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ON THE CONSISTENCY OF SOME PARTITION THEOREMS FOR CONTINUOUS COLORINGS, AND THE STRUCTURE OF $\aleph_1$-DENSE REAL ORDER TYPES

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We present some techniques in c.c.c. forcing, and apply them to prove consistency results concerning the isomorphism and embeddability relations on the family of $\aleph_1$-dense sets of real numbers. In this direction we continue the work of Baumgartner [2] who proved the axiom BA stating that every two $\aleph_1$-dense subsets of $\mathbb{R}$ are isomorphic, is consistent. We e.g. prove Con(BA+$(2^{\aleph_0}>\aleph_2)$). Let $\langle K^H, \leqslant\rangle$ be the set of order types of $\aleph_1$-dense homogeneous subsets of $\mathbb{R}$ with the relation of embeddability. We prove that for every finite model $\langle L, \leqslant\rangle$: Con(MA+ $\langle K^H, \leqslant\rangle \simeq \langle L, \leqslant\rangle$) iff L is a distributive lattice. We prove that it is consistent that the Magidor–Malitz language is not countably compact. We deal with the consistency of certain topological partition theorems. E.g. We prove that MA is consistent with the axiom OCA which says: "If X is a second countable space of power $\aleph_1$, and $\{U_0, \ldots, U_{n-1}\}$ is a cover of $D(X) \overset{\text{def}}{=} X\times X - \{\langle x, x\rangle \mid x \in X\}$ consisting of symmetric open sets, then X can be partitioned into $\{X_i \mid i \in \omega\}$ such that for every $i \in \omega$ there is $l<n$ such that $D(X_i) \subseteq U_l$". We also prove that MA+OCA $\Rightarrow 2^{\aleph_0}=\aleph_2$.

Introduction

The purpose of this paper is to prove consistency results about partitions of second countable spaces of power $\aleph_1$, and about the relations of embeddability and isomorphism between sets of real numbers of power $\aleph_1$.

Our intention is not only to prove new results, but also to present the techniques used. Because of this reason, in the first sections, we tried as much as possible to present applications in which the proofs were technically simple, and in which only one technique was being used at a time. Thus we sometimes had to repeat ourselves, and in one case we chose to reprove a theorem from [1], though in a different way. On the other hand we sometimes omit the proof of some details which resemble previous arguments.

The starting point of this paper is the theorem of Baumgartner [2] that *the axiom* BA, *which says that every two* $\aleph_1$*-dense sets of real numbers are order-isomorphic, is consistent.* Baumgartner in fact proved that MA+BA is consistent. The isomorphization of two $\aleph_1$-dense sets of real numbers was done by means of a c.c.c. forcing set. This suggested that maybe $\mathrm{MA}_{\aleph_1}$ already implies BA.

The negative answer to the above question was found by Shelah. He invented two techniques: the club method and the explicit contradiction method. Using these methods Shelah [1] proved that $\mathrm{MA}_{\aleph_1}$ was consistent with the existence of an entangled set (see Section 7), thus showing that $\mathrm{MA}_{\aleph_1} \not\Rightarrow \mathrm{BA}$.

Avraham [1] then found another way to refute BA. By means of the club method he constructured a universe V satisfying MA and a set of real numbers of power $\aleph_1$, $A \in V$, such that every 1-1 uncountable $g \subseteq A \times A$ contained an uncountable order preserving function. Such an A is not isomorphic to $A^* \stackrel{\text{def}}{=} \{-a \mid a \in A\}$, thus $V \models \neg\mathrm{BA}$.

Answering a question of Avraham, Shelah [1] proved that it is consistent that every 1-1 $g \subseteq \mathbb{R} \times \mathbb{R}$ of power $\aleph_1$ can be represented as the union of countably many monotonic functions. The proof involved a new trick: The preassignment of colors (see Section 3).

The club method

The club method plays the most central role in this paper. We explain in what context one can try to use this method. Let $|A| = \aleph_1$, and let R be a binary relation on A. Suppose $R = \bigcup_{i \in \omega} B_i \times C_i$, (in this case we say that R has a countable semibase). By the club method one can try to construct a c.c.c. forcing set which adds to V an uncountable subset of A^n which has various homogeneity properties with respect to R. E.g. one might want to add an uncountable $g \subseteq A \times A$ such that for every $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in g$, $\langle a_1, a_2 \rangle \in R$ iff $\langle b_1, b_2 \rangle \in R$. (This is the case of adding an order preserving function.) Note that if X is a second countable space and $R \subseteq X \times X$ is open, then R has a countable semibase, hence $<_R$ and ${}_R>$ have countable semibases.

The club method makes the problem of isomorphizing two $\aleph_1$-dense subsets of $\mathbb{R}$ just one special option in a wide spectrum of possibilities.

In the beginning, we knew to apply the club method only when the ground model satisfied CH. After understanding the exact role of CH it was possible to replace it by an axiom denoted by A1 which may holds also in the absence of CH. A1 has the property that if $V \models \mathrm{A1}$ and P is a c.c.c. forcing set of power $<2^{\aleph_1}$, then V^P also satisfies A1. Hence one can carry out a finite support iteration of length $2^{\aleph_1}$ consisting of general c.c.c. forcing sets and 'club method' forcing sets. In this way we obtain the consistency of $\mathrm{BA} + (2^{\aleph_0} > \aleph_2)$ which could not have been obtained by the method of [2].

The other techniques described in this paper are easily combined with the club method in many different ways, thus yielding a rich variety of consistency results.

Summary of results

1. The club method and the semiopen coloring axiom

In this section we present the club method by means of an application. Let X denote a second countable space of power $\aleph_1$, let U be a symmetric open subset of $X \times X$, and for a set A let $D(A) = A \times A - \{\langle a, a\rangle \mid a \in A\}$. The semi open coloring axiom (SOCA) says: "For every X and U as above there is an uncountable $A \subseteq X$ such that either $D(A) \subseteq U$ or $D(A) \cap U = \emptyset$". In Section 1 we prove that MA+SOCA is consistent. This is probably the simplest application of the club method.

In addition we prove in Section 1 the consistency of a certain strengthening of SOCA, we prove some corollaries of SOCA, and bring some counter-examples.

2. The explicit contradiction method and the increasing set axiom

A set $S \subseteq \mathbb{R}$ of cardinality $\aleph_1$ is called an increasing set if for every $n \in \omega$ and a set $\{\langle a(\alpha, 0), \ldots, a(\alpha, n-1)\rangle \mid \alpha < \aleph_1\} \subseteq A^n$ of pairwise disjoint n-tuples there are $\alpha, \beta < \aleph_1$ such that for every $i < n$, $a(\alpha, i) < a(\beta, i)$.

Suppose $A \in V$ is increasing, and we want to construct a universe $W \supseteq V$ which satisfies MA and in which A retains its increasingness. The problem is that when we iterate c.c.c. forcing sets in order to take care of MA it may happen (and indeed it does happen if $V \models$ CH) that some of the iterands P_i force that A is not increasing. The way in which this difficulty is overcome, is that we construct a c.c.c. forcing set Q such that $\Vdash_Q$ (P_i is not c.c.c) $\wedge$ (A is increasing). Hence forcing through Q retains the increasingness of A and frees us from forcing through P_i. The particular method in which this is done is called the explicit contradiction method.

Section 2 is devoted to the proof that $MA_{\aleph_1}$ is consistent with the existence of an increasing set. Indeed $MA_{\aleph_1} \Rightarrow A$ is increasing iff every uncountable 1-1 $g \subseteq A \times A$ contains an uncountable order preserving function. Thus what we prove in Section 2 coincides with Theorem 2 of [1]. However, since this is the simplest application of the explicit contradiction method, and since the proof we present can be used to retain also other properties of A, we take the liberty to reprove Theorem 2 of [1].

3. The open coloring axiom, and how to preassign colors

Let X denote a second countable space of power $\aleph_1$. An open cover $\mathcal{U} = \{U_0, \ldots, U_{n-1}\}$ of $D(X)$ consisting of symmetric sets is called an open coloring of

X. $A \subseteq X$ is $\mathcal{U}$-homogeneous if for some $i<n$, $D(A) \subseteq U_i$. Let OCA be the axiom: "For every X and $\mathcal{U}$ as above X can be partitioned into countably many $\mathcal{U}$-homogeneous subsets". Let ISA be the axiom: "There exists an increasing set".

Trying to strengthen SOCA, and Theorem 6 of [1], we prove that MA+SOCA+OCA+ISA is consistent. The new element in the proof is a use of the so-called preassignment of colors. Let X, $\mathcal{U}$ be as above, and let $A \subseteq \mathbb{R}$ be an increasing set. We want to partition X into countably many $\mathcal{U}$-homogeneous subsets without destroying the increasingness of A. There is a method to assign to each $a \in X$ a color $i(a)<n$ such that there is a c.c.c forcing set P which partitions X into countably many $\mathcal{U}$-homogeneous sets, in this partition every $a \in X$ belongs to a set with color $i(a)$, and P does not destroy increasingness of A. The preassignment of colors resembles Theorem 6 of [1], but here we have one additional trick devised in order to retain the increasingness of A.

OCA can be generalized to colorings of n-tuples rather than colorings of pairs. For $\nu, \xi \in {}^{\omega>}2$ let $\nu \wedge \xi$ denote the maximal common initial segment of ν and ξ. For $A \subseteq {}^{\omega}2$ let

$$T[A] = \{\nu \wedge \xi \mid \nu, \xi \in A \text{ and are distinct}\}.$$

For $\nu, \xi \in {}^{\omega>}2$ and $l=0, 1$, let $\nu <_l \xi$ denote the fact that $\nu^\frown\langle l\rangle$ is an initial segment of ξ. Let σ, τ be finite subsets of ${}^{\omega>}2$. $\sigma \sim \tau$ means that $\langle \sigma, <_0, <_1\rangle \cong \langle \tau, <_0, <_1\rangle$.

Let TCAm be the axiom saying: "If $\langle C_0, \ldots, C_{k-1}\rangle$ is a partition of the unordered m-tuples of ${}^{\omega>}2$, and $A \subseteq {}^{\omega}2$ is of power $\aleph_1$, then there is a partition of A $\{A_i \mid i \in \omega\}$ such that for every $i \in \omega$ and two subsets σ_1, σ_2 of A_i of power $m+1$: if $T[\sigma_1] \sim T[\sigma_2]$, then there is $j<k$ such that $T[\sigma_1], T[\sigma_2] \subseteq C_j$".

In Section 3 we prove that Ma$+\bigwedge_{m \in \omega}TCAm$ is consistent. TCA1 is implied by OCA, and MA+TCA1 $\Rightarrow$ OCA.

TCAm has also a topological equivalent but its formulation is not very transparent. The more direct and stronger generalization of OCA remains open.

We conclude Section 3 with another axiom concerning partitions. Let X, Y be a second countable spaces such that $|X| = \aleph_1$, and Y does not contain isolated points; let f be a symmetric continuous function from $D(X)$ to Y. Let NWDA be the axiom which says: "If X, Y, f are as above, then there is a partition $\{A_i \mid i \in \omega\}$ of X such that for every $i, j \in \omega$ $f(A_i \times A_j - \{\langle a, a\rangle \mid a \in X\})$ is nowhere dense". We prove that MA+NWDA is consistent.

We did not investigate the relationship of NWDA with other axioms and its possible generalizations.

4. *The semi open coloring axiom does not imply the open coloring axiom; the tail method*

In Section 4 we prove that SOCA+MA+$(2^{\aleph_0} = \aleph_2) \not\Rightarrow$ OCA. Indeed, in Section 5 we prove that MA+SOCA is consistent with $2^{\aleph_0} > \aleph_2$, and in Section 11 we prove that MA$+2^{\aleph_0}>\aleph_2 \Rightarrow \neg$OCA, hence the result of Section 4 becomes less interesting. But the proof serves well in demonstrating an additional trick called the tail method. This trick is used also in Sections 9 and 10, but there, the technical details are somewhat more complicated.

5. *Enlarging the continuum beyond $\aleph_2$*

In Baumgartner's proof of the consistency of BA, the construction of a c.c.c. forcing set which isomorphizes $\aleph_1$-dense sets of real numbers, is done under the assumption of CH. So in the universe satisfying BA the continuum had to be $\aleph_2$. The substitute for CH in the application of the club method was found by Shelah. This immediately implied that BA is consistent with $2^{\aleph_0}>\aleph_2$. In this section we demonstrate this method by proving that MA+SOCA+$(2^{\aleph_0}>\aleph_2)$ is consistent.

6. MA, OCA *and the embeddability relation on $\aleph_1$-dense real order types*

Let $K=\{A\subseteq\mathbb{R} \mid A\neq\emptyset, A$ has no endpoints and every interval of A has cardinality $\aleph_1\}$. For $A, B\in K$ let $A\preccurlyeq B$ and $A\cong B$ respectively mean that $\langle A,<\rangle$ is embeddable or isomorphic to $\langle B,<\rangle$. Let $A\in K$. A is homogeneous if for every $a,b\in A$ there is an automorphism f of $\langle A,<\rangle$ such that $f(a)=b$. Let $K^H=\{A\in K \mid A$ is homogeneous$\}$. Let $N(A,B)$ mean that there is $C\in K$ such that $C\preccurlyeq A$ and $C\preccurlyeq B$; $A\perp B\equiv\neg N(A,B)$ and $A \perp\!\!\perp B\equiv A\perp B\wedge A\perp B^*$. Let NA be the axiom: $(\forall A,B\in K)\ N(A,B)$.

A great part of this work was motivated by questions about the possible structure of K and K^H. Since SOCA easily implies $(\forall A,B\in K)\ (N(A,B)\cup N(A,B^*))$ it was natural to ask whether it also implied NA. Since "A is increasing" implies $\neg N(A,A^*)$, this question was answered in Section 3. There was still another reason why MA+OCA+ISA was interesting. Shelah proved the consistency of the following axiom: "There are $A,B\in K^H$ such that: $A\cong A^*$, $B\cong B^*$, $A\perp\!\!\perp B$, $A\cup B\in K^H$ and for every $C\in K^H$ either $C\cong A$ or $C\cong B$ or $C\cong A\cup B$".

It was of interest to us to find whether in this axiom one can make the modification that $A\perp A^*$ and $A^*\cong B$. In Section 6 we indeed show that this modified axiom follows from MA+OCA+ISA.

In fact MA+OCA almost determines the structure of K^H and K. If MA+OCA is conjuncted with ISA, then K^H is as above, if MA+OCA is conjuncted with $\neg$ISA, then BA holds.

7. *Relationship with the weak continuum hypothesis*

The weak continuum hypothesis WCH is the statement that $2^{\aleph_0}<2^{\aleph_1}$. In Section 7 we first show that BA$\Rightarrow\neg$WCH. The question that naturally arises is what happens if BA is weakened and is replaced by NA. We prove that unlike BA, NA is consistent with WCH.

This automatically implies that NA $\not\Rightarrow$ BA. The fact that MA+NA $\not\Rightarrow$ BA follows from the results of Section 9.

In the proof of NA+WCH we introduce a forcing which makes two members of K near. This is a simple version of a forcing set which isomorphizes two members of K.

One can consider the following strengthening of NA. Let DNA be the following axiom: "If $A, B \in K$, then there is an uncountable order preserving function $g \subseteq A \times B$ such that $\mathrm{Dom}(g)$, $\mathrm{Rng}(g) \in K$ and are dense in A and B respectively. Section 7 is concluded with a proof that NA $\Rightarrow$ DNA.

8. A weak form of Martin's axiom, the consistency of the incompactness of the Magidor-Malitz quantifiers.

Let MML denote the Magidor–Malitz language. In [7] Magidor and Malitz proved that $\diamondsuit_{\aleph_1} \Rightarrow$ "MML is countably compact". This suggested the following question: "Construct a universe in which MML is not countably compact". A first solution to this problem was found by Shelah (unpublished) using methods of Avraham. Shelah's solution involves properties of Suslin trees which are expressible by MML sentences. The result of Shelah is that the countable incompactness of MML is consistent with CH.

In Section 8 we bring a simpler solution to this question, here we obtain a universe in which MA+$(\aleph_1 < 2^{\aleph_0})$+(MML is not countably compact) holds.

Let $A \in K$ and $k \in \omega$, A is k-entangled if for every sequence $\{\langle a(\alpha, 0), \ldots, a(\alpha, k-1)\rangle \mid \alpha < \aleph_1\} \subseteq A^k$ of pairwise disjoint 1-1 sequences, and for every $\langle \varepsilon(0), \ldots, \varepsilon(k-1)\rangle \in \{0, 1\}^k$ there are $\alpha_0, \alpha_1 < \aleph_1$ such that for every $i < k$, $a(\alpha_{\varepsilon(i)}, i) < a(\alpha_{1-\varepsilon(i)}, i)$. The k-entangleness of A can be expressed by an MML sentence and $\mathrm{MA}_{\aleph_1} \Rightarrow \neg(\exists A \in K)(\forall k \in \omega)$ (A is k-entangled). Let $W \models \mathrm{MA}_{\aleph_1} + (\forall k \in \omega)(\exists A \in K)$ (A is k-entangled). Hence in W MML is not countably compact.

