Advanced Topics in Complexity – Ex. 1

Due date: 22.4.02

Question 1

Let $b \in \{0,1\}$. A b-fooling set of size m for a function f is a sequence $(x_1, y_1), \ldots, (x_m, y_m)$ such that:

1. $f(x_i, y_i) = b$ for every $i = 1, \ldots, m$.
2. $f(x_i, y_j) = 1 - b$ or $f(x_j, y_i) = 1 - b$ for every $i \neq j$.

Part 1. Prove that there is a 1-fooling set of size n for the function IP_n.

Part 2. Prove that every 1-fooling set for the function IP_n has size at most n.
Hint: Prove that the vectors $x_1, \ldots, x_m$ must be linearly independent over $\mathcal{Z}_2$.

Part 3. Prove that every 0-fooling set for the function IP_n has size at most $n + 1$.

Question 2

Part 1. Let A and B be matrices. Prove that $\mathrm{rank}(A \cdot B) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$.

Part 2. Let $\mathrm{IP'}_n$ be the matrix for the inner-product as defined in the class (where $\mathrm{IP'}_n(x, y) \in \{-1, 1\}$). Prove that $\mathrm{IP}'_n \cdot \mathrm{IP}'_n = 2^n I$ (where I is the identity matrix with 2^n rows).

Part 3. Use the Part 1 and Part 2 to prove that $\mathrm{rank}(\mathrm{IP}_n) = 2^n - 1$. (Recall that a different proof was given in the class.)

Question 3

Part 1. Assume that a function f has a 1-fooling set of size m. Prove that $N^1(f) \geq \log m$.

Part 2. Prove that $N^1(\overline{\mathrm{GT}}_n) \geq n$.

Question 4

Define $\mathrm{rank}_2(M)$ as the rank of M over the field $\mathcal{Z}_2$.

Part 1. Prove that for every Boolean matrix $\mathrm{rank}_2(M) \leq \mathrm{rank}(M)$.

Part 2. Prove that $D(f) \leq \mathrm{rank}_2(M_f) + 1$.

Part 3. Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be such that for every $x_1, x_2 \in \{0,1\}^n$, where $x_1 \neq x_2$, there is some y such that $f(x_1, y) \neq f(x_2, y)$. Prove that $D(f) \geq \log n$.