Department of Computer Science
Frontiers in Computer Science

Assignment 1, Semester 2, 2001

Deadline: April 15

Introduction

This exercises focuses on several aspects of Boolean functions as used when performing program analysis where Boolean functions are used to describe program states. Solve at least one of the questions.

Background

A Boolean function on the variables $x_1, \ldots, x_n$ is a mapping from

$$\varphi : \{true, false\}^n \rightarrow \{true, false\}$$

Recall the notions of an *interpretation* and of a *model*: Let φ be a Boolean function on variables $\{x_1, \ldots, x_n\}$. An *interpretation* is an assignment of truth values to $x_1, \ldots, x_n$. An interpretation μ is *model* for φ if $\varphi(\mu(x_1), \ldots, \mu)x_n)) = true$. Recall also that we can represent φ as a propositional formula. For example, the formula $x \wedge y$ (on variables x, y) represents the function that maps $x = true, y = true$ to $true$ and all other assignments (on x, y) to $false$.

We say that a Boolean function φ is positive if the interpretation which assigns the value $true$ to all variables is a model of $varphi$. Namely, $\varphi(ture, \ldots, true) = true$. We denote by Pos the set of positive Boolean functions.

We sometimes specify a model as a set of variables — those mapped to true and denote by $models(\varphi)$ the sets of models of the Boolean function φ. For example, consider the Boolean functions on $\{x, y\}$ represented by the formula $x \vee y$ and $y \rightarrow x$. Their sets of models are respectively $\{\{x\}, \{y\}, \{x, y\}\}$ and $\{\{\ \}, \{x\}, \{x, y\}\}$.

We denote by Def the set of positive Boolean formula whose sets of models are closed under intersection. For example, $y \rightarrow x$ is in Def but $x \vee y$ is not.

Questions

1. Show that using only propositional variables and the connectives $\{\wedge, \vee, \leftrightarrow\}$ we can represent exactly the set of positive Boolean functions.

2. Show that using only propositional variables and the connectives $\{\wedge, \rightarrow\}$ we can represent exactly the set of positive Boolean functions.

3. What is the length of the longest chain $\varphi_1, \varphi_2, \ldots, \varphi_k$ of Boolean functions (on n variables) in Def such that $models(\varphi_1) \subset models(\varphi_2) \subset \cdots \subset models(\varphi_k)$. Show such a chain.