The notion of entangledness was defined by Shelah in [1]. There, it is proved that for every $k \in \omega$, $\mathrm{MA}_{\aleph_1} + (\exists A \in K)$ (A is k-entangled) is consistent. It is somewhat more complicated to prove that $\mathrm{MA}_{\aleph_1} + (\forall k \in \omega)(\exists A \in K)$ (A is k-entangled) is consistent. We prove this fact in Section 8.

The other question considered in Section 8 is whether iterating forcing sets obtained by means of the club method can yield a universe satisfying $\mathrm{MA}_{\aleph_1}$. To prove that this is not so we define a property, denoted by s.c.c., and stronger than the countable chain condition, such that every forcing set gotten from the club method has this property. On the other hand we prove that a finite support iteration of s.c.c. forcing sets does not destroy Suslin trees. We hence obtain that OCA, SOCA, NA, etc. are consistent with the existence of a Suslin tree.

9. The isomorphizing forcing, and more on the possible structure of K

In this section we first construct for $A, B \in K$ a c.c.c. forcing set P such that $\Vdash_P A \cong B$. This construction is a basic tool for results concerning the possible structure of K. This construction can be carried out under assumptions weaker than CH, hence we can prove that BA is consistent with $2^{\aleph_0} > \aleph_2$. The other important property of this construction is that it enables to isomorphize two sets leaving some other sets far. E.g., we prove that if $A, B, C \perp\!\!\!\perp D$, then there is a c.c.c. P which isomorphizes A and B and keeps $C \perp\!\!\!\perp D$.

$A \in K$ is rigid if $\langle A, < \rangle$ has no automorphisms other than the identity. Let

$$\mathrm{RHA} \equiv (\forall A \in K)\,(\exists B, C \in K)\,((B, C \subseteq A) \wedge (B \text{ is rigid}) \wedge (C \text{ is homogeneous})).$$

Note that $\mathrm{RHA} \Rightarrow \neg\mathrm{CH}$.

Combining the construction of isomorphizing forcing sets with the explicit contradiction method and the tail method we prove the consistency of MA+RHA.

10. *The structure of K and K^H when K^H is finite*

In Section 6 we prove that $\mathrm{MA}_{\aleph_1} \Rightarrow K^H/{\cong}$ is partially ordered by $\leqslant$. Clearly $*$ is an automorphism of $\langle K^H/{\cong}, \leqslant \rangle$. Let $K^{HZ} = (K^H/{\cong}) \cup \{\emptyset\}$, $\langle K^{HZ}, \leqslant, * \rangle$ is a partially ordered set with an involution. In Section 10 we prove the following theorem: Let $\langle L, \leqslant, * \rangle$ be a finite partially ordered set with an involution: Then MA+$(K^{HZ} \cong L)$ is consistent iff L is a finite distributive lattice with an involution.

This theorem was preceded by the following result by Shelah: It is consistent that $K^{HZ} = \{0, a, b, c\}$ where $a \wedge b = 0$, $a^* = a$, $b^* = b$ and $c = a \vee b$. Avraham and Rubin then showed (Section 3, 6) that K^{HZ} may by $\{0, a, b, c\}$ where $a \wedge b = 0$, $a = b^*$ and $c = a \vee b$.

Some results in the same direction were proved by Rubin for the class $K^{IIH} \stackrel{\mathrm{def}}{=} \{A \in K^H \mid A \text{ is of the second category}\}$.

We also prove in Section 10 some results about the possible infinite K^{HZ}'s.

11. MA+OCA *implies* $2^{\aleph_0} = \aleph_2$.

Until the writing of this paper had been almost finished, we believed that the method to enlarge $2^{\aleph_0}$ beyond $\aleph_2$ worked for all applications of the club method. We realized that CH was used not only in the application of the club method, but also in order to e.g., get from $A, B \in K$ $A', B' \in K$ such that $A' \subseteq A$, $B' \subseteq B$ and $A' \perp\!\!\!\perp B'$. However this could be done too without assuming CH. Finally we noticed that, indeed, we did not know to preassign colors (Section 3) without CH, and we did not know how to prove the consistency of SOCA1 (Section 1) and the results of Section 10 without assuming CH in the intermediate models.

Shelah then found that at least in the case of OCA, the failure to prove the consistency of OCA+MA+$(2^{\aleph_0} > \aleph_2)$ followed from the fact that this axiom was false. He found a c.c.c. forcing set P of power $\aleph_2$, and $\aleph_2$ dense subsets of P, such that if V contains a filter of P which intersects all these dense sets, then V contains an open coloring of a set $A \subseteq {}^{\omega}2$ of power $\aleph_1$ for which there is no partition of A into countably many homogeneous sets. Section 11 contains this result.

Whether the results of Section 10 and SOCA1 are consistent with MA+ $(2^{\aleph_0} > \aleph_2)$ remains open.

Main open problems

In the paper we mention many open problems, they appear in the relevant context. Let us mention here those problems which, we believe, require new techniques.

(1) (Baumgartner) Is it consistent that every two $\aleph_2$-dense sets are isomorphic? More generally, are the axioms appearing in this paper consistent when we replace $\aleph_1$ by $\aleph_2$?

(2) The axioms mentioned in this paper are all consistent with MA. We do not know how to prove the consistency of similar axioms which contradict MA. E.g., is the following axiom consistent: $\neg\text{BA} + (\forall A, B \in K)(A \leqslant B)$? Is the following axiom consistent: $\text{OCA} + 2^{\aleph_0} > \aleph_2$?

(3) Let OCA(m, k) be the following axiom: "For every second countable space X of power $\aleph_1$ and every finite open cover $\mathcal{U}$ of X^m, there is a partition $\{X_i \mid i \in \omega\}$ of X such that for every $i \in \omega$, X_i^m intersects at most k members of $\mathcal{U}$." Does there exist a k for which OCA(m, k) is consistent? In fact we do not know the answer even for $m = 3$, and even if the axiom is weakened to require only the existence of one uncountable subset A of X such that A^m intersects at most k members of $\mathcal{U}$.

(4) Are some of the axioms mentioned consistent with the existence of a second category subset of $\mathbb{R}$ of power $\aleph_1$? E.g. are $\text{NA} + (\exists A \in K)$ (A is of the second category) and $\text{SOCA} + (\exists A \in K)$ (A is of the second category) consistent?

Historical remarks

The club method, explicit contradiction method, the method to enlarge $2^{\aleph_0}$ beyond $\aleph_2$ are due to Shelah. The tail method is due to Rubin. The method of preassigning colors is due to Shelah, but an additional trick was added by Avraham and Rubin. Section 1 dealing with SOCA is mainly the work of Avraham and Rubin. Section 2 is another proof of a theorem by Avraham and Shelah in [1]. The axiom OCA appearing in Section 3 and its corollaries concerning the structure of K appearing in Section 6 are due to Avraham and Rubin. The axiom TCAm which generalizes OCA is due to Shelah and the axiom NWDA is due to Rubin. The proof that SOCA$\not\Rightarrow$OCA appearing in Section 4 is due to Rubin. Section 5 dealing with how to enlarge $2^{\aleph_0}$ beyond $\aleph_2$ is due to Shelah. Section 7 dealing with the relationship with WCH is due to Shelah. The weak Martin's axiom appearing in Section 8 and the proof that it is consistent with the existence of Suslin trees is due to Avraham and Rubin. The proof that MML may be countably incompact is due to Rubin. This theorem was first proved by Shelah using other methods. The proof was a slight improvement of a theorem of Shelah in [1].

The isomorphizing forcing in Section 9 is due to Shelah. BA1 as well as RHA are due to Rubin. RHA uses the tail method as well as an important lemma essentially due to Shelah. This lemma states that if $A, B \perp\!\!\!\perp C, D$ then it is possible to isomorphize A and B keeping $C \perp\!\!\!\perp D$. Section 10 which deals with the structure of K and K^H when K^H is finite is due to Rubin. The theorem stating that $MA+OCA \Rightarrow 2^{\aleph_1}=\aleph_2$ appearing in Section 11 is due to Shelah.

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For the reader's convenience we include here an index of axioms and some notations used in this work.

1. The club method and the semiopen coloring axiom

In this section we present the club method which is the main technique in this paper. We prove a theorem in which the club method is used. This theorem is perhaps the simplest application of this method.

For a set A let $D(A)=A\times A-\{\langle a, a\rangle \mid a \in A\}$. A function f from the set of unordered pairs of a set A to $\{0, 1\}$ is called a coloring of A in two colors. We regard f as a symmetric function from $D(A)$ to $\{0, 1\}$. A subset $B \subseteq A$ is called f-homogeneous, or in short homogeneous, if $f \restriction D(B)$ is a constant function; we say that B is of color l, or in short B is l-colored if the value of $f \restriction D(B)$ is l.

From now on X denotes a second countable topological Hausdorff space of cardinality $\aleph_1$. Let f be a coloring of X in two colors, f is called a semiopen coloring (SOC), if $f^{-1}(1)$ is open in $X \times X$.

Let the semiopen coloring axiom be the following axiom.

Axiom SOCA. For every X and a SOC f of X, X contains an uncountable f-homogeneous subset.

Theorem 1.1. SOCA *is consistent with* ZFC.

Proof. We prove the following claim. Let $V \models \mathrm{CH}$, and let f be a SOC of X such that X has no uncountable homogeneous subset of color 0; then there is a c.c.c. forcing set of power $\aleph_1$, $P = P_{X,f}$, such that in V^P, X contains an uncountable homogeneous set of color 1.

By the method of Solovey and Tenenbaum [9], this claim suffices in order to prove the theorem. More specifically we start with a universe satisfying CH+ $(2^{\aleph_1} = \aleph_2)$ and carry out an iteration with direct limits $\{P_\alpha \mid \alpha < \aleph_2\}$, in which each $(\alpha+1)$st iterand is the P_α-name of some forcing set of the form $P_{X,f}$.

We thus turn to the construction of $P_{X,f}$ assuming that CH holds, and X and f are given. We first need a model of the form $\langle \aleph_1, <, \ldots, \rangle$ that includes the information about X and f, and that encompasses enough set theory. In order not to repeat the same definition over and over, we shall at this point fix a model that will serve us also in the future. Let $H(\aleph_1)$ be the set of hereditarily countable sets. By CH, $|H(\aleph_1)| = \aleph_1$. We choose a 1-1 correspondence h between $H(\aleph_1)$ and $\aleph_1$. Let $M^0 = \langle \aleph_1, <, h, \in_1 \rangle$ where $\alpha \in_1 \beta$ iff $h(\alpha) \in h(\beta)$. In order not to have two belonging relation symbols we shall denote $\in_1$ by $\in$ and will refrain from using "$\alpha \in \beta$" to mean the usual belonging relation between countable ordinals; instead we shall write "$\alpha < \beta$". We reserve M^0 to mean the above model throughout this paper.

W.l.o.g. $X \subseteq \aleph_1$. Let $M = \langle M^0, X, f, T \rangle$; by this we mean that we expand M^0 by adding to it a unary predicate to represent X, a binary function symbol to represent f, and some binary relation symbol to represent some fixed countable base for X. T can be defined in the following way: let $\{U_i \mid i \in \omega\} \stackrel{\mathrm{def}}{=} \mathcal{U}$ be a countable base for X; $T = \{\langle i, \alpha \rangle \mid i \in \omega \text{ and } \alpha \in U_i\}$.

For $\alpha < \aleph_1$, let M_α denote the submodel of M whose universe is α. Let $C_M = \{\alpha \mid M_\alpha \prec M\}$. C_M is a closed unbounded set (club).

A subset $A \subseteq \aleph_1$ is called C_M-separated or in short separated, if for every $\alpha, \beta \in A$ such that $\alpha < \beta$ there are $\gamma_1, \gamma_2 \in C_M$ such that $\gamma_1 \le \alpha < \gamma_2 \le \beta$.

Let λ be a cardinal and A be a set; we denote $P_\lambda(A) = \{B \subseteq A \mid |B| < \lambda\}$. Let $P_{X,f} = \{\sigma \in P_{\aleph_0}(X) \mid \sigma$ is homogeneous of color 1 and σ is separated$\}$. The partial ordering on $P_{X,f}$ is set inclusion.

We show that $P_{X,f}$ is c.c.c. Suppose by contradiction that it is not; then it is easy to see that there is $\Gamma_1 = \{\sigma^i \mid i < \aleph_1\} \subseteq P_{X,f}$ such that:

(1) for every $i < j < \aleph_1$, $\sigma^i \cup \sigma^j$ is not homogeneous of color 1;

(2) for every $i < j < \aleph_1 : |\sigma^i| = |\sigma^j|$, and $\sigma^i \cup \sigma^j$ is C_M-separated.

Let $\{\alpha^i_1, \dots, \alpha^i_n\}$ be an enumeration of σ^i in an increasing order. Since σ^i is homogeneous of color 1 and since f is a SOC, there are $U^i_1, \dots, U^i_n \in \mathcal{U}$ such that for every $k \neq l$, $\sigma^i_k \in U^i_k$ and $f(U^i_k \times U^i_l) = \{1\}$. Let Γ be an uncountable subset of Γ_1 such that for every $i, j \in \Gamma$ and for every $1 \leq k \leq n$, $U^i_k = U^j_k \overset{\text{def}}{=} U_k$. By reindexing we can assume that $\Gamma = \{\sigma^i \mid i < \aleph_1\}$. We thus conclude

(*) For every $i, j < \aleph_1$ and $1 \leq k \neq l \leq n$, $f(\alpha^i_k, \alpha^j_l) = 1$.

The next step which we call 'the duplication argument' is one of the central arguments in this paper. For a subset A of a topological space X, let $\mathrm{cl}(A)$ be the topological closure of A in X.

$\Gamma \subseteq X^n$ and X^n is second countable, hence for some countable $\Gamma_0 \subseteq \Gamma$, $\mathrm{cl}(\Gamma_0) = \mathrm{cl}(\Gamma)$. Let $\gamma \in C_M$ be such that $\Gamma_0 \in |M_\gamma|$. (More precisely we mean that $h(\Gamma_0) < \gamma$, but we shall always make this abuse of notation.) Note also that $\mathcal{U} \subseteq M_\alpha$ for every $\alpha \in C_M$. There is a formula in the language of M_γ and with the parameter Γ_0, $\varphi(x_1, \dots, x_n)$, which says that $\langle x_1, \dots, x_n \rangle \in \mathrm{cl}(\Gamma_0)$. Let $i < \aleph_1$ be such that $\gamma < \alpha^i_1$. We want to define by a downward induction a sequence of certain formulas $\varphi_l(x_1, \dots, x_l)$, $l = 0, \dots, n$, where $\varphi_n = \varphi$ and where $M \models \varphi_l[\alpha^i_1, \dots, \alpha^i_l]$. For the sake of clarity we first show how to get φ_{n-1}. Let $\delta \in C_M$ and $\alpha^i_{n-1} < \delta < \alpha^i_n$. For every $\alpha \in |M_\delta|$, $M \models \psi[\alpha^i_1, \dots, \alpha^i_{n-1}, \alpha]$ where $\psi(x_1, \dots, x_{n-1}, x) \equiv (\exists x_n > x)\, \varphi(x_1, \dots, x_n)$; for one can take x_n to be α^i_n. Since $M_\delta \prec M$, $M_\delta \models \psi[\alpha^i_1, \dots, \alpha^i_{n-1}, \alpha]$. Hence $M_\delta \models \forall x\, \psi[\alpha^i_1, \dots, \alpha^i_{n-1}, x]$, hence M satisfies the same formula. This means that $L \overset{\text{def}}{=} \{\beta \mid \langle \alpha^i_1, \dots, \alpha^i_{n-1}, \beta \rangle \in \mathrm{cl}(\Gamma_0)\}$ is unbounded and thus uncountable. We assumed that X did not contain uncountable homogeneous sets of color 0, thus there are $\beta_1, \beta_2 \in L$ such that $f(\beta_1, \beta_2) = 1$. Let $U^n_1, U^n_2 \in \mathcal{U}$ be disjoint sets such that $\beta_l \in U_l$ and $f(U^n_1 \times U^n_2) = \{1\}$. Let

$$\varphi_{n-1}(x_1, \dots, x_{n-1}) \equiv \exists x^n_1\, \exists x^n_2 \left(\bigwedge_{l=1}^{2} (x^n_l \in U^n_l \wedge \varphi(x_1, \dots, x_{n-1}, x^n_l)) \right).$$

Clearly $M \models \varphi_{n-1}[\alpha^i_1, \dots, \alpha^i_{n-1}]$. Suppose φ_m has been defined and $M \models \varphi_m[\alpha^i_1, \dots, \alpha^i_m]$. Repeating the same argument as before, there are disjoint $U^m_1, U^m_2 \in \mathcal{U}$ and $\beta_l \in U^m_l$, $l = 1, 2$, such that $f(U^m_1 \times U^m_2) = \{1\}$, and $M \models \varphi_m[\alpha^i_1, \dots, \alpha^i_{m-1}, \beta_l]$. Let

$$\varphi_{m-1} = \exists x^m_1\, \exists x^m_2 \left(\bigwedge_{l=1}^{2} (x^m_l \in U^m_l \wedge \varphi_m(x_1, \dots, x_{m-1}, x^m_l)) \right).$$

Now we start with φ_0 and inductively pick β^j_l, $l = 1, 2$, $j = 1, \dots, n$. Since $M \models \varphi_0$ there are $\beta^1_l \in U^1_l$, $l = 1, 2$, such that $M \models \varphi_1[\beta^1_l]$. Suppose $\beta^1_l, \dots, \beta^m_l$, $l = 1, 2$, were defined so that $\beta^j_l \in U^j_l$ and $M \models \varphi_m[\beta^1_l, \dots, \beta^m_l]$, $l = 1, 2$. Hence β^{m+1}_l can be chosen to satisfy the same induction hypotheses. The fact that $M \models \varphi_n[\beta^1_l, \dots, \beta^n_l]$ means that $\langle \beta^1_l, \dots, \beta^n_l \rangle \in \mathrm{cl}(\Gamma_0)$. Since U^m_l is a neighborhood of β^m_l, there are $\boldsymbol{\alpha}_l \in \Gamma_0 \cap U^1_l \times \cdots \times U^n_l$. By (*) and the choice of the U^j_l's, $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ is homogeneous of color 1, a contradiction. We have thus proved that $P_{X,f}$ is c.c.c.

