomial size monotone span program has a family of $\mathcal{NC}^2$ circuits. Since any monotone span program with m rows that computes a function f has an equivalent monotone span program with no more than m columns, we can deduce the existence of a family of $\mathcal{NC}^2$ circuits that computes f. However, we want a uniform family of circuits. Since any transformation from a monotone span program with an arbitrary number of columns to an equivalent program with a smaller number of columns has to go over all the columns of the big original program, we cannot use the generic span program for $f_{\widehat{P}}$, as presented in §3.4. In this section we show a monotone span program, with a linear number of both rows and columns, that accepts Acc and rejects Rej. We show that the span program can be generated in space $O(\log n)$, and we ensure the uniformity of the $\mathcal{NC}^2$ circuits.

Let $\mathcal{R}_{\mathrm{LZI}}$ be the monochromatic 1-mod-p cover of M_{LZI} described in Corollary 4.7, and consider the following monotone span program $\widehat{S}$:[10] The program $\widehat{S}$ has a column for each k-tuple $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle \in (\mathrm{GF}(p)^{2k})^k$ where each $\vec{v}_i$ is a vector with a leading 1 from $\mathrm{GF}(p)^{2k}$. Thus, the number of columns in $\widehat{S}$ is smaller than the number of rectangles in $\mathcal{R}_{\mathrm{LZI}}$, and hence is linear in the number of variables. Intuitively, the columns of $\widehat{S}$ are a basis to the columns of the program $\widehat{P}$ from §4.

Recall that in $\mathcal{R}_{\mathrm{LZI}}$ there are two types of rectangles:

0**-rectangles.** We associated a 0-rectangle for every k-tuple of vectors $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle \in (\mathrm{GF}(p)^{2k})^k$, each with a leading 1.

1**-rectangles.** We associated a 1-rectangle $R = X \times Y$ with any pair

$$\langle \langle \vec{v}_1, \ldots, \vec{v}_{i-1} \rangle, U \rangle$$

such that $\langle \vec{v}_1, \ldots, \vec{v}_{i-1} \rangle$ is a tuple of $i-1$ vectors with a leading 1 from $\mathrm{GF}(p)^{2k}$, where $1 \leq i \leq k$, and $U \in V_k^{2k}(p)$ is a subspace.

Every rectangle is assigned a row in $\widehat{S}$. Let R be a rectangle in $\mathcal{R}_{\mathrm{LZI}}$, and let c be a column in $\widehat{S}$ labeled with the tuple $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle$. Then the value of the entry $\widehat{S}[R, c]$ is defined as follows:

[10]We do not know if the function computed by the monotone span program $\widehat{S}$ is the same as the function from Theorem 4.9.

For a 0-rectangle R, let $\langle \vec{u}_1, \ldots, \vec{u}_k \rangle$ be the k tuple of vectors associated with R. We set $\widehat{S}[R, c] = 1$ if $\vec{u}_j = \vec{v}_j$ for every $j \in [k]$. Otherwise, $\widehat{S}[R, c] = 0$.

For a 1-rectangle R, let $\langle \langle \vec{u}_1, \ldots, \vec{u}_{i-1} \rangle, U_i \rangle$ be the $(i-1)$-tuple of vectors and the subspace associated with R. In this case set $\widehat{S}[R, c] = 1$ if $\vec{u}_j = \vec{v}_j$ for every $j \in [i-1]$ and $v_i \notin U_i$. Otherwise $\widehat{S}[R, c] = 0$.

By putting the rows corresponding to 0-rectangles in the upper part of $\widehat{S}$, the upper block of $\widehat{S}$ is in fact the unit matrix I. To compute an entry in the lower part of $\widehat{S}$, we only have to check if a vector in $\mathrm{GF}(p)^{2k}$ belongs to a subspace, where $k = O(\sqrt{\log n})$. This can be easily done in space $O(\log n)$. Thus, $\widehat{S}$ can be generated in log-space. To construct a circuit that simulates the span program, we need a circuit that tests the rank of a matrix over $\mathrm{GF}(p)$. This can also be done in space $O(\log n)$ [7, 21, 10, 18]. We next prove that the function computed by $\widehat{S}$ can be used for obtaining our lower bounds. That is, the program $\widehat{S}$ accepts every $\vec{z}_x \in \mathrm{Acc}$ and rejects every $\vec{w}_y \in \mathrm{Rej}$. This fact is proved in the following two claims:

CLAIM 5.2. *The program $\widehat{S}$ accepts every $\vec{z}_x \in \mathrm{Acc}$.*

Proof. Let $\vec{z}_x \in \mathrm{Acc}$. Throughout the proof we view the characteristic vector $\vec{z}_x$ as the set of rectangles it represents. We show that the vector $\vec{1}$ is the sum of the rows labeled by rectangles $R \in \vec{z}_x$, where the computations are done over $\mathrm{GF}(p)$.