The union of all elements of a generic subset of $P_{X,f}$ is a homogeneous subset of X of color 1. It remains to show that this union is indeed uncountable.

It suffices to show that for every $\sigma \in P_{X,f}$, $\{\alpha \mid \sigma \cup \{\alpha\} \in P_{X,f}\}$ is unbounded. Suppose by contradiction $\sigma = \{\alpha_1, \ldots, \alpha_n\}$ is a counterexample to this claim. Let $Qx\,\varphi(x)$ mean: "there are unboundedly many x's satisfying φ". Using the fact that there is $\delta \in C_M$ such that $\alpha_{n-1} < \delta < \alpha_n$, it is easy to see that $M \models Qx\,\varphi[\alpha_1, \ldots, \alpha_{n-1}, x]$ where $\varphi(x_1, \ldots, x_n) \equiv (\{x_1, \ldots, x_n\}$ is homogenesous of color 1$) \wedge (\{y \mid \{x_1, \ldots, x_n, y\}$ is homogeneous of color 1$\}$ is bounded). For every β satisfying $\varphi(\alpha_1, \ldots, \alpha_{n-1}, x)$ let ν_β be a bound as assured by φ. Let $\{\beta_i \mid i < \aleph_1\}$ be a separated set such that for every $i < j < \aleph_1$, $M \models \varphi[\alpha_1, \ldots, \alpha_{n-1}, \beta_i]$ and $\beta_j > \nu_{\beta_i}$. The set $\{\{\alpha_1, \ldots, \alpha_{n-1}, \beta_i\} \mid 0 < i < \aleph_1\}$ is an uncountable antichain in $P_{X,f}$, a contradiction. □

The use of topological terminology and especially the use of the Hausdorff condition in Theorem 1.1 was redundant; we did not lose however any generality. We now give an equivalent formulation of the theorem that does not involve topology. Let $|A| = \aleph_1$ and f be a coloring of A in two colors. A *semibase* for f is a family $\{\langle C_i, D_i \rangle \mid i < \alpha\}$ such that $f^{-1}(1) = \bigcup_{i<\alpha} C_i \times D_i$.

Theorem. *It is consistent with* ZFC, *that for every coloring f of $\aleph_1$ in two colors which has a countable semibase, $\aleph_1$ contains an uncountable f-homogeneous subset.*

Theorem 1.2 (Consequences of SOCA). *Assume* SOCA, *then:*

(a) *If $f \subseteq \mathbb{R} \times \mathbb{R}$ is a* 1-1 *uncountable function, then there is a monotonic uncountable $g \subseteq f$.* (A model satisfying this property was built in [1].)

(b) *If $f \subseteq \mathbb{R} \times \mathbb{R}$ is a* 1-1 *uncountable function, then there is an uncountable $g \subseteq f$ such that g or g^{-1} is a Lipschiz function.*

(c) *If $A \subseteq P(\omega)$ is uncountable, then either A contains an uncountable chain, or A contains an uncountable set of pairwise incomparable elements. If B is an uncountable Boolean algebra, then B contains an uncountable set of pairwise incomparable elements.* (A model satisfying this axiom was built by Baumgartner in [3].)

(d) *Let $R \subseteq D(X)$ be open, then there is an uncountable $A \subseteq X$ such that either $D(A) \subseteq R$, or $D(A) \cap R = \emptyset$ or $R \restriction A$ is a linear ordering on A.*

Proof. (a) Since $f \subseteq \mathbb{R} \times \mathbb{R}$, f is equipped with a second countable topology. Let c be the following coloring of $f: c(\boldsymbol{a}_1, \boldsymbol{a}_2) = 0$ if $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is an order preserving function, and otherwise $c(\boldsymbol{a}_1, \boldsymbol{a}_2) = 1$. Since f is 1-1 both $c^{-1}(0)$ and $c^{-1}(1)$ are open, hence the claim of (a) follows.

(b) We regard f as a topological subspace of $\mathbb{R} \times \mathbb{R}$. For $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in f$ let $c(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) = 1$ if $|(b_2 - b_1)/(a_2 - a_1)| < 1$, otherwise $c(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) = 0$. Clearly c is a SOC, hence (b) follows.

(c) The relation of incomparability on $P(\omega)$ has a countable semibase since $\tau, \sigma \in P(\omega)$ are incomparable iff for some distinct $n, m \in \omega$, $n \in \tau \not\ni m$ and $n \notin \sigma \ni m$. Hence the first part of (c) follows.

Let B be an uncountable Boolean algebra. If B does not contain a countable dense subset, then by a theorem of Baumgartner [3], B contains an uncountable set of pairwise incomparable elements. Hence w.l.o.g. B contains a countable dense subset, so B is embeddable in $p(\omega)$. By the first part of (c), B contains a chain or an anti-chain. If the latter happens, then our claim is true; otherwise let C be an uncountable chain in B. A subset of $P(\omega)$ which is a chain must be embeddable in $\langle \mathbb{R}, < \rangle$, since the lexicographic ordering between the characteristic functions of the elements of C is identical with the containment relation on C, and on the other hand $p(\omega)$ together with its lexicographic order is isomorphic to a Cantor set.

Let $d \in C$ be such that $C^1 \stackrel{\text{def}}{=} \{c \in C \mid c \subseteq d\}$ and $C^2 \stackrel{\text{def}}{=} \{c \in C \mid d \subseteq c\}$ are uncountable. Let $f: C^1 \to C^2$ be a 1-1 function. By (a) there is an uncountable monotonic $g \subseteq f$. If g is order reversing let $D = \{c \cup (g(c) - d) \mid c \in \mathrm{Dom}(g)\}$, then D is an uncountable set of pairwise incomparable elements. If g is order preserving let $D = \{(d-c) \cup (g(c) - d) \mid c \in \mathrm{Dom}(g)\}$; again, D is as required.

(d) Let $R'(x, y) \equiv R(y, x)$, hence R' is open in $X \times X$. Let $f(x, y) = 1$ if $R(x, y)$ or $R'(x, y)$ holds; and otherwise $f(x, y) = 0$. Hence f is a SOC. Let A be an uncountable f-homogeneous subset of X. If A has color 0, then $D(A) \cap R = \emptyset$, and thus A is as required. Otherwise, for every distinct $a, b \in A$, $R(a, b)$ or $R'(a, b)$ holds. Let $g: D(A) \to \{0, 1\}$ be defined as follows: $g(a, b) = 1$ if $R(a, b)$ and $R'(a, b)$ hold; and otherwise $g(a, b) = 0$. g is a SOC, hence let B be an uncountable g-homogeneous subset of A. If the color of B is 1, then $D(B) \subseteq R$, hence B is as required; otherwise $R \restriction B$ is an antisymmetric connected relation on B. Let $<$ be a linear ordering of B such that $\langle B, < \rangle$ is embeddable in $\langle \mathbb{R}, < \rangle$. Let $c: D(B) \to \{0, 1\}$ be defined as follows: $c(a, b) = 1$ iff $a < b \Leftrightarrow R(a, b)$. Obviously c has a countable open semibase, and R is a linear ordering on any c-homogeneous set. $\square$

Strengthenings of SOCA

Proposition 1.3. SOCA+MA *is consistent with* ZFC.

Proof. In the proof of the consistency of SOCA we iterated c.c.c. forcing sets. We had the freedom to include in the iteration any c.c.c iterands, and SOCA would have still held. So we interlace in the iteration all $P_{X,f}$'s and all c.c.c. forcing sets of power $\aleph_1$. If P is the forcing set gotten as the limit of such an iteration, then $V^P \models \mathrm{SOCA} + \mathrm{MA}$. $\square$

Proposition 1.4. *Suppose* $V \models \mathrm{SOCA} + \mathrm{MA}$. *Let f be a SOC of a space X such that X does not contain uncountable 0-colored sets, then X is a countable union of 1-colored sets.*

Proof. Let P be the following forcing set.

$$P = \{f \mid \mathrm{Dom}(f) \in P_{\aleph_0}(X),\ \mathrm{Rng}(f) \subseteq \omega, \text{ and for every } i \in \omega,$$
$$f^{-1}(i) \text{ is a homogeneous set of color } 1\}.$$

It suffices to show that P is c.c.c. Let $\{f_\alpha \mid \alpha<\aleph_1\}\subseteq P$. W.l.o.g. for every $\alpha\neq\beta$, $\mathrm{Dom}(f_\alpha)\cap\mathrm{Dom}(f_\beta)=\emptyset$, and $\langle a_\alpha(0,0),\ldots,a_\alpha(0,m_0),\ldots,a_\alpha(n,0),\ldots,a_\alpha(n,m_n)\rangle$ is a 1-1 enumeration of $\mathrm{Dom}(f_\alpha)$ such that for every $i=0,\ldots,n$ and $j=0,\ldots,m_i$, $f_\alpha(a_\alpha(i,j))=1$. We can further assume that for every $i=0,\ldots,n$, $0\leqslant j<k\leqslant m_i$ and $\alpha,\beta<\aleph_1$, $f(a_\alpha(i,j),a_\beta(i,k))=1$. Recalling that X does not contain uncountable 0-colored sets, we apply successively SOCA to the subsets $\{a_\alpha(i,j)\mid\alpha<\aleph_1\}$ of X. Hence we obtain an uncountable subset $A\subseteq\aleph_1$ such that for every distinct $\alpha,\beta\in A$ and for every i and j, $f(a_\alpha(i,j),a_\beta(i,j))=1$. Hence every finite subset of $\{f_\alpha\mid\alpha\in A\}$ is compatible. □

Remark. Note that we needed a rather weak form of MA since P has the property that every uncountable subset of P contains an uncountable set of finitely compatible elements.

We do not know whether the analogue of Proposition 1.4 for the color 0 is true.

Question. Is conjunction of the following axioms consistent? MA+SOCA+ "There is a pair $\langle X,f\rangle$ such that f is a SOC of X, X does not contain uncountable 1-colored sets but X is not a countable union of 0-colored sets".

We can still say something about the analogue of 1.4. Let SOCA1 be the axiom which says that for every pair $\langle X,f\rangle$ such that f is a SOC of X: X contains an uncountable homogeneous set, and if for some $l\in\{0,1\}$, X does not contain uncountable l-colored sets, then X is a countable union of $(1-l)$-colored sets.

Theorem 1.5. MA+SOCA1 *is consistent.*

Proof. The proof is as the proof of Theorem 1.1 except that the first claim in Theorem 1.1 has to be strengthened as follows.

Claim (CH). *Let f be a* SOC *of X, and X is not a countable union of* 0*-colored sets. Then there is a c.c.c forcing set $P^1_{X,f}=P$ of power $\aleph_1$ such that $\Vdash_P$ "X contains a* 1*-colored uncountable set".*

Proof. Assume X is not a countable union of 0-colored sets. Let $\{F_i\mid i<\aleph_1\}$ be an enumeration of all 0-colored closed subsets of X. Choose by induction a sequence $\{x_i\mid i<\aleph_1\}\subseteq X$ such that for every i, $x_i\notin\bigcup_{j\leqslant i}F_j\cup\{x_j\mid j<i\}$; this choice is possible since X is not a countable union of 0-colored sets. Let $Y=\{x_i\mid i<\aleph_1\}$: clearly Y is a second countable Hausdorff space of power $\aleph_1$ and $f\restriction Y$ is a SOC of Y. We show that Y does not contain a 0-colored uncountable subset. Suppoe it did, and let A be such an example. $\mathrm{cl}(A)$ is also homogeneous of color 0, hence for some $i<\aleph_1$, $\mathrm{cl}(A)=F_i$. Since A is uncountable, for some $j>i$, $A\ni x_j$. This contradicts the definition of $\{x_j\mid j<\aleph_1\}$.

Let $P^1_{X,f}=P_{Y,f\restriction Y}$, clearly $P^1_{X,f}$ is as desired. □

To prove Theorem 1.5, we start with a universe V satisfying CH $+(2^{\aleph_1}=\aleph_2)$. We make a list of tasks which includes all possible names of pairs $\langle X, f\rangle$ and all possible names of c.c.c forcing sets of power $\aleph_1$. Let this list be $\{R_\alpha \mid \alpha<\aleph_2\}$. We define $\{P_\alpha \mid \alpha\leqslant\aleph_2\}$ as follows: P_0 is a trivial forcing set, and for limit δ $P_\delta=\bigcup_{\alpha<\delta} P_\alpha$. Suppose P_α has been defined. If R_α is a P_α-name of a c.c.c. forcing set we define $P_{\alpha+1}=P_\alpha * R_\alpha$. If R_α is a name of a pair $\langle X, f\rangle$ such that X is not a countable union of 0-colored sets, then $P_{\alpha+1}=P_\alpha * P^1_{X,f}$. In all other cases $P_{\alpha+1}=P_\alpha$. This concludes the proof of 1.5. □

Some easy counter-examples

One can try to strengthen SOCA in various ways.

(1) Increase the number of colors, namely consider f's from X to ω in which for every $i\in\omega$, $f^{-1}(i)$ is open.

(2) Consider colorings of unordered n-tuples rather than coloring of pairs.

(3) Consider colorings f in which for every i, $f^{-1}(i)$ is a Borel set.

(4) Try to decompose X into countably many homogeneous sets.

Appropriate versions of (2) and (4) are consistent, this will be proved in Section 3. (1) and (3) are inconsistent. We give counter-examples to (1)–(4).

Example 1.6. There is an open coloring f of the unordered pairs of ${}^{\omega}2$ in $\aleph_0$ colors, such that ${}^{\omega}2$ does not contain an uncountable homogeneous subset.

For distinct $\eta, \nu\in{}^{\omega}2$ let $f(\eta, \nu)$ be the maximal common segment of η and ν.

Example 1.7 (Blass [4]). There is an open coloring f of the unordered triples from ${}^{\omega}2$ in 2 colors such that ${}^{\omega}2$ does not contain an uncountable homogeneous subset.

Let $\eta, \nu, \xi\in{}^{\omega}2$ be distinct and $\eta<\nu<\xi$ lexicographically ordered. $f(\eta, \nu, \xi)=0$ if the maximal common initial segment of ξ and η is a proper initial segment of the maximal common initial segment of ν and η. Otherwise $f(\eta, \nu, \xi)=1$.

Example 1.8. There is $X\subseteq\mathbb{R}\times\mathbb{R}$ of power $\aleph_1$ and a SOC f of X such that X is not the countable union of homogeneous subsets.

Let $A\subseteq\mathbb{R}$ be a power $\aleph_1$, $X=A\times A$ and $f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle)=1$ iff $\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\}$ is a strictly order preserving function, and otherwise the value of f is 0.

Clearly f is a SOC of X. For $B\subseteq X$ let $D_B=\{a\in A \mid$ there are distinct $b_1, b_2\in A$ such that $\langle a, b_1\rangle, \langle a, b_2\rangle\in B\}$. If B is a 1-colored homogeneous set, then $D_B=\emptyset$. If B is 0-colored, then it is easily seen that $|D_B|\leqslant\aleph_0$. Let $\{B_i \mid i\in\omega\}$ be a family of homogeneous subsets of X, and let $a\in A-\bigcup_{i\in\omega} D_{B_i}$, hence $\{b \mid \langle a, b\rangle\in\bigcup_{i\in\omega} B_i\}$ is at most countable. Thus $\bigcup_{i\in\omega} B_i\neq X$.

Questions about Borel partitions of subsets of $\mathbb{R}$ of power $\aleph_1$, are equivalent to questions about general partitions of $\aleph_1$. Galvin and Shelah deal with such questions in [6]. This fact is expressed in the following observation, which is due independently to K. Kunen, B.V. Rao, and J. Silver.

Observation 1.9. *Let R be an n-place relation on $\aleph_1$. Then there is a G_δ relation S on the Cantor set C and a subset A of C such that $\langle A, S\restriction A\rangle \cong \langle\aleph_1, R\rangle$.*

Proof. For the sake of simplicity we take an R which is binary symmetric and irreflexive. We represent C as ${}^{\omega}5$. Let $\{a^\alpha \mid \alpha<\aleph_1\}$ be a family of almost disjoint infinite subsets of ω. For every $\alpha<\aleph_1$, let $\{a^\alpha_\beta \mid \beta\leqslant\alpha\}$ be a family of pairwise disjoint subsets of ω such that for every $\beta\leqslant\alpha$ the symmetric difference of a^α_β and a^β is finite. Let $\{b^\alpha \mid \alpha<\aleph_1\}$ be a family of infinite subsets of ω such that for every $\alpha<\beta<\aleph_1$, $b^\alpha-b^\beta$ is finite and $b^\beta-b^\alpha$ is infinite. For every $\alpha<\aleph_1$ we define $\eta_\alpha\in{}^{\omega}5$. $\eta_\alpha(2i+1)=1$ if $i\in b^\alpha$ and otherwise $\eta_\alpha(2i+1)=0$. $\eta_\alpha(2i)=2$ if $i\in a^\alpha_\alpha$, $\eta_\alpha(2i)=3$ if for some $\beta<\alpha$, $i\in a^\alpha_\beta$ and $\langle\beta,\alpha\rangle\in R$. Otherwise $\eta_\alpha(2i)=4$. Let $S_1=\{\langle\eta,\nu\rangle \mid \eta,\nu\in{}^{\omega}5,\ \{i \mid \nu(i)=1$ and $\eta(i)=0\}$ is infinite and $\{i \mid \nu(i)=3$ and $\eta(i)=2\}$ is infinite$\}$, and let $S=S_1\cup S_1^{-1}$. Let $A=\{\eta_\alpha \mid \alpha<\aleph_1\}$; clearly S is a G_δ set and $\langle A, S\restriction A\rangle\cong\langle\aleph_1, R\rangle$. □

Question 1.10. Using oracle forcing it is easy to construct a model of set theory in which $\mathbb{R}$ contains a second category set of power $\aleph_1$, and in which for every second countable space Y of the second category and every SOC of Y, Y contains an uncountable f-homogeneous subset. We do not know whether

$$\mathrm{SOCA} + (\exists X\subseteq\mathbb{R})\,(|X|=\aleph_1 \text{ and } X \text{ is of the second category})$$

is consistent.

Question 1.11. If in SOCA one replaces everywhere $\aleph_1$ by $\aleph_2$ is the resulting axiom still consistent?

Question 1.12. If in observation 1.9 one replaces $\aleph_1$ by $\aleph_2$ is the resulting statement consistent with ZFC?

2. The explicit contradiction method, and the increasing set axiom

Suppose that we want to construct a model of $\mathrm{MA}+\aleph_1<2^{\aleph_0}$ or of $\mathrm{MA}_{\aleph_1}$, and at the same time we want to preserve a certain property Φ of a certain set A. There is a problem when we encounter a c.c.c. forcing set P which ruins property Φ, that is, in V^P, A does not satisfy Φ anymore. In such a case we shall find a c.c.c. forcing set Q such that in V^Q, P is not c.c.c., and A still has property Φ. We call the particular method in which we do this 'the explicit contradiction method'.

We take the liberty to explain this method by an application which yields a known result. We do so in order not to start with applications that involve more than one tehcnique.

2 **Definition.** Let $A \subseteq \mathbb{R}$ be of power $\aleph_1$; A is called an *increasing set*, if in every uncountable set of pairwise disjoint finite sequences from A there are two sequences $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_n \rangle$ having the same length such that $a_1 < b_1, \ldots, a_n < b_n$.

Axiom ISA. These exists an increasing set.

The following theorem is due to Avraham and Shelah [1]. It follows from Theorem 2 there, together with the discussion preceding it.

Theorem 2.1. $\mathrm{MA}_{\aleph_1}$ + ISA *is consistent.*

Remark. The proof in [1] is slightly different from ours and does not use the explicit contradiction method. Instead, there, a model V^P is constructed such that $V^P \models$ "Every uncountable 1-1 function from A to A contains an uncountable OP subfunction". This implies that there is no c.c.c. $Q \in V^P$ such that in V^P, $\Vdash_Q A$ is not increasing.

This slight difference between the proof becomes essential, if one wants at the same time to carry out some task that requires CH in the intermediate stages.

E.g., **Theorem.** MA *is consistent with the existence of a rigid increasing set.* ('Rigid' means there are no order automorphisms except for the identity.)

Proof. Let V be any universe. Let us add to V a set A of $\aleph_1$ Cohen reals. It is easy to see that in this Cohen extension of V the set A is increasing. (This fact and more appears in [1, §5 Remark 21].) Hence we can w.l.o.g. assume that this is our universe V and there is an increasing set A.

As usual we will define a finite support iteration $\{P_i \mid i < 2^{\aleph_1}\}$ in which all possible c.c.c. forcing sets of power $\aleph_1$ are considered. For each single step in the iteration we need the following lemma in which the explicit contradiction method is used.

Lemma 2.2. *Let A be an increasing set in V, P be a c.c.c. forcing set $p \in P$, and $p \Vdash_P$ "A is not increasing". Then there is a c.c.c. forcing set $Q = Q_{P,p,A}$ of power $\aleph_1$ such that $\Vdash_Q$ "A is increasing and P is not c.c.c.".*

Proof. Let $\tilde{B}$ be a P-name of a set of pairwise disjoint 1-1 sequences of length n such that $p \Vdash_P$ "$\tilde{B}$ is a counterexample to the increasingness of A". Since P is c.c.c. it is easy to pick a sequence $\{\langle p_i, \boldsymbol{b}^i \rangle \mid i < \aleph_1\}$ such that: (1) for $i : p_i \geq p$ and $p_i \Vdash_p \boldsymbol{b}^i \in \tilde{B}$ and (2) let $\boldsymbol{b}^i = \langle b_1^i, \ldots, b_n^i \rangle$, then for every $i \neq j$, $\{b_1^i, \ldots, b_n^i\} \cap \{b_1^j, \ldots, b_n^j\} = \emptyset$.

Let $\boldsymbol{a} = \langle a_1, \ldots, a_n \rangle$, $\boldsymbol{b} = \langle b_1, \ldots, b_n \rangle \in \mathbb{R}^n$ be distinct, we say that $\{\boldsymbol{a}, \boldsymbol{b}\}$ is order preserving (OP), if for every $1 \leq k \leq l \leq n$ $a_k < b_k \Leftrightarrow a_l < b_l$. We say that p_i, p_j are explicitly contradictory if $\{b^i, b^j\}$ is OP. The main point is that if p_i and p_j are explicitly contradictory, we indeed know that they are incompatible in P; for if $r \geq p_i, p_j$, then $r \geq p$, hence $r \Vdash_P$ "$\tilde{B}$ is a counterexample to the increasingness of A, and $b^i, b^j \in B$". This is of course a contradiction. Recall that we are looking for a Q that will add an uncountable anti-chain to P. Hence our choice for Q is obvious. Let $\sigma \in P_{\aleph_0}(\aleph_1)$ and $q_\sigma \overset{\text{def}}{=} \{p_i \mid i \in \sigma\}$. Let $Q' = \{q_\sigma \mid \sigma \in p_{\aleph_0}(\aleph_1)$ and for every $i \neq j \in \sigma$, p_i and p_j are explicity contradictory$\}$. $q_\tau \leq q_\sigma$ if $\tau \subseteq \sigma$.

Obviously a Q'-generic set adds an antichain D to P. Once we show that Q' is c.c.c., there is a standard way to find some $q_0 \in Q'$ such that $q_0 \Vdash_{Q'}$ "D is uncountable". Hence we shall take Q to be $\{q \in Q' \mid q_0 \leq q\}$.

We thus show that Q' is c.c.c. Let $\{q_{\sigma_i} \mid i < \aleph_1\}$ be an uncountable subset of Q'. W.l.o.g. $\{\sigma_i \mid i < \aleph_1\}$ is a Δ-system and for every i, $\sigma_i = \{\alpha^1, \ldots, \alpha^k, \alpha^{i,1}, \ldots, \alpha^{i,l}\}$ where $\alpha^1 < \cdots < \alpha^k < \alpha^{i,1} < \cdots < \alpha^{i,l}$. Let $c = b^{\alpha^1} \frown \cdots \frown b^{\alpha^k} \overset{\text{def}}{=} \langle c_1, \ldots, c_m \rangle$, and $c^i = b^{\alpha^{i,1}} \frown \cdots \frown b^{\alpha^{i,l}} \overset{\text{def}}{=} \langle c^i_1, \ldots, c^i_r \rangle$. For every β let $U^\beta_1, \ldots, U^\beta_r$ be rational neighborhoods of $c^\beta_1, \ldots, c^\beta_r$ respectively such that for every $1 \leq i, j \leq r$ for every $d_i \in U^\beta_i$ and $d_j \in U^\beta_j$: $c^\beta_i < c^\beta_j \Leftrightarrow d_i < d_j$. By choosing a subsequence, we can assume that for every i, U^β_i is independent of β.

Let β, γ be such that $\{c^\beta, c^\gamma\}$ is OP. Hence for every $1 \leq i \leq k$, $\{b^{\alpha^{\beta,i}}, b^{\alpha^{\gamma,i}}\}$ is OP. If $i \neq j$, then since $\{b^{\alpha^{\beta,i}}, b^{\alpha^{\beta,j}}\}$ is OP, and since we uniformized the U^β_i's, also $\{b^{\alpha^{\beta,i}}, b^{\alpha^{\gamma,j}}\}$ is OP. Hence $\{q_{\sigma_\beta} \cup q_{\sigma_\gamma}\} \in Q'$. So Q' is c.c.c.

Our next goal is to show that $\Vdash_{Q'}$ "A is increasing". The proof is very similar to the proof that Q' is c.c.c. Suppose by contradiction $\tilde{B}$ is a Q'-name, $q_0 \in Q'$ and $q_0 \Vdash_{Q'}$ "$\tilde{B}$ is a name of a counterexample to the increasingness of A". Let $\{\langle q_{\sigma_\alpha}, a^\alpha \rangle \mid \alpha < \aleph_1\}$ be a sequence such that for every α, $q_0 \leq q_{\sigma_\alpha}$, $q_{\sigma_\alpha} \Vdash a^\alpha \in \tilde{B}$, and for every $\alpha \neq \beta$, a^α and a^β are disjoint. As in the previous argument we assume that the σ_α's form a Δ-system, and we choose $U^\alpha_i = U_i$ with the same properties. Define the c^α's as in the previous argument, and find β, γ such that $\{c^\beta \frown d^\beta, c^\gamma \frown d^\gamma\}$ is OP; then $q_{\sigma_\beta} \cup q_{\sigma_\gamma} \in Q'$, $\{d^\beta, d^\gamma\}$ is OP, a contradiction. □

Continuation of the proof of Theorem 2.1. It follows from Lemma 2.2 that if P is a c.c.c. forcing set such that $\Vdash_P$ "A is increasing", and if $\tilde{Q}$ is a P-name of a c.c.c. forcing set, then there is a P-name $\tilde{R} = \tilde{R}_{\tilde{Q}}$ such that $\Vdash_P$ ($\tilde{R}$ is a c.c.c. forcing set and $\Vdash_{\tilde{R}}$ "A is increasing"), and for every $p \in P$: if $p \Vdash_P$ ($\Vdash_{\tilde{Q}} A$ is increasing), then $p \Vdash_P \tilde{R} = \tilde{Q}$; and if $p \Vdash_P (\exists q \in \tilde{Q})$ ($q \Vdash_Q A$ is not increasing), then $p \Vdash_P (\Vdash_{\tilde{R}} \tilde{Q}$ is not c.c.c.).

Let $\{N_i \mid i < 2^{\aleph_1}\}$ be an enumeration of $P_{\aleph_2}(2^{\aleph_1})$. We define by induction an increasing sequence of forcing sets. P_0 is the trivial forcing set, and if δ is a limit ordinal, then $P_\delta = \bigcup_{i<\delta} P_i$. Suppose P_i has been defined; if $\nVdash_{P_i}$ "A is increasing" or if $\nVdash_{P_i}$ "N_i is a c.c.c. forcing set", then let $P_{i+1} = P_i$; otherwise let $P_{i+1} = P_i * \tilde{R}_{N_i}$.

We first show that for every i, $\Vdash_{P_i}$ "A is increasing". By our definition if this happens for P_i, then it happens for P_{i+1}.

Let $\mathrm{cf}(\delta) > \aleph_0$, and suppose for every $i < \delta$, $\Vdash_{P_i}$ "A is increasing". Suppose by contradiction that G is a P_δ-generic set and $B \in V[G]$ is a counterexample to the increasingness of A. Let $G_i = G \cap P_i$. There is $i<\delta$ such that the closure of B in A^n, $\bar{B}$, belongs to $V[G_i]$. It is easy to see that there is $B' \in V[G_i]$ such that B' is an uncountable subset of $\bar{B}$ consisting of pairwise disjoint sequences. Hence there cannot be two sequences in B' which form an OP pair. Hence A is not increasing in $V[G_i]$, a contradiction.

Let $\mathrm{cf}(\delta) = \aleph_0$, suppose our claim is true for every $i<\delta$. Let G be a P_δ-generic set and let $B \subseteq A^n$ and $B \in V[G]$. Then there are $\{B_i \mid i \in \omega\}$ such that $\bigcup_{i\in\omega} B_i = B$ and for every i there is $\gamma_i<\delta$ such that $B_i \in V[G_{\gamma_i}]$. Hence one of the B_i's is uncountable, hence if B is a counterexample to the increasingness of A, there is such an example belonging to a previous $V[G_i]$, and by the induction hypothesis this is impossible.

Let $P = P_{2^{\aleph_1}}$, the argument showing that $\mathrm{MA}_{\aleph_1}$ holds in V^P is standard. □

Remark. Note that if $\mathrm{MA}_{\aleph_1}$ holds, then A is increasing iff for every 1-1 uncountable $f \subseteq A \times A$ there is an uncountable OP function $g \subseteq f$.

3. The open coloring axiom, and how to preassign colors

In [1] it was shown (Theorem 6) that it is consistent with ZFC that every 1-1 $f \subseteq \mathbb{R}\times\mathbb{R}$ of power $\aleph_1$ is the union of countably many monotonic functions. This fact is a special case of the open coloring axiom (OCA) to be defined below. (S. Todorčević proved that, under MA, OCA is a consequence of this fact.)

Let X be a second countable Hausdorff space of power $\aleph_1$. An *open coloring* of X is finite cover $\mathcal{U} = \{U_0, \ldots, U_{n-1}\}$ of $D(X)$ such that for every l, $U_l = \{\langle y, x\rangle \mid \langle x, y\rangle \in U_l\}$. $A \subseteq X$ is $\mathcal{U}$-homogeneous if for some color l, $D(A) \subseteq U_l$. A $\mathcal{U}$-homogeneous partition of X is a countable partition $\{X_i \mid i \in \omega\}$ of X consisting of $\mathcal{U}$-homogeneous sets.

The open coloring axiom is as follows.

Axiom OCA. For every X and every open coloring $\mathcal{U}$ of X, X has a $\mathcal{U}$-homogeneous partition.

It turns out that in a universe V satisfying MA+OCA+ISA, the set of real order types of power $\aleph_1$ has nice properties, e.g. there are exactly three homogeneous such order types; so our first goal is to prove the consistency of the conjunction of these three axioms. As seen in the following theorem we do a little more, and add to the above axioms also SOCA.

Theorem 3.1. MA+OCA+SOCA+ISA *is consistent.*

Later in the section we shall prove a generalization of OCA.

Proof of Theorem 3.1. We start with a universe V satisfying $\mathrm{CH}+(2^{\aleph_1}=\aleph_2)$ and with an increasing set $A\in V$. We construct a finite support iteration $\{P_i \mid i\leqslant\aleph_2\}$, according to a list of tasks of length $\aleph_2$ which is prepared in advance. In each atomic step of the iteration we deal with one of the following tasks.

(1) For a given c.c.c. forcing set Q of power $\aleph_1$, we have to find a c.c.c. forcing set $P=P_Q$ of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and either Q is not c.c.c. or there is a Q-generic filter over V.

(2) For a given X and a SOC f of X we have to find a c.c.c. forcing set $P=P_{X,f}$ of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and X contains an uncountable f-homogeneous subset.

(3) For a given X and an open coloring $\mathcal{U}$ of X we have to find a c.c.c. forcing set $P=P_{X,\mathcal{U}}$ of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and X has a $\mathcal{U}$-homogeneous partition.

We expect the reader to known how to define the list of tasks, how to define the iteration and why $V[P_{\aleph_2}]$ satisfies all the four axioms. We shall concentrate only on the atomic steps of the iteration.

The existence of $P=P_Q$ satisfying the requirements of (1) was proved in the previous section (Lemma 2.2).

We start with task (3) where the additional trick of preassigning colors is used. This method appears also in [1, Theorem 6]. There, a special case of OCA is proved. In the present application there is an additional complication, since at the same time we want to preserve the increasingness of A.

Lemma 3.2. *Suppose $V\models$"CH, $A\in V$ is increasing" and $\mathcal{U}=\{U_0,\ldots,U_{n-1}\}$ is an open coloring of X. Then there is a c.c.c. forcing set P of power $\aleph_1$ such that $\Vdash_P$ "A is increasing", and X has a $\mathcal{U}$-homogeneous partition.*

Proof. W.l.o.g. $A, X\subseteq\aleph_1$. As in Theorem 1.1 we form a model M with universe $\aleph_1$ that includes enough set theory, and includes also A, X and $\mathcal{U}$ as predicates. Let M_α be the submodel of M whose universe is α, and let $C=C_M=\{\alpha \mid M\cap\alpha\prec M\}$.