Since $\vec{z}_x \in \mathrm{Acc}$, it is the characteristic vector of the set of all the rectangles in $\mathcal{R}_{\mathrm{LZI}}$ covering the row x of M_{LZI}. Let $\langle X_1, \ldots, X_k \rangle$ be the k-tuple of subspaces from V_k^{2k} labeling the row x in M_{LZI}. Then the rectangles in $\vec{z}_x$ are of two types:

(i) 0-rectangles, labeled by $\langle \vec{x}_1, \ldots, \vec{x}_k \rangle$ where $\vec{x}_j \in X_j$ for every $j \in [k]$.

(ii) 1-rectangles, labeled by $\langle \langle \vec{x}_1, \ldots, \vec{x}_{i-1} \rangle, X_i \rangle$ where $\vec{x}_j \in X_j$ for every $j \in [i-1]$.

Let c be a column in $\widehat{S}$. Assume that c is labeled by $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle$. We show that the sum of the rows labeled by rectangles from $\vec{z}_x$, in the column c is 1. More specifically, we show that there is exactly one row labeled by $\vec{z}_x$ that is 1 in the column c. We consider two different cases:

(i) $\vec{v}_j \in X_j$ for every $j \in [k]$. We divide the rectangles in $\vec{z}_x$ into three:

1. The unique 0-rectangle $R \in \vec{z}_x$ labeled by $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle$. According to the definition of $\widehat{S}$, we have $\widehat{S}[R, c] = 1$.

2. Other 0-rectangles. Since the upper block of $\widehat{S}$ is the unit matrix I, we have $\widehat{S}[R, c] = 0$ for any such rectangle.

3. 1-rectangles. If R is a 1-rectangle labeled by $\langle \langle \vec{x}_1, \ldots, \vec{x}_{i-1} \rangle, X_i \rangle$ then we have that $\widehat{S}[R, c] = 0$ since $\vec{v}_i \in X_i$.

Thus, there is exactly one rectangle $R \in \vec{z}_x$ such that $\widehat{S}[R, c] = 1$, and hence the sum of the rows labeled by rectangles from $\vec{z}_x$, in the column c is 1.

(ii) Otherwise, there exists an index $\ell \in [k]$ such that $\vec{v}_j \in X_j$ for every $j \in [\ell - 1]$ and $\vec{v}_\ell \notin X_\ell$. In this case, for every 0-rectangle $R \in \vec{z}_x$, it holds that $\widehat{S}[R, c] = 0$, since for every such rectangle $\vec{x}_\ell \in X_\ell$, while $\vec{v}_\ell \notin X_\ell$, and thus $\vec{x}_\ell \neq \vec{v}_\ell$. Let $R \in \vec{z}_x$ be a 1-rectangle labeled by $\langle \langle \vec{x}_1, \ldots, \vec{x}_{i-1} \rangle, X_i \rangle$, for some $i \in [k]$, where $\vec{x}_j \in X_j$ for every $j \in [i-1]$. We have to check 3 cases:

Case I: $i < \ell$. In this case $\vec{v}_i \in X_i$, because $i \in [\ell - 1]$, and thus $\widehat{S}[R, c] = 0$.

Case II: $i > \ell$. In this case $\vec{x}_\ell \in X_\ell$, since $\ell \in [i-1]$. On the other hand, $\vec{v}_\ell \notin X_\ell$, and thus $\vec{v}_\ell \neq \vec{x}_\ell$, with $\ell \in [i-1]$, and so $\widehat{S}[R, c] = 0$.

Case III: $i = \ell$. In this case the only rectangle $R \in \vec{z}_x$ satisfying $\widehat{S}[R, c] = 1$ is the rectangle labeled by $\langle \langle \vec{v}_1, \ldots, \vec{v}_{i-1} \rangle, X_i \rangle$.

Therefore, again there is exactly one rectangle $R \in \vec{z}_x$ such that $\widehat{S}[R, c] = 1$, and the sum of the rows labeled by rectangles from $\vec{z}_x$, in the column c is 1.
Therefore, $\widehat{S}$ accepts Acc. ∎

It is left to prove that $\widehat{S}$ rejects Rej. This part is a little more complicated than in the generic case discussed in Lemma 3.3.