We know that each element of X should be put into one of a countable set of homogeneous subsets of X, and our first aim is to decide in advance what will be the color of the homogeneous set to which each element a of X should belong. Let $\{\alpha_i \mid i<\aleph_1\}$ be an isomorphism between $\langle\aleph_1,<\rangle$ and $\langle C,<\rangle$, let $E_i=\{\beta \mid \alpha_i\leqslant\beta<\alpha_{i+1}\}$, and let $\mathscr{E}=\{E_i \mid i<\aleph_1\}$; we call $\mathscr{E}$ the set of C-slices. For every $i<\aleph_1$ let $\{a_l^i \mid l\in\omega\}$ be an enumeration of $X\cap E_i$ such that $a_0^i=\min(X\cap E_i)$. Let $\varphi(x)=\varphi(x_1,\ldots,x_l)$ be a formula in the language of M and possibly with parameters from $|M|$, $Qx\,\varphi(x)$ abbreviates the following formula $\forall\alpha\,(\exists x_1>\alpha)\cdots(\exists x_l>\alpha)\,\varphi(x)$. Let $\delta=\langle\delta_0,\ldots,\delta_{l-1}\rangle\in{}^l n$ be a sequence, and $\varphi(x,y)=$

$\varphi(x_0, \ldots, x_{l-1}, y_1, \ldots, y_m)$ be a formula with parameter from $|M|$; we denote

$$\psi_{\varphi,\delta} \equiv \varphi(\mathbf{x}, \mathbf{y}) \wedge \varphi(\mathbf{x}', \mathbf{y}') \wedge \left(\bigwedge_{t=0}^{l-1} x_t, x'_t \in X\right) \wedge \left(\bigwedge_{t=1}^{m} y_t, y'_t \in A\right)$$

$$\wedge \left(\bigwedge_{t=0}^{l-1} \langle x_t, x'_t\rangle \in U_{\delta_t}\right) \wedge (\{\mathbf{y}, \mathbf{y}'\} \text{ is OP})$$

where $\mathbf{x}'$, $\mathbf{y}'$ are disjoint sequences of distinct variables disjoint from $\mathbf{x}$ and $\mathbf{y}$.

Claim 1. *Let $i < \aleph_1$. Then for every $l \in \omega$ there is $\boldsymbol{\delta} \in {}^l n$ such that for every $m \in \omega$ and every $\varphi(\mathbf{x}, \mathbf{y}) \equiv \varphi(x_0, \ldots, x_{l-1}, y_1, \ldots, y_m)$ with parameters from $|M_{\alpha_i}|$: if there are $b_1, \ldots, b_m \in A \cap (|M| - |M_{\alpha_i}|)$ such that $M \models \varphi[a^i_0, \ldots, a^i_{l-1}, b_1, \ldots, b_m]$, then $M \models Q\mathbf{x}, \mathbf{y}\, Q\mathbf{x}', \mathbf{y}'\, \psi_{\varphi,\delta}$.*

Proof. Suppose by contradiction the claim is not true, so for every $\boldsymbol{\delta} \in {}^l n$ let $\varphi_{\boldsymbol{\delta}}(\mathbf{x}, \mathbf{y}^{\boldsymbol{\delta}})$ be a formula showing that $\boldsymbol{\delta}$ is not as required in the claim. We assume that the $\mathbf{y}^{\boldsymbol{\delta}}$'s are pairwise disjoint sequences of variables, and that their concatenation is $\mathbf{y} = \langle y_1, \ldots, y_m\rangle$. Let

$$\varphi(\mathbf{x}, \mathbf{y}) \equiv \bigwedge_{\boldsymbol{\delta} \in {}^l n} \varphi_{\boldsymbol{\delta}}, \quad \text{and} \quad \chi(\mathbf{x}, \mathbf{y}) \equiv \varphi(\mathbf{x}, \mathbf{y}) \wedge \left(\bigwedge_{t=1}^{m} y_t \in A\right).$$

By the choice of the $\varphi_{\boldsymbol{\delta}}$'s there are $b_1, \ldots, b_m \in A \cap (|M| - |M_{\alpha_i}|)$ such that $M \models \varphi[a^i_0, \ldots, a^i_{l-1}, b_1, \ldots, b_m]$, hence

(1) $M \models Q\mathbf{x}, \mathbf{y}\, \chi(\mathbf{x}, \mathbf{y})$.

On the other hand it is clear that for every $\boldsymbol{\delta}$

(2) $M \models \neg Q\mathbf{x}, \mathbf{y}\, Q\mathbf{x}', \mathbf{y}'\, \psi_{\varphi,\boldsymbol{\delta}}$.

Hence there is $\beta^0 < \aleph_1$ such that for every $\boldsymbol{a} \in |M|^l$ and $\boldsymbol{b} \in |M|^m$ if $\beta^0 < \boldsymbol{a}, \boldsymbol{b}$ then there is $\beta = \beta(\boldsymbol{a}, \boldsymbol{b})$ such that for every $\boldsymbol{\delta} \in {}^l n$ and $\boldsymbol{a}', \boldsymbol{b}' > \beta$, $M \models \neg\psi_{\varphi,\boldsymbol{\delta}}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}', \boldsymbol{b}']$.

We define by induction on $j < \aleph_1$, $\boldsymbol{a}^j \in X^l$ and $\boldsymbol{b}^j \in A^m$, our induction hypothesis is that for every $j < \aleph_1$, $\boldsymbol{a}^j, \boldsymbol{b}^j > \beta_0$. Suppose $\boldsymbol{a}^k$, $\boldsymbol{b}^k$ have been defined for every $k < j$. Let $\beta_j > \beta^0 \cup \bigcup_{k<j} \beta(\boldsymbol{a}^k, \boldsymbol{b}^k)$. Let $\boldsymbol{a}^j \in X^l$, $\boldsymbol{b}^j \in A^m$ be such that $\boldsymbol{a}^j, \boldsymbol{b}^j > \beta_j$, and $M \models \varphi[\boldsymbol{a}^j, \boldsymbol{b}^j]$. This choice is possible by (1).

By the increasingness of A there are $k < j$ such that $\{\boldsymbol{b}^k, \boldsymbol{b}^j\}$ is OP. Let $\boldsymbol{a}^j = \langle a^{j,0}, \ldots, a^{j,l-1}\rangle$ and $\boldsymbol{a}^k = \langle a^{k,0}, \ldots, a^{k,l-1}\rangle$, and let δ_t be such that $\langle a^{j,t}, a^{k,t}\rangle \in U_{\delta_t}$. Let $\boldsymbol{\delta} = \langle \delta_0, \ldots, \delta_{l-1}\rangle$. $M \models \chi[\boldsymbol{a}^k, \boldsymbol{b}^k]$, and $M \models \chi[\boldsymbol{a}^j, \boldsymbol{b}^j]$; however since $\boldsymbol{a}^j, \boldsymbol{b}^j > \beta_j$, $M \models \neg\psi_{\varphi,\boldsymbol{\delta}}[\boldsymbol{a}^k, \boldsymbol{b}^k, \boldsymbol{a}^j, \boldsymbol{b}^j]$. This is a contradiction, and the claim is proved. □

Let $i < \aleph_1$; for every $l \in \omega$ let $\boldsymbol{\delta}^i_l$ be the least element in ${}^l n$ according to the lexicographic order of ${}^l n$, which satisfies the requirements of Claim 1. Recalling that for every $l \in \omega$, $a^i_0 \leq a^i$, it is easy to see that if $k < l$, then $\boldsymbol{\delta}^i_k$ is an initial segment of $\boldsymbol{\delta}^i_l$. Let $\langle \delta^i_0, \delta^i_1, \ldots\rangle = \bigcup_{l \in \omega} \boldsymbol{\delta}^i_l$. If $a \in X$ and $a \geq \alpha_0$, then for some i and l, $a = a^i_l$; we denote $\delta(a) = \delta^i_l$ and call $\delta(a)$ the color of a. We have thus assigned a color to every a in $X-\alpha_0 \stackrel{\text{def}}{=} X'$, and when we construct P we shall put each $a \in X'$ in a homogeneous set of color $\delta(a)$.

We are ready to define the forcing set P which satisfies the requirements of the lemma.

Let $\{n_i \mid i \in \omega\}$ be an enumeration of the set of colors n such that for every $l<n$, $\{i \mid n_i = l\}$ is infinite. Let P be the set of finite approximations of a homogeneous partition $\{X_i \mid i \in \omega\}$ of X' which respects the preassigned colors, and in which X_i has the color n_i. More precisely, $P=\{f \mid \mathrm{Dom}(f) \in P_{\aleph_0}(X'),\ \mathrm{Rng}(f) \subseteq \omega$, and for every $a, b \in \mathrm{Dom}(f)$ if $f(a)=f(b)=i$, then $\langle a, b\rangle \in U_{n_i}$ and $\delta(a)=\delta(b)=n_i\}$.

Clearly $\Vdash_P$ "X has a U-homogeneous partition". We have to show that P is c.c.c., and that $\Vdash_P$ "A is increasing". The proofs of these two facts are similar, we thus skip the first, and assuming that we already know that P is c.c.c., we prove that $\Vdash_P$ "A is increasing".

Suppose by contradiction there is $p^0 \in P$ and $m \in \omega$ such that $p^0 \Vdash_P$ "There is a family $\{\boldsymbol{b}^\alpha \mid \alpha<\aleph_1\} \subseteq A^m$ of pairwise disjoint sequences such that for no $\alpha \neq \beta$, $\{\boldsymbol{b}^\alpha, \boldsymbol{b}^\beta\}$ is OP". Let $\tilde{B}$ be a name for this family. Let $\{\langle p_\alpha, \boldsymbol{b}^\alpha\rangle \mid \alpha<\aleph_1\}$ be such that (1) $p_\alpha \geqslant p^0$; (2) $p_\alpha \Vdash_P \boldsymbol{b}^\alpha \in \tilde{B}$; and (3) if $\alpha \neq \beta$, then $\boldsymbol{b}^\alpha$ and $\boldsymbol{b}^\beta$ are pairwise disjoint.

W.l.o.g. the p_α's form a Δ-system, and they all have the same structure. More precisely, we need the following uniform behavior of the $\langle p_\alpha, \boldsymbol{b}^\alpha\rangle$'s.

(1) $\{\mathrm{Dom}(p_\alpha) \mid \alpha<\aleph_1\}$ is a Δ-system.

(2) Let $\mathrm{Dom}(p_\alpha)=\{a_{\alpha,0}, \ldots, a_{\alpha,l}\}$ where $a_{\alpha,0}<\cdots<A_{\alpha,l}$ and the first r elements form the kernel of $\{\mathrm{Dom}(p_\alpha) \mid \alpha<\aleph_1\}$ and $\boldsymbol{b}^\alpha=\langle b_{\alpha,1}, \ldots, b_{\alpha,m}\rangle$. Then for every $\alpha, \beta<\aleph_1$ for every $i, j \leqslant l$ and $1 \leqslant k$, $t \leqslant m: \delta(a_{\alpha,i})=\delta(a_{\beta,i})$, $p_\alpha(a_{\alpha,i})=p_\beta(a_{\beta,i})$, $b_{\alpha,k}=a_{\alpha,i}$ iff $b_{\beta,k}=a_{\beta,i}$ and $b_{\alpha,k} \leqslant b_{\alpha,t}$ iff $b_{\beta,k} \leqslant b_{\beta,t}$.

We can assume that for every $\alpha<\aleph_1$, $b_{\alpha,1}<\cdots<b_{\alpha,m}$.

Since X is second countable, we can further uniformize the $\langle p_\alpha, \boldsymbol{b}^\alpha\rangle$'s in the following way.

(3) There are open sets $V_0, \ldots, V_l \subseteq X$ such that for every $\alpha<\aleph_1$ and distinct $i, j \in\{0, \ldots, l\}: a_{\alpha,i} \in V_i$, if $p_\alpha(a_{\alpha,i})=p_\alpha(a_{\alpha,j})$ then $V_i \times V_j \subseteq U_{\delta(a_{\alpha,i})}$.

Let $\boldsymbol{d}_\alpha=\langle a_{\alpha,0}, \ldots, a_{\alpha,l}, b_{\alpha,1}, \ldots, b_{\alpha,m}\rangle$, and let D be the topological closure of $\{\boldsymbol{d}_\alpha \mid \alpha<\aleph_1\}$ in $X^{l+1} \times A^m$. Since X is second countable D is the closure of a countable set, hence it is definable by a parameter d in M. Let $\gamma \in C_M$ be such that $d, a_{\alpha,0}, \ldots, a_{\alpha,r-1} \in |M_\gamma|$, (recall that $\{a_{\alpha,0}, \ldots, a_{\alpha,r-1}\}$ is the kernel of $\{\mathrm{Dom}(p_\beta) \mid \beta<\aleph_1\}$.) We choose α such that $a_{\alpha,r}, \ldots, a_{\alpha,l}, b_{\alpha,1}, \ldots, b_{\alpha,l} \notin |M_\gamma|$. We intend to apply the duplication argument to $\boldsymbol{d}_\alpha$.

Let $\boldsymbol{d}_0=\langle a_0, \ldots, a_l, b_1, \ldots, b_m\rangle$, $\boldsymbol{a}=\langle a_0, \ldots, a_l\rangle$ and $\boldsymbol{b}=\langle b_1, \ldots, b_m\rangle$. Let $E^1, \ldots, E^k$ be those C_M-slices E for which there is a_i, $r \leqslant i \leqslant l$, such that $a_i \in E$, or there is b_j, $1 \leqslant j \leqslant m$, such that $b_j \in E$. Let $\boldsymbol{a}=\boldsymbol{a}^0 \frown \cdots \frown \boldsymbol{a}^k$ where $\boldsymbol{a}^0=\langle a_0, \ldots, a_{r-1}\rangle$ is the sequence of those elements of $\boldsymbol{a}$ which belongs to $|M_\gamma|$, and for $i>0$, $\boldsymbol{a}^i$ is the sequence of those elements of $\boldsymbol{a}$ which belong to E^i. Let $\boldsymbol{b}=\boldsymbol{b}^1 \frown \cdots \frown \boldsymbol{b}^k$ where $\boldsymbol{b}^i$ is the sequqnce of those elements of $\boldsymbol{b}$ which belong to E^i. Let β_i be the minimal element of E^i.

We define by a downward induction formulas $\varphi_k, \ldots, \varphi_0$. The induction hypotheses are: (1) the parameters of φ_i belong to $|M_\gamma|$, and (2) $M \models \varphi_i[a^1, \ldots, a^i, b^1, \ldots, b^i]$. Let $\varphi_0 \equiv \langle a_0, \ldots, a_{r-1} \rangle \frown \cdots \frown x^k \frown y^1 \frown \cdots \frown y^k \in D$. Suppose that $\varphi_{i+1}(x^1, \ldots, x^{i+1}, y^1, \ldots, y^{i+1})$ has been defined. Let s be the length of b^{i+1}, let $a^{i+1} = \langle a'_1, \ldots, a'_t \rangle$, and let $\delta_j = \delta(a'_j)$, $j = 1, \ldots, t$. The formula $\varphi_{i+1}(a^1, \ldots, a^i, x^{i+1}, b^1, \ldots, b^i, y^{i+1})$ has parameters from $|M_{\beta_{i+1}}|$, hence by the definition of δ there are $c^l = \langle c^l_1, \ldots, c^l_t \rangle$ and d^l, $l = 1, 2$, such that: (1) $M \models \varphi_{i+1}[a^1, \ldots, a^i, c^l, b^1, \ldots, b^i, d^l]$, $l = 1, 2$; (2) for every $j = 1, \ldots, t$, $\langle c^1_j, c^2_j \rangle \in U_{\delta_j}$; and (3) $\{d^1, d^2\}$ is OP.

For $l = 1, 2$, $j = 1, \ldots, t$ let $V^{i+1,l}_j$ be basic open sets in X such that $\langle c^1_j, c^2_j \rangle \in V^{i+1,1}_j \times V^{i+1,2}_j \subseteq U_{\delta_j}$; and let $V^{i+1,l} = V^{i+1,l}_1 \times \cdots \times V^{i+1,l}_t$. For $l = 1, 2$ let $W^{i+1,l}$ be basic open sets in A^s such that $d^l \in W^{i+1,l}$, $l = 1, 2$, and for every $d_1 \in W^{i+1,1}$ and $d_2 \in W^{i+1,2}$, $\{d_1, d_2\}$ is OP. Let $u^l = \langle u^l_1, \ldots, u^l_t \rangle$, $v^l = \langle v^l_1, \ldots, v^l_s \rangle$ be sequences of variables. Let

$$\varphi_i(x^1, \ldots, x^i, y^1, \ldots, y^i) \equiv \exists u^1, u^2, v^1, v^2 \left(\bigwedge_{l=1}^{2} \varphi_{i+1}(x^1, \ldots, x^i, u^l, y^1, \ldots, y^i, v^l) \right) \wedge \left(\bigwedge_{l=1}^{2} u^l \in V^{i+1,l} \right) \wedge \left(\bigwedge_{l=1}^{2} v^l \in W^{i+1,l} \right).$$

Clearly φ_i satisfies the induction hypotheses. We have thus defined φ_0.