CLAIM 5.3. *The program $\widehat{S}$ rejects every $\vec{w}_y \in$ Rej.*

Proof. Let $\vec{w}_y \in$ Rej. Throughout the proof we view the characteristic vector $\vec{w}_y$ as the set of rectangles it represents. Then there exists a column labeled by $\langle Y_1, \ldots, Y_k \rangle$ in M_{LZI}, such that $\vec{w}_y$ is the set of all the rectangles in $\mathcal{R}_{\text{LZI}}$ that *do not* cover this column. The rectangles in $\vec{w}_y$, i.e. rectangles not covering the column labeled by $\langle Y_1, \ldots, Y_k \rangle$, are of the following types:

0-rectangles. If R is a 0-rectangle labeled by $\langle \vec{x}_1, \ldots, \vec{x}_k \rangle$ not covering the column $\langle Y_1, \ldots, Y_k \rangle$, then there exist an index $i \in [k]$ such that $\vec{x}_i \notin Y_i$.

1-rectangles. If R is a 1-rectangle labeled by $\langle \langle \vec{x}_1, \ldots, \vec{x}_{i-1} \rangle, X_i \rangle$ and not covering $\langle Y_1, \ldots, Y_k \rangle$, then either there exists an index $j \in [i-1]$ such that $\vec{x}_j \notin Y_j$ or $\dim(X_i \cap Y_i) > 0$.

Assume toward contradiction that the vector $\vec{1}$ is a linear combination of the rows labeled by rectangles from $\vec{w}_y$. Denote by C_y the set of columns of $\widehat{S}$, labeled by a k-tuple of vectors $\langle \vec{y}_1, \ldots, \vec{y}_k \rangle$ such that $\vec{y}_i \in Y_i$, and $\vec{y}_i$ has a leading 1 for every $i \in [k]$. We will use the sub-matrix of $\widehat{S}$ defined by the rows of $\vec{w}_y$ and the columns C_y, to contradict the existence of the above linear combination. We claim that for every $R \in \vec{w}_y$, the sum of the entries in the row labeled by R, over the columns in C_y, is 0. That is,

CLAIM 5.4. *For every $R \in \vec{w}_y$*

$$\sum_{c \in C_y} \widehat{S}[R, c] = 0.$$

Proof. If $R \in \vec{w}_y$ is a 0-rectangle labeled by $\langle \vec{x}_1, \ldots, \vec{x}_k \rangle$, and c is a column in C_y, labeled by $\langle \vec{y}_1, \ldots, \vec{y}_k \rangle$, then there is an index $i \in [k]$ such that $\vec{x}_i \notin Y_i$, and since $\vec{y}_j \in Y_j$ for every $j \in [k]$, we get that $\vec{x}_i \neq \vec{y}_i$ and thus $\widehat{S}[R, c] = 0$. Therefore, $\sum_{c \in C_y} \widehat{S}[R, c] = 0$.

If $R \in \vec{w}_y$ is a 1-rectangle labeled by $\langle \langle \vec{x}_1, \ldots, \vec{x}_{i-1} \rangle, X_i \rangle$, then either there exists an index $j \in [i-1]$ such that $\vec{x}_j \notin Y_j$ or $\dim(X_i \cap Y_i) > 0$. If the former is true, then for every column $c \in C_y$, labeled by $\langle \vec{y}_1, \ldots, \vec{y}_k \rangle$ we have $\vec{x}_j \notin Y_j$ and $\vec{y}_j \in Y_j$, and thus $\vec{x}_j \neq \vec{y}_j$. Since $j \in [i-1]$, this leads to $\widehat{S}[R, c] = 0$.

The only case left to discuss is when $R \in \vec{w}_y$ is a 1-rectangle that is labeled by $\langle \langle \vec{x}_1, \ldots, \vec{x}_{i-1} \rangle, X_i \rangle$, such that $\vec{x}_j \in Y_j$ for every $j \in [i-1]$, and $\dim(X_i \cap Y_i) \neq 0$. We get that $\widehat{S}[R, c] = 1$ for every column $c \in C_y$ labeled by

$$\langle \vec{x}_1, \ldots, \vec{x}_{i-1}, \vec{y}_i, \ldots, \vec{y}_k \rangle,$$

where $\vec{y}_i \notin X_i$. The number of choices for a vector $\vec{y}_i$ with a leading 1 such that $\vec{y}_i \in Y_i$ and $\vec{y}_i \notin X_i$ is the number of vectors with a leading 1 in the set $Y_i \backslash X_i = Y_i \backslash (Y_i \cap X_i)$. Since both Y_i and $Y_i \cap X_i$ are linear subspaces, the number of vectors with a leading 1 is equivalent to 1 modulo p in both of them. Thus the number choices for such $\vec{y}_i$ is equivalent to 0 modulo p. To get the number of columns $c \in C_y$ such that $\widehat{S}[R, c] = 1$, we have to multiply the number of ways to choose $\vec{y}_i$ by the number of ways to choose $\vec{y}_{i+1}, \ldots, \vec{y}_k$, but the result is still equivalent to 0 modulo p. ∎(Claim 5.4)