As it was done in Theorem 1.1 starting with φ_0 we can choose two sequences $a^{1,l} \frown \cdots \frown a^{k,l} \frown b^{1,l} \frown \cdots \frown b^{k,l} \overset{\text{def}}{=} e^l$, $l = 1, 2$, such that $a^0 \frown e^1, a^0 \frown e^2 \in D$, for every $i = 1, \ldots, k$, $a^{i,1} \frown a^{i,2} \in V^{i,1} \times V^{i,2}$ and $b^{i,1} \frown b^{i,2} \in W^{i,1} \times W^{i,2}$. $D_0 = \{d_\alpha \in \alpha < \aleph_1\}$ was dense in D, hence there are $d_{\alpha(l)} \in D_0$, $l = 1, 2$, such that

$$d_{\alpha(l)} \in X^{r-1} \times V^{1,l} \times \cdots V^{k,l} \times W^{1,l} \times \cdots W^{k,l}.$$

It is easy to see that $p_{\alpha(1)} \cup p_{\alpha(2)} \in P$ and $\{b^{\alpha(1)}, b^{\alpha(2)}\}$ is OP. This contradicts the assumption that $p^0 \Vdash_P$ "$\bar{B}$ is a counter-example to the increasingness of A". Hence Lemma 3.2 is proved. □

We turn now to the last kind of tasks that we have got to carry out.

Lemma 3.3 (CH). *Let $A \in V$ be an increasing set, and let f be a SOC of X, then there is a c.c.c. forcing set P of power $\aleph_1$ such that $\Vdash_P$ "A is increasing, and X contains an f-homogeneous uncountable subset".*

Proof. Let us first assume that (*): there is no $n \geq 0$ and an uncountable 1-1 h such that $\mathrm{Dom}(h) \subseteq X$, $\mathrm{Rng}(h) \subseteq A^n$, every two distinct elements in $\mathrm{Rng}(h)$ are disjoint and whenever $x, y \in \mathrm{Dom}(h)$ and $f(x, y) = 1$, $\{h(x), h(y)\}$ is not OP. (Note that for $n = 0$ this means that there are no uncountable 0-colored homogeneous sets.)

Let M be a model including enough set theory and including X, f, A. Let

$$P=\{\sigma\in P_{\aleph_0}(X)\in\sigma \text{ is homogeneous of color 1 and is } C_M\text{-separated}\}.$$

Clearly by the proof of 1.1, $\Vdash_P$ "X contains an uncountable homogeneous subset". Suppose by contradiction there is $p\in P$ such that $p\Vdash_P$ "A is not increasing". Let $\tilde{B}$ be a name of a subset of A^n such that p forces that $\tilde{B}$ is a counter-example to the increasingness of A. Let $\{\langle p_\alpha, \boldsymbol{b}^\alpha\rangle \mid \alpha<\aleph_1\}$ be such that for every α, $p_\alpha\geqslant p$, $p_\alpha\Vdash \boldsymbol{b}^\alpha\in\tilde{B}$, and for every $\alpha\neq\beta$, $\boldsymbol{b}^\alpha$, $\boldsymbol{b}^\beta$ are pairwise disjoint. Let $p^\alpha=\{a_0^\alpha,\ldots,a_{m_\alpha-1}^\alpha\}$ where $a_0^\alpha<\cdots<a_{m_\alpha-1}^\alpha$, let $\boldsymbol{a}^\alpha=\langle a_0^\alpha,\ldots,a_{m_\alpha-1}^\alpha\rangle$ and $\boldsymbol{b}^\alpha=\langle b_1^\alpha,\ldots,b_n^\alpha\rangle$. W.l.o.g. (1) for every α, $m_\alpha=m$; (2) $\{p_\alpha\mid\alpha<\aleph_1\}$ is a Δ-system with kernel $\{a_0,\ldots,a_{r-1}\}$ where for every $\alpha<\aleph_1$ and $i\leqslant r-1$, $a_i^\alpha=a_i$ and for every α and β, $p_\alpha\cup p_\beta$ is C_M-separated; for every $0\leqslant i<j\leqslant m$ and $\alpha<\beta<\aleph_1$, $f(a_i^\alpha, a_j^\beta)=1$; and (4) for every α, $b_1^\alpha<\cdots<b_n^\alpha$. Let $D_0=\{\boldsymbol{a}^\alpha{}^\frown\boldsymbol{b}^\alpha\mid\alpha<\aleph_1\}$. Hence $D_0\subseteq X^m\times A^n$, let D be the topological closure of D_0. Let $\gamma<\aleph_1$ be such that D is definable in M by a parameter belonging to $|M_\gamma|$, and let α be such that $p_\alpha\cap|M_\gamma|=\{a_0,\ldots,a_{r-1}\}$ and $\mathrm{Rng}(\boldsymbol{b}^\alpha)\cap|M_\gamma|=\emptyset$.

We shall now duplicate $\boldsymbol{a}^\alpha{}^\frown\boldsymbol{b}^\alpha$. Let $\boldsymbol{a}^\alpha=\boldsymbol{a}=\langle a_0,\ldots,a_{m-1}\rangle$, $\boldsymbol{b}^\alpha=\boldsymbol{b}=\langle b_1,\ldots,b_n\rangle$. Let $E^1,\ldots,E^m$ be the set of all those C_M-slices which intersect $\{a_r,\ldots,a_{m-1}\}\cup\{b_1,\ldots,b_n\}$. We represent $\boldsymbol{a}$ as $\boldsymbol{a}_0{}^\frown\cdots{}^\frown\boldsymbol{a}_k$ and $\boldsymbol{b}$ as $\boldsymbol{b}_1{}^\frown\cdots{}^\frown\boldsymbol{b}_k$ where $\boldsymbol{a}_0=\langle a_0,\ldots,a_{r-1}\rangle$, and for $i>0$, $\boldsymbol{a}_i$, $\boldsymbol{b}_i$ are respectively the subsequences of $\boldsymbol{a}$ and $\boldsymbol{b}$ consisting of those elements which belong to E^i. Note that since p_α is C_M-separated, then for every $i>0$, $\boldsymbol{a}_i$ is either empty or consists of one element.

We define by a downward induction formulas φ_i, $i=k,\ldots,0$, with parameters in $|M_\gamma|$ such that $M\vDash\varphi_i[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_i,\boldsymbol{b}_1,\ldots,\boldsymbol{b}_i]$.

$$\varphi_k\equiv\boldsymbol{a}_0{}^\frown\boldsymbol{x}_1{}^\frown\cdots{}^\frown\boldsymbol{x}_k{}^\frown\boldsymbol{y}_1{}^\frown\cdots{}^\frown\boldsymbol{y}_k\in D.$$

Suppose φ_{i+1} has been defined, and we want to define φ_i. There are two cases: (1) $\boldsymbol{x}_i$ consists of one variable and (2) $\boldsymbol{x}_i$ is an empty sequence. Since $M\vDash\varphi_{i+1}[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{i+1},\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{i+1}]$, it follows that

$$M\vDash Q\boldsymbol{x}_{i+1},\boldsymbol{y}_{i+1}\,\varphi_{i+1}[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_i,\boldsymbol{x}_{i+1},\boldsymbol{b}_1,\ldots,\boldsymbol{b}_i,\boldsymbol{y}_{i+1}].$$

By (*) in case (1), and by the increasingness of A in case (2), there are $\boldsymbol{c}^l$, $\boldsymbol{d}^l$, $l=1,2$, such that $M\vDash\varphi_{i+1}[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_i,\boldsymbol{c}^l,\boldsymbol{b}_1,\ldots,\boldsymbol{b}_i,\boldsymbol{d}^l]$, $\{\boldsymbol{d}^1,\boldsymbol{d}^2\}$ is OP, and if $\boldsymbol{c}^l=\langle c^l\rangle$, $l=1,2$, then $f(c^1,c^2)=1$. Let V_{i+1}^l be basic open sets in X such that $(c^l\in V_{i+1}^l$ and $l=1,2$, and $f(V_{i+1}^1\times V_{i+1}^2)=\{1\}$, and let W^l, $l=1,2$, be a product of basic open sets in A such that $\boldsymbol{d}^l\in W^l$, $l=1,2$, and for every $\boldsymbol{e}^1\in W^1$, $\boldsymbol{e}^2\in W^2$, $\{\boldsymbol{e}^1,\boldsymbol{e}^2\}$ is OP. Let

$$\varphi_1\equiv\exists\boldsymbol{x}_{i+1}^1,\boldsymbol{x}_{i+1}^2,\boldsymbol{y}_{i+1}^1,\boldsymbol{y}_{i+1}^2\left(\bigwedge_{l=1}^{2}\varphi_{i+1}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_i,\boldsymbol{x}_{i+1}^l,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_i,\boldsymbol{y}_{i+1}^l)\right.$$
$$\left.\wedge(\{\boldsymbol{y}_{i+1}^1,\boldsymbol{y}_{i+1}^2\}\text{ is OP})\wedge f(x_{i+1}^1,x_{i+1}^2)=1\right).$$

The last conjunct is added only in case 1.

Starting now from φ_0 and using successively $\varphi_1, \ldots, \varphi_k$ we can construct $c^l = \langle c_1^l, \ldots, c_{m-1}^l \rangle$, and d^l, $l = 1, 2$, such that: $\langle a_0, \ldots, a_{r-1} \rangle \wedge c^l \wedge d^l \in D$, $\{d^1, d^2\}$ is OP, and for every $i = 0, \ldots, m-1$, $f(c_i^1, c_i^2) = 1$. Since D_0 is dense in D, there are $\beta^1, \beta^2 < \aleph_1$ such that $a^{\beta^l} \wedge b^{\beta^l}$ is close enough to $\langle a_0, \ldots, a_{r-1} \rangle \wedge c^l \wedge d^l$, $l = 1, 2$; but then $p_{\beta^1} \cup p_{\beta^2} \in P$ and $\{b^{\beta^1}, b^{\beta^2}\}$ is OP. A contradiction and hence P is as desired.

So far we have dealt with the case when (*) holds. Consider now the case when $\neg$(*) holds. So, there is a sequence $\{\langle a_\alpha, b^\alpha \rangle \mid \alpha < \aleph_1\}$ such that the a_α's are distinct and belong to X, the b^α belong to A^n and they are pairwise disjoint, and whenever $f(a_\alpha, a_\beta) = 1$, $\{b^\alpha, b^\beta\}$ is not OP. If $n = 0$, then $\{a_\alpha \mid \alpha < \aleph_1\}$ is already an uncountable homogeneous set, so P can be chosen to be the trivial forcing. Suppose $n > 0$. We color distinct b^α, b^β in two colors according to whether $\{b^\alpha, b^\beta\}$ is OP or not. This is an open coloring hence by Lemma 3.2 there is a c.c.c. forcing set P of power $\aleph_1$ which does not destroy the increasingness of A and decomposes $\{b^\alpha \mid \alpha < \aleph_1\}$ into countably many homogeneous sets. We show that P adds an uncountable homogeneous set to X. Let $\{b^\alpha \mid \alpha \in \Gamma\} = B$ be an uncountable homogeneous set added by P. Since P did not destroy the increasingness of A, for every $\alpha, \beta \in \Gamma$, $\{b^\alpha, b^\beta\}$ is OP, hence $f(a_\alpha, a_\beta) = 0$, and hence $\{a_\alpha \mid \alpha \in \Gamma\}$ is f-homogeneous of color 0. We have thus proved Lemma 3.3, and since we skip the details of the iteration this concludes the proof of Theorem 3.1. □

Question 3.4. Can SOCA be replaced by SOCA1 in Theorem 3.1?

In the remainder of this section we try to generalize OCA to colorings of n-tuples rather than just colorings of pairs. Example 1.7 shows that the most direct generalization of OCA is inconsistent. However, the following axiom generalizing OCA might still be consistent with ZFC.

Axiom OCA(m, k). If X is a second countable Hausdorff space of power $\aleph_1$, and $\mathcal{U}$ is a finite open cover of X^m, then X can be partitioned into $\{X_i \mid i \in \omega\}$ such that for every $i \in \omega$, $(X_i)^m$ intersects at most k elements of $\mathcal{U}$.

Question 3.5. Is it true that for every m there exists a k such that OCA(m, k) is consistent?

In fact we do not even know the answer to the following weakened version of the above question. Is there k such that the following axiom is consistent: "If X is a Hausdorff second countable space and $\mathcal{U}$ is a finite open cover of X^3, then there is an uncountable $A \subseteq X$ such that A^3 intersects at most k elements of $\mathcal{U}$".

At this point it is worthwhile to mention the following theorem of A. Blass [4]. If $\mathcal{U}$ is a symmetric partition of the n-tuples of ${}^\omega 2$ into finitely many open sets, then ${}^\omega 2$ contains a perfect subset in which at most $(n-1)!$ colors appear.

We will prove a weaker generalization of OCA; however rather than formulating this new axiom in topological terms, we translate it into an equivalent statement on colorings of the binary tree.

We first introduce some terminology. Let $\bar{T}=\langle {}^{\omega\geq}2, \leqslant\rangle$ be the tree of binary sequences of length $\leqslant\omega$; let $T={}^{\omega>}2$ and $L={}^{\omega}2$. L is regarded as the set of branches of T. For $\nu, \eta \in \bar{T}$, $\nu < \eta$ denotes that ν is a proper initial segment of η, $\nu \wedge \eta$ denotes the maximal common initial segment of ν and η, Λ denotes the empty sequence, if $\nu \leqslant \eta$, then $[\nu, \eta]$, (ν, η) denote respectively the closed and open intervals with endopoints ν and η, and $[\nu]$, $[\nu)$ denote respectively $[\Lambda, \nu]$ and $[\Lambda, \nu)$. If $A \subseteq L$ let $T[A]=\{\nu \wedge \eta \mid \nu, \eta \in A \text{ and } \nu \neq \eta\}$; note that $T[A]$ is closed under $\wedge$. For $B \subseteq T$ let $B^{[m]}=\{\sigma \subseteq B \mid |\sigma| = m \text{ and } \sigma \text{ is closed under } \wedge\}$. Let $\nu <_L \eta$ denote that $\nu^\frown\langle 0\rangle \leqslant \eta$ and $\nu <_R \eta$ mean that $\nu^\frown\langle 1\rangle \leqslant \eta$. If $\sigma, \tau \in T^{[m]}$ then $\sigma \sim \tau$ means that $\langle \sigma, <_L, <_R\rangle \cong \langle \tau, <_L, <_R\rangle$. A function $f: T^{[m]} \to n$ is called an m-coloring of T; $B \subseteq T$ is f-homogeneous if for every $\sigma, \tau \in B^{[m]}$ such that $\sigma \sim \tau: f(\sigma) = f(\tau)$; $A \subseteq L$ is f-homogeneous if $T[A]$ is.

Let the tree m-coloring axiom be as follows.

Axiom TCA*m*. For every $A \subseteq L$ of power $\aleph_1$ and for every m-coloring f of T, A can be partitioned into countably many f-homogeneous subsets.

Let $\mathrm{TCA} = \bigwedge_{m\in\omega} \mathrm{TCA}m$.

We shall later present a toplogical formulation equivalent to TCA. For the time being the reader can check the following proposition.

Proposition 3.6. (a) OCA$\Rightarrow$TCA1.
(b) $\mathrm{MA}_{\aleph_1}$+TCA1$\Rightarrow$OCA.

Our next goal is the following theorem.

Theorem 3.7. TCA+MA *is consistent.*

Lemma 3.8 (CH). *Let $A \subseteq L$ be of power $\aleph_1$, and let $\mathscr{D} = \{D_i \mid i \in \omega\}$ be a partition of the levels of T into finite intervals, that is, D_i can be written as $[n_i, n_{i+1})$ where $n_0 = 0$ and $n_i < n_{i+1}$. Then there is a c.c.c. forcing set $P = P_{A,\mathscr{D}}$ of power $\aleph_1$ such that after forcing with P, A can be partitioned into countably many sets $\{A_j \mid j \in \omega\}$ such that for every j, $T[A_j]$ intersects each D_i in at most one point.*

Proof. Let M be a model with universe $\aleph_1$ which encodes T, A and $\mathscr{D}$, and let $C = C_M$. P will consist of all finite approximations of the desired partition $\{A_j \mid j \in \omega\}$ in which each A_j intersects each C-slice in at most one point. To be more precise let $\{\alpha_i \mid i < \aleph_1\}$ be an order preserving enumeration of C, let $E_i = [\alpha_i, \alpha_{i+1})$, and let $\mathscr{E} = \{E_i \mid i < \aleph_1\}$. $\mathscr{E}$ is called the set of C-slices. $P = \{f \mid \mathrm{Dom}(f) \in P_{\aleph_0}(A),\ \mathrm{Rng}(f) \subseteq \omega$ and for every $j \in \omega$: for every D_i, $|T[f^{-1}(j)] \cap D_i| \leqslant 1$ and for every E_i, $|f^{-1}(j) \cap E_i| \leqslant 1\}$.

By the standard duplication method one can easily show that P is c.c.c., and clearly $\Vdash_P$ "A can be partitioned into $\{A_j \mid j \in \omega\}$ such that for every j and i, $|T[A_j] \cap D_i| \leq 1$". $\square$

Let $\sigma \in \bigcup_{m\in\omega} T^{[m]}$ and $\nu \in T$, $\sigma < \nu$ denotes that $\max(\{\nu \wedge \xi \mid \xi \in \sigma\}) \in \sigma$. Note that (1) if $\sigma < \nu$, then $\sigma \cup \{\nu\} \in \bigcup_{m\in\omega} T^{[m]}$ and (2) σ can be written as $\{\xi_1, \ldots, \xi_m\}$ where for each $i < m$, $\{\xi_1, \ldots, \xi_i\} \in T^{[i]}$ and $\{\xi_1, \ldots, \xi_i\} < \xi_{i+1}$. Let $\sigma \in T^{[m]}$ and $\nu, \eta \in T$; $\nu \sim_\sigma \eta$ if there is an isomorphism between $\langle \sigma \cup \{\nu\}, <_L, <_R\rangle$ and $\langle \sigma \cup \{\eta\}, <_L, <_R\rangle$ which is the identity on σ. For $\nu \in T$ let $\mathrm{lth}(\nu)$ be the length of ν. Let $\boldsymbol{n} = \{n_i \mid i \in \omega\}$ be a strictly increasing sequence of natural numbers, let $\sigma \subseteq T$ and $\nu \in T$; we say that σ, ν are $\boldsymbol{n}$-separated if for some i: for every $\eta \in \sigma$, $\mathrm{lth}(\eta) < n_i$ and $\mathrm{lth}(\nu) \geq n_i$.

Lemma 3.9 (CH). *Let $f: T^{[m+1]} \to n$ be an $m+1$-coloring of T and $A \subseteq L$ be uncountable. Then there is a c.c.c. forcing set $P = P^1_{A,f}$ of power $\aleph_1$ such that after forcing with P we have the following situation: there is a strictly increasing sequence $\boldsymbol{n}$ with $n_0 = 0$ and an uncountable $B \subseteq A$ such that for every $\sigma \in T[B]^{[m]}$ and $\nu, \eta \in T[B]$: if $\sigma < \nu$, η; $\nu \sim_\sigma \eta$ and σ, ν are $\boldsymbol{n}$-separated, then $f(\sigma \cup \{\nu\}) = f(\sigma \cup \{\eta\})$.* (We call such B a prehomogeneous set.)

Moreover, (*) there is a countable $A' \subseteq A$ such that for every $a \in A - A'$ there is $p \in P$ such that $p \Vdash_P a \in B$.

Before proving Lemma 3.9, let us see how Theorem 3.7 follows from Lemmas 3.8 and 3.9.

Proof of Theorem 3.7. As usual we deal just with the atomic step in the iteration. So, given a subset $A \subseteq L$ of power $\aleph_1$ and an m-coloring f of T we have to find a c.c.c. forcing set of power $\aleph_1$ such that after forcing with it A can be partitioned into countably many f-homogeneous subsets. We prove this by induction on m.

The case $m = 1$ follows from the proof of the consistency of OCA.

Suppose by induction for every m-coloring f of T and every $A \subseteq L$ of power $\aleph_1$ there is a c.c.c. forcing set $P = P_{A,f}$ of power $\aleph_1$ such that $\Vdash_P$ "A can be partitioned into countably many f-homogeneous subsets".

Let V be a universe satisfying CH, $A \subseteq L$ be of power $\aleph_1$ and f be an $m+1$-coloring of T. Let $\{Q_i \mid i \in \omega\}$, $\{P_i \mid i \in \omega\}$ be a finite support iteration, Q_0 is trivial, P_i is a Q_i-name for the forcing set $P^1_{A,f}$ from Lemma 3.9 in the universe V^{Q_i}, and $Q_{i+1} = Q_i * P_i$. Let $P^2_{A,f} = \bigcup_{i\in\omega} Q_i$. We denote $P^2_{A,f}$ by Q. In V^Q we have a family $\{B_i \mid i \in \omega\}$ of prehomogeneous subsets of A, and corresponding to each B_i we have a sequence $\boldsymbol{n}^i$. By (*) of 3.9 it is easy to check that $|A - \bigcup_{i\in\omega} B_i| \leq \aleph_0$. Let $D^i_j = \{\nu \in T \mid n^i_j \leq \mathrm{lth}(\nu) < n^i_{j+1}\}$ and $\mathscr{D}^i = \{D^i_j \mid j \in \omega\}$. Let R be the Q-name of the following forcing set. R is gotten by a finite support iteration of $P_{A,\mathscr{D}^i}$ of Lemma 3.8. After forcing with R each B_i is partitioned into countably many sets which we denote by $\{B_{ij} \mid j \in \omega\}$. It is easy to see that for every B_{ij}: if $\sigma \in T[B_{ij}]^{[m]}$, $\nu, \eta \in T[B_{ij}]$, $\sigma < \nu, \eta$ and $\nu \sim_\sigma \eta$, then $f(\sigma \cup \{\nu\}) = f(\sigma \cup \{\eta\})$.

We can now define an m-coloring on each $T[B_{ij}]$. The color $f_{ij}(\sigma)$ where $\sigma\in T[B_{ij}]^{[m]}$ is the sequence of colors of the form $f(\sigma\cup\{\nu\})$ where the ν's belong to $T[B_{ij}]$ and they represent all equivalence classes of $\sim_\sigma$ in which $\sigma<\nu$. More precisely for every $\sigma\in T[B_{ij}]^{[m]}$ let $\nu_1^\sigma,\dots,\nu_{k_\sigma}^\sigma>\sigma$ be such that for every $\nu>\sigma$ there is a unique i such that $\nu\sim_\sigma\nu_i^\sigma$. Moreover we pick the ν_i^σ's in such a way that if $\tau\sim\nu$, then for every i there is an isomorphism between $\langle\tau\cup\{\nu_i^\tau\},<_L,<_R\rangle$ and $\langle\sigma\cup\{\nu_i^\sigma\},<_L,<_R\rangle$ which maps τ onto σ. We define $f_{ij}(\sigma)=\langle f(\sigma\cup\{\nu_1^\sigma\}),\dots,f(\sigma\cup\{\nu_{k_\sigma}^\sigma\})\rangle$.

By the induction hypothesis there is a c.c.c. forcing set S of power $\aleph_1$ such that $\Vdash_S$ "Each B_{ij} can be partitioned into countably many f_{ij}-homogeneous sets". It is easy to see that if $B\subseteq B_{ij}$ is f_{ij}-homogeneous, then B is f-homogeneous. Hence after forcing with $Q*R*S$, A can be partitioned into countably many f-homogeneous subsets. This completes the proof of Theorem 3.7. □

Proof of Lemma 3.9. For $B\subseteq T$ and $a\in T$, let $B^{[m,a]}=\{\sigma\in B^{[m]}\mid \text{there is } \nu\in\sigma$ such that $\nu<a\}$. Let $f:T^{[m+1]}\to n$ and A be as in 3.9, let M be a model with universe $\aleph_1$ which has T, f and A as predicates. Let $\{\alpha_i\mid i<\aleph_1\}$ be an order preserving enumeration of C_M, $M_i=M\restriction\alpha_i$, $E_i=[\alpha_i,\alpha_{i+1})$ and $A''=A\cap[\alpha_0,\aleph_1)$. For every $a\in A''$ we define a coloring $f_a:T^{[m,a]}\to n$. Suppose $a\in E_i$, for every finite subset $C\subseteq T$ there is a function $g_C:C^{[m,a]}\to n$ such that for every formula $\varphi(x)$ in the language of M, and with parameters from M_i: if $M\models\varphi[a]$, then for every $\alpha<\aleph_1$ there are $b,c\in A$ such that $b,c>\alpha$, $M\models\varphi[b]\wedge\varphi[c]$ and for every $\sigma\in C^{[m,a]}$, $\sigma<b\wedge c$ and $f(\sigma\cup\{b\wedge c\})=g_C(\sigma)$. The existence of such g_C is proved, as in the analogous argument in the proof of the consistency of OCA. By Konig's lemma we can choose the g_C's to be pairwise compatible. Let $f_a=\bigcup\{g_C\mid C\in P_{\aleph_0}(T)\}$.

We are ready to define the forcing set $P=P^1_{A,f}$ of Lemma 3.9. An element p of P is an object of the form $\langle \boldsymbol{n},C\rangle$ where $\boldsymbol{n}=\langle n_0,\dots,n_{k-1}\rangle$ is a strictly increasing finite sequence of natural numbers with $n_0=0$, C is a finite C_M-separated subset of A'' and the following conditions hold: (1) for every distinct $a,b\in C$, $\mathrm{lth}(a\wedge b)<n_{k-1}$, denote by $n_{a,b}$ the maximal n_i such that $n_i\le\mathrm{lth}(a\wedge b)$; (2) let $f_a\restriction k$ abbreviate $f_a\restriction\{\nu\in T\mid\mathrm{lth}(\nu)<k\}^{[m,a]}$, then for every distinct $a,b\in C$, $f_a\restriction n_{a,b}=f_b\restriction n_{a,b}$; and (3) for every distinct $a,b\in C$ and for every $\sigma\in\mathrm{Dom}(f_a\restriction n_{a,b})$, $f(\sigma\cup\{a\wedge b\})=f_a(\sigma)$. We denote $\boldsymbol{n}=\boldsymbol{n}^p$, $n_i=n_i^p$, $C=C_p$ and $n_{a,b}=n^p_{a,b}$.

Let $p,q\in P$, then $p\le q$ if $\boldsymbol{n}^p\trianglelefteq\boldsymbol{n}^q$ and $C_p\subseteq C_q$.

We prove that P is c.c.c. Let $\{p_\alpha\in\alpha<\aleph_1\}\subseteq P$. W.l.o.g. (1) for every $\alpha,\beta<\aleph_1$, $\boldsymbol{n}^{p_\alpha}=\boldsymbol{n}^{p_\beta}=\boldsymbol{n}=\langle n_0,\dots,n_{k-1}\rangle$, and $\{C_{p_\alpha}\mid\alpha<\aleph_1\}$ is a Δ-system; (2) for every $\alpha<\beta<\aleph_1$, $C_{p_\alpha}=\{a_{\alpha,0},\dots,a_{\alpha,r-1},a_{\alpha,r},\dots,a_{\alpha,s-1}\}$ where $\{a_{\alpha,0},\dots,a_{\alpha,r-1}\}$ is the kernel of the Δ-system, and $a_{\alpha,0}<\cdots<a_{\alpha,s-1}<a_{\beta,r}$; (3) for every $\alpha,\beta<\aleph_1$ and $i<s$, $a_{\alpha,i}\restriction n_{k-1}=a_{\beta,i}\restriction n_{k-1}$ and $f_{a_{\alpha,i}}\restriction n_{k-1}=f_{a_{\beta,i}}\restriction n_{k-1}$.

We regard each C_{p_α} as an element of L^s. We use the usual topology on L and define D to be the topological closure of $\{C_{p_\alpha}\mid\alpha<\aleph_1\}$ in L^s. D is definable from some parameter e in M. Let i_0 be such that $e,a_{\alpha,0},\dots,a_{\alpha,r-1}<\alpha_{i_0}$, and let p_α be such that $\alpha_{i_0}\le a_{\alpha,r},\dots,a_{\alpha,s-1}$.

We apply the duplication argument to p_α. Let us denote $p_\alpha = p$, $C_p = C$ and $a_{\alpha,i} = a_i$. We define by a downward induction formulas $\varphi_{s-1}(x_r, \ldots, x_{s-1}), \ldots, \varphi_r(x_r), \varphi_{r-1}$ with parameters $<\alpha_{i_0}$ such that $M \models \varphi_i[a_r, \ldots, a_i]$. $\varphi_{s-1}(x_r, \ldots, x_{s-1})$ is the formula saying that $\langle a_0, \ldots, a_{r-1}, x_r, \ldots, x_{s-1}\rangle \in D$. Suppose φ_{i+1} has been defined. By the definition of f_a there are b^1, b^2 such that $M \models \varphi_{i+1}[a_r, \ldots, a_i, b^j]$, $j = 1, 2$, $\mathrm{lth}(b^1 \wedge b^2) \geq n_{k-1}$ and for every $\sigma \in \mathrm{Dom}(f_{a_{i+1}} \upharpoonright n_{k-1})$, $f(\sigma \cup \{b^1 \wedge b^2\}) = f_{a_{i+1}}(\sigma)$. Let $\nu_{i+1} = b^1 \wedge b^2$ and

$$\varphi_i(x_r, \ldots, x_i) \equiv \exists y^1, y^2 \left(\bigwedge_{j=1}^{2} \varphi_{i+1}(x_1, \ldots, x_i, y^j) \wedge (y^1 \wedge y^2 = \nu_{i+1}) \right).$$

Next we construct by induction sequences $\langle b^j_r, \ldots, b^j_{s-1}\rangle = \boldsymbol{b}^j$, $j = 1, 2$ such that: $\langle a_0, \ldots, a_{r-1}\rangle \frown \boldsymbol{b}^j \in D$, and for every $i = r, \ldots, s-1$, $b^1_i \wedge b^2_i = \nu_i$. Since $\{C_{p_\alpha} \mid \alpha < \aleph_1\}$ is dense in D there are $\alpha, \beta < \aleph_1$ such that for every $i = r, \ldots, s-1$, $a_{\alpha,i} \wedge a_{\beta,i} = \nu_i$. Let $n_k > \max(\{\mathrm{lth}(\nu_i) \mid i = r, \ldots, s-1\})$; recalling that for every i, $f_{a_{\alpha,i}} \upharpoonright n_{k-1} = f_{a_{\beta,i}} \upharpoonright n_{k-1}$, it is easy to see that $\langle \boldsymbol{n} \frown \langle n_k\rangle, C_{P_\alpha} \cup C_{P_\beta}\rangle \in P$. Hence P is c.c.c.

P is not yet as required in Lemma 3.9, since if G is P-generic, $\bigcup\{C_p \mid p \in G\}$ need not be uncountable. However, by a standard argument, it is easy to find a countable set $A' \subseteq A$ such that if $P' = \{p \in P \mid C_p \cap A' = \emptyset\}$, then for every P'-generic filter G $\bigcup\{C_p \mid p \in G\}$ is uncountable. P' is obviously as required in 3.9. □

This concludes the proof of Theorem 3.7.

Remark. As in Lemma 3.2 we can also prove that TCA+MA+ISA is consistent.

Question 3.10. Prove that TCA$m \not\Rightarrow$ TCA$m+1$.

Our next goal is to find a generalization of OCA which is equivalent to TCA.

Let $D_m(A)$ be the set of 1-1 m-tuples from A. Let X be second countable and of power $\aleph_1$. An open m-coloring of X is a finite open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of $D_m(X)$ such that each U_i is symmetric. We next define what it means for an open m-coloring to be strongly open. We define by a downward induction $\varphi_i(V_1, \ldots, V_i)$ where φ_i is a property of i-tuples of open sets from X. $\varphi_m(V_1, \ldots, V_m) \equiv \exists U_i\ (V_1 \times \cdots \times V_m \subseteq U_i)$. Suppose φ_{i+1} has been defined;

$$\varphi'_i(V_1, \ldots, V_i) \equiv (\forall x_1, x_2 \in V_i)\ (x_1 \neq x_2 \rightarrow (\exists V^1, V^2)\ (x_1 \in V^1 \wedge x_2 \in V^2 \wedge (V^1, V^2 \text{ are open}) \wedge \varphi_{i+1}(V_1, \ldots, V_{i-1}, V^1, V^2))),$$

$$\varphi_i(V_1, \ldots, V_i) \equiv \bigwedge \{\varphi'_i(V_{\pi(1)}, \ldots, V_{\pi(i)}) \mid \pi \text{ is a permutation of } i\}.$$

Definition 3.11. Let $\mathcal{U}$ be an open m-coloring of X. $\mathcal{U}$ is *strongly open* if $\varphi_1(X)$ holds.

Let t_m be the number of isomorphism types of models of the form $\langle \sigma, <_L, <_R\rangle$

where $\sigma \in T^{[m]}$. Let $\mathcal{U}$ be an m-coloring of X. $A \subseteq X$ is $\mathcal{U}$-homogeneous if there is a subset $\mathcal{U}' \subseteq \mathcal{U}$ such that $|\mathcal{U}'| \leq t_m$ and $D_m(A) \subseteq \bigcup\{U \mid U \in \mathcal{U}'\}$.

Axiom OCA*m*. If X is second countable and of power $\aleph_1$ and $\mathcal{U}$ is a strongly open m-coloring of X, then X can be partitioned into countably many $\mathcal{U}$-homogeneous subsets.

Theorem 3.12. (a) OCA$m+1 \Rightarrow$ TCAm.
(b) TCA$m + MA_{\aleph_1} \Rightarrow$ OCA$m+1$.