The number of columns in C_y is the product of the number of vectors with a leading 1 in Y_i for $i \in [k]$. Since each such number is 1 modulo p, the number of columns in C_y is equivalent to 1 modulo p.

Recall that we assumed that $\vec{1}$ is a linear combination of the rows corresponding to the rectangles in $\vec{w}_y$. Therefore, we can write

$$\sum_{R\in\vec{w}_y} \alpha_R \cdot \widehat{S}_R = \vec{1},$$

where for each $R \in \vec{w}_y$, the constant α_R is in $\mathrm{GF}(p)$, and $\widehat{S}_R$ is the row in $\widehat{S}$ corresponding to R. We compute the sum

$$\sum_{R\in\vec{w}_y} \alpha_R \sum_{c\in C_y} \widehat{S}[R,c]$$

in two different ways. Since for every column c it holds that

$$\sum_{R\in\vec{w}_y} \alpha_R \widehat{S}[R,c] = 1,$$

we get

$$\sum_{R\in\vec{w}_y} \alpha_R \sum_{c\in C_y} \widehat{S}[R,c] = \sum_{c\in C_y} \sum_{R\in\vec{w}_y} \alpha_R \widehat{S}[R,c] = \sum_{c\in C_y} 1 = |C_y| = 1 \bmod p.$$

On the other hand, according to Claim 5.4, the sum over any row $R \in \vec{w}_y$ of the entries in the columns of C_y is equivalent to 0 modulo p, and we get that

$$\sum_{R\in\vec{w}_y} \alpha_R \sum_{c\in C_y} \widehat{S}[R,c] = \sum_{R\in\vec{w}_y} \alpha_R \cdot 0 = 0 \bmod p.$$

A contradiction. Thus $\vec{1}$ is not a linear combination of the rows of $\widehat{S}$ labeled by $\vec{w}_y$, and hence $\widehat{S}$ rejects $\vec{w}_y$. $\square$(Claim 5.3)

5.2. Span Programs and Secret Sharing Schemes. Secret sharing schemes, introduced by Blakley [8], Shamir [25], and Ito, Saito, and Nishizeki [16], are a cryptographic tool allowing a dealer to share a secret between a set of parties such that only some pre-defined authorized subsets of parties can reconstruct the secret from their shares. The reader is referred to [26] and [28] for a more formal and detailed discussion on secret sharing schemes.

The authorized sets in a secret sharing scheme are described by a monotone Boolean function $f : \{0,1\}^n \to \{0,1\}$, where n is the number of parties and the authorized subsets are the subsets with their characteristic vectors in $f^{-1}(1)$. Most of the known secret sharing schemes are linear schemes, that is, schemes in which the shares are a linear combination of the secret and some random field elements. Linear schemes are equivalent to monotone span programs where the total size of the shares is the size of the corresponding monotone span program. Beimel and Ishai [4] showed functions that, under plausible assumptions, have no efficient linear secret sharing scheme but yet have an efficient non-linear secret sharing scheme. However, prior to this work, no secret sharing schemes were proved more powerful than linear schemes, without any assumptions.

A quasi-linear secret sharing scheme [4] is obtained by composing linear secret sharing schemes, possibly over different fields. Beimel and Ishai [4] have shown that under the assumption that the power of monotone span programs over different fields is incomparable, quasi-linear schemes are super-polynomially stronger than linear schemes. Their proof is very similar to the proof of Theorem 5.1. That is, the functions described in Theorem 5.1 have, by definition, a small quasi-linear secret sharing scheme but cannot have a small linear scheme.

THEOREM 5.5. *There is an explicit family of functions* $\{f_n\}_{n\in\mathcal{N}}$ *such that the complexity of every linear secret sharing scheme for the family is* $n^{\Omega(\sqrt{\log n})}$, *and yet the family has a polynomial quasi-linear secret sharing scheme.*

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