Proof. (a) Assume OCA$m+1$, and let $f: T^{[m]} \rightarrow n$ be an m-coloring of T and $A \subseteq L$ be of power $\aleph_1$. For every $i \in N$ and $\sigma \in T^{[m]}$ we define a symmetric open subset of $D_{m+1}(A)$:

$$U_{\sigma,i} = \{\langle a_0, \ldots, a_m\rangle \in D_m(A) \mid \langle T[\{a_0, \ldots, a_m\}], <_L, <_R\rangle \cong \langle \sigma, <_L, <_R\rangle \text{ and } f(T[\{a_0, \ldots, a_m\}] = i)\}.$$

Clearly $\mathcal{U} \stackrel{\text{def}}{=} \{U_{\sigma,i} \mid \sigma \in T^{[m]} \text{ and } i \in n\}$ is a finite open cover of $D_{m+1}(A)$, and it is easy to check that $\mathcal{U}$ is a strongly open $(m+1)$-coloring of A. Applying OCA$m+1$ to A and $\mathcal{U}$ one gets a countable partition of A into $\mathcal{U}$-homogeneous subsets. It is easy to check that these sets are in fact f-homogeneous.

(b) Assume $\text{MA}_{\aleph_1}$+TCAm. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a strongly open $(m+1)$-coloring of a second countable space X of power $\aleph_1$. W.l.o.g. X is Hausdorff. Let $\mathcal{B}$ be a countable open base of X. A tree approximation of $\mathcal{U}$ is a function g such that: $\text{Dom}(g) \subseteq T$, $\text{Rng}(g) \subseteq \mathcal{B}$ and (1) if $\eta < \nu \in \text{Dom}(g)$, then $\eta \in \text{Dom}(g)$; (2) let $\nu, \eta \in \text{Dom}(g)$; then if ν and η are incomparable with respect to $\leqslant$, then $g(\nu) \cap g(\eta) = \emptyset$, and if $\nu \leqslant \eta$, then $g(\nu) \supseteq g(\eta)$; and (3) if $i \leqslant m$ and $\nu_0, \ldots, \nu_i \in \text{Dom}(g)$ are incomparable in T, then $\varphi_{i+1}(g(\nu_0), \ldots, g(\nu_i))$ holds. Let g be an approximation of $\mathcal{U}$, and let $B \subseteq X$; we say that g is an approximation of $\mathcal{U}$ on B if: (1) for every $b \in B$ there is a branch t_b of Dom(g) such that $b \in \bigcap\{g(\nu) \mid \nu \in t_b\}$; and (2) the function mapping b to t_b is 1-1.

Using $\text{MA}_{\aleph_1}$ it is easy to see that there is a family $\{\langle g_i, B_i\rangle \mid i \in \omega\}$ such that g_i is an approximation of $\mathcal{U}$ on B_i and $\bigcup_{i \in \omega} B_i = X$. W.l.o.g. $\text{Dom}(g_i) = T$. Let $A_i = \{t_b \mid b \in B_i\}$, hence $A_i \subseteq L$. For every i we now define an m-coloring f_i of T. Let $\sigma \in T^{[m]}$; if there is no $C \subseteq A_i$ such that $T[C] = \sigma$, define $f_i(\sigma) = 0$; otherwise, let $t_{b_0}, \ldots, t_{b_m}$ be such that $T[\{t_{b_0}, \ldots, t_{b_m}\}] = \sigma$ and let $f(\sigma) = i$ where $\langle b_0, \ldots, b_m\rangle \in U_i$. Clearly, $f(\sigma)$ does not depend on the choice of $b_0, \ldots, b_m$. One can easily check that if $A \subseteq A_i$ is f_i-homogeneous, then $\{b \mid t_b \in A\}$ is $\mathcal{U}$-homogeneous, hence OCA$m+1$ follows. □

Remarks. (a) We did not mention the polarized versions of TCA, however the proof that they are consistent resembles the proof that TCA is consistent.

(b) The consistency of OCAm or TCA implies by absoluteness a special case of Blass' theorem [4], namely that if $\mathcal{U}$ is a strongly open coloring of ${}^{\omega}2$, then ${}^{\omega}2$

contains a perfect set in which at most t_m colors appear. The existence of such a perfect set is a Σ^1_1 statement, and since it holds in some extension it must exist in the ground model.

The main question in this matter is whether our consistency result can be strengthened to include all open colorings as in Blass' theorem.

An open coloring can be regarded as a continuous function from $X \times X$ to the set of colors equipped with its descrete topology. It seems thus natural to examine partition theorems for general continuous functions. We did not investigate these questions thoroughly, however here is one example of such a theorem.

Let the nowhere denseness axiom be as follows

Axiom NWDA2. If X and Y are second countable Hausdorff spaces, $|X| = \aleph_1$ and Y is regular and does not contain isolated points, and if $f: D_2(X) \to Y$ is a symmetric continuous function, then X can be partitioned into $\{A_i \mid i \in \omega\}$ such that for every $i, j \in \omega$, $f(A_i \times A_j)$ is nowhere dense.

Note that even the weakest form of NWDA does not follow from ZFC, for if $A \subseteq \mathbb{R}$ is an uncountable Lusin set (i.e. its intersection with every nowhere dense set is countable) and $f(a, b) = a + b$, then for every uncountable $B \subseteq A$, $f(B \times B)$ if of the second cateogry.

Question 3.13. Does NWDA2 follow from $MA_{\aleph_1}$?

Theorem 3.14. MA+NWDA2 *is consistent.*

Proof. We deal with the atomic step in the interation, and we assume CH in every intermediate stage. Let X, Y, f be as in the axiom, let $\mathscr{B}$ and $\mathscr{C}$ be countable bases of X and Y respectively, and let M be a model whose universe is $\aleph_1$ and which encodes f, X, Y, $\mathscr{B}$ and $\mathscr{C}$. Let $\{E_i \mid i < \aleph_1\}$ be an enumeration of the C_M-slices in an increasing order. Let $U \subseteq Y$ be a finite union of elements of $\mathscr{C}$, and let $a \in E_\alpha$. We say that U is permissible for a, if for every formula $\varphi(x)$ with parameters in $\bigcup_{\beta<\alpha} E_\beta$: if $M \models \varphi[a]$, then there are distinct b, c such that $M \models \varphi[b]$, $M \models \varphi[c]$ and $f(b, c) \notin \mathrm{cl}(U)$. Let

$$P = P^1_{X,Y,f} = \{\langle \sigma, U\rangle \mid \sigma \in P_{\aleph_0}(X),\ \sigma \text{ is } C_M\text{-separated},$$
$$f(D_2(\sigma)) \cap \mathrm{cl}(U) = \emptyset, \text{ and for every } a \in \sigma,\ U \text{ is permissible for } a\}.$$

$\langle \sigma_1, U_1\rangle \leq \langle \sigma_2, U_2\rangle$ if $\sigma_1 \subseteq \sigma_2$ and $U_1 \subseteq U_2$.

One can easily check that if U is permissible for a and $V \subseteq Y$ is open and non-empty, then there is $U_1 \supseteq U$ such that $U_1 \cap V \neq \emptyset$ and Y_1 is permissible for a. It is easy to check by the duplication argument that P is c.c.c. Let G be P-generic and $A = \bigcup \{\sigma \mid \exists U\, (\langle \sigma, U\rangle \in G\}$. Then $f(D_2(A))$ is nowhere dense. Let $P^2_{X,Y,f}$ be the forcing set gotten by iterating $P^1_{X,Y,f}$ ω times with finite support. It is easy to check that if G is $P^2_{X,Y,f}$ generic, then in $V[G]$, X has a partition $\{A_i \mid i \in \omega\}$ such that for every $i \in \omega$, $f(D_2(A_i))$ is nowhere dense.

The proof will be completed if we show the following claim. □

Claim. *Let $\{A_i \mid i \in \omega\}$ be a family of second countable spaces of power $\aleph_1$, Y be a second countable space without isolated points, and for every $i \leqslant j$ let $f_{ij} : A_i \times A_j \to Y$ be a continuous function. Then there is a c.c.c. forcing set P of power $\aleph_1$ such that after forcing with P each A_i can be partitioned into $\{A_{ij} \mid j \in \omega\}$ such that for every distinct $\langle i_1, j_1\rangle$, $\langle i_2, j_2\rangle$, $f_{i_1 i_2}(A_{i_1 j_1} \times A_{i_2 j_2})$ is nowhere dense.*

We leave it to the reader to construct such P. (Here one does not have to assume CH in the ground model.)

Question 3.15. Let NWDAm denote the axiom analogous to NWDA2 where m-place functions replace 2-place functions. Prove that NWDAm is consistent.

4. The semi open coloring axiom does not imply the open coloring axiom; the tail method

In this section we present another trick called the "tail method". This method is used in the proof of the following theorem.

Theorem 4.1. $\mathrm{MA} + \mathrm{SOCA} + \neg\mathrm{OCA} + 2^{\aleph_0} = \aleph_2$ *is consistent.*

Indeed in Section 11 we prove that $\mathrm{MA}_{\aleph_1} + \mathrm{OCA} \Rightarrow 2^{\aleph_0} = \aleph_2$ and in Section 5 we prove that $\mathrm{MA}_{\aleph_1} + \mathrm{SOCA} + 2^{\aleph_0} > \aleph_2$ is consistent, hence this means that $\mathrm{MA}_{\aleph_1} + \mathrm{SOCA} + \neg\mathrm{OCA} + 2^{\aleph_0} > \aleph_2$ is consistent.

Still Theorem 4.1 adds some information, but more importantly it is a simple application of the tail method and thus will well serve in presenting this method.

The consistency of $\mathrm{MA}_{\aleph_1}$ with the existence of an entangled set which is proved in [1], implies that $\mathrm{MA}_{\aleph_1} + \neg\mathrm{SOCA} + \neg\mathrm{OCA}$ is consistent.

However we were unsuccessful in proving or disproving the following.

Question 4.2. Does OCA imply SOCA? Does MA + OCA imply SOCA?

Proof of Theorem 4.1. We give a detailed description of the proof, but skip the details which are standard; we also skip some formalities in order to simplify notations.

Definition. $\langle X, f\rangle$ is a *SOC pair* if f is a SOC of X; it is called an *OC pair* if in addition $f^{-1}(0)$ is open in $D(X)$.

We want to construct a universe W in which MA+SOCA+$\neg$OCA holds. To do this we start with a universe V and an OC pair $\langle Y, g\rangle \in V$ such that $V \models$ "CH$+2^{\aleph_1}=\aleph_2$ and Y does not contain uncountable g-homogeneous subsets". W is gotten from V by a finite support iteration of forcing sets $\{P_\alpha \mid \alpha \leq \aleph_2\}$, and we want that $\langle Y, g\rangle$ will be a counter-example to OCA in W. So we prepare in advance a list of tasks which will enumerate all possible SOC pairs $\langle X, f\rangle$ and all possible c.c.c. forcing sets of power $\aleph_1$. In addition we prepare a 1-1 enumeration $\{y_\beta \mid \beta < \aleph_1\}$ of Y. We define by induction on $\alpha < \aleph_2$ a forcing set P_α and a club $C_\alpha \subseteq \aleph_1$. Let $Y_\alpha = \{y_\beta \mid \beta \in C_\alpha\}$. We call Y_α the αth tail of Y. Our induction hypothesis is that $\Vdash_{P_\alpha} Y_\alpha$ does not contain uncountable g-homogeneous subsets.

Let $P = P_{\aleph_2}$. It is clear that the induction hypothesis assures that $\Vdash_P$ "Y is not a countable union of homogeneous sets".

If δ is a limit ordinal, then $P_\delta = \bigcup_{\alpha<\delta} P_\alpha$. We choose a club $C_\delta \subseteq \aleph_1$ such that for every $\alpha < \delta$, $|C_\delta - C_\alpha| \leq \aleph_0$. We want to check that the induction hypothesis holds.

Case 1. $\mathrm{cf}(\delta) = \aleph_1$. Suppose by contradiction for some P_δ-generic filter G there is $A \in V[G]$ such that A is an uncountable g-homogeneous subset of Y_δ. Since $\langle Y, g\rangle$ is an OC pair we can assume that A is closed; and since Y is second countable there is $\alpha < \delta$ such that $A \in V(G \cap P_\alpha]$. $A \cap Y_\alpha$ is an uncountable homogeneous subset of Y_α belonging to $V[G \cap P_\alpha]$, and this contradicts the induction hypothesis.

Case 2. $\mathrm{cf}(\delta) = \aleph_0$. Suppose by contradiction that there is a P_δ-generic G and $A \in V[G]$ such that A is an uncountable homogeneous subset of Y_δ. Let $\{\alpha_i \mid i \in \omega\}$ be an increasing sequence converging to δ. Then there are A_i, $i \in \omega$, such that $A = \bigcup_{i\in\omega} A_i$ and $A_i \in V[G \cap P_{\alpha_i}]$. Some A_i is uncountable, hence $A_i \cap Y_{\delta_i}$ is uncountable. This again contradicts the induction hypothesis.

Let us see how to define $P_{\alpha+1}$, $C_{\alpha+1}$ in the successor stage. If our αth task is to deal with a c.c.c. forcing set P_α, we will use a version of the explicit contradiction method, this will be explained later.

We first deal with the case when the αth task is a P_α-name of a SOC pair $\langle X_\alpha, f_\alpha\rangle$. For a SOC pair $\langle Z, h\rangle$ such that Z does not contain uncountable 0-colored sets, let $M(Z, h, Y, g) = M$ be a model whose universe is $\aleph_1$, and which encompasses enough set theory, and Z, h, Y and g. Let C_M be the club of initial elementary submodels of M. Let $P(Z, h, Y, g)$ be the forcing set consisting of all C_M-separated finite 1-colored subsets of Z.

Suppose X_α does not contain uncountable 0-colored sets. We want to add to X_α an uncountable 1-colored set without destroying the induction hypothesis. For this we need the following lemma.

Lemma 4.3 (CH). *Let $\langle Y, g\rangle$ be an OC pair which does not contain uncountable homogeneous subsets, and let $\{y_\beta \mid \beta < \aleph_1\}$ be a* 1-1 *enumeration of Y. Let $\langle X, f\rangle$ be a SOC pair which does not contain uncountable* 0*-colored subsets. Then there is a club $C \subseteq \aleph_1$ an uncountable $X' \subseteq X$ and a c.c.c. forcing set $P_{X',f} = P$ of power $\aleph_1$ such that $\Vdash_P$ "$\{y_\beta \mid \beta \in C\}$ does not contain uncountable homogeneous subsets".*

Let $\{\beta_\gamma \mid \gamma<\aleph_1\}$ be an order preserving enumeration of C_α. In order to define $P_{\alpha+1}$, $C_{\alpha+1}$ we apply Lemma 4.3 to $\langle Y_\alpha, g\rangle$, $\langle X, f\rangle$ and the enumeration $\{y_{\beta_\gamma} \mid \gamma<\aleph_1\}$ of Y_α. Let X', C be respectively the subspace of X and the club whose existence is assured in 4.3. We define $P_{\alpha+1}$ to be $P_\alpha * P_{X',f}$ and $C_{\alpha+1}=\{\beta_\gamma \mid \gamma \in C\}$. It is clear that $P_{\alpha+1}$, $C_{\alpha+1}$ satisfy the induction hypothesis.

Lemma 4.3 is broken into two claims.

Lemma 4.4 (CH). *Let $\langle X, f\rangle$ be a SOC pair, $\langle Y, g\rangle$ be an OC pair and $\{y_\beta \mid \beta<\aleph_1\}$ be a 1-1 enumeration of Y. Suppose X does not contain uncountable 0-colored subsets, and Y does not contain uncountable homogeneous subsets, then there are uncountable $X'\subseteq X$ and a club $C\subseteq\aleph_1$ such that letting Y' be $\{y_\beta \mid \beta\in C\}$, for every uncountable 1-1 function $h\subseteq X'\times Y'$ and every $l\in\{0,1\}$, there are $x_1, x_2\in \mathrm{Dom}(h)$ such that $f(x_1,x_2)=1$ and $g(h(x_1),h(x_2))=l$.*

Lemma 4.5 (CH). *Let $\langle X', f\rangle$, $\langle Y', g\rangle$ be as assured by Lemma 4.4, and let $P=P(X',f,Y',g)$, then $\Vdash_P$ "Y' does not contain uncountable homogeneous subsets".*

Proof of Lemma 4.4. We first prove the following claim.

Claim 1. *Let $\langle X, f\rangle$, $\langle Y, g\rangle$ and $\{y_\beta \mid \beta<\aleph_1\}$ be as in 4.4, let $F\subseteq X\times Y$ and $l\in\{0,1\}$, and suppose that for every $x_1, x_2\in X$ and $y_1, y_2\in Y$: if $x_1\neq x_2$, $y_1\neq y_2$, $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle\in F$ and $f(x_1,x_2)=1$, then $g(y_1,y_2)=l$. Then there are at most countably many x's in X for which $|\{y \mid \langle x,y\rangle\in F\}|>\aleph_0$, and there are at most countably many y's in Y for which $|\{x \mid \langle x,y\rangle\in F\}|>\aleph_0$.*

Proof. Let $F(x)=\{y \mid \langle x,y\rangle\in F\}$ and $F^{-1}(y)=\{x \mid \langle x,y\rangle\in F\}$. Suppose by contradiction that $A\overset{\text{def}}{=}\{x \mid |F(x)|=\aleph_1\}$ is uncountable. Since Y does not contain uncountable l-colored sets, for every $x\in A$ there are $y_x^1, y_x^2\in F(x)$ such that $g(y_x^1, y_x^2)=1-l$, and the choice of the y_x^i's can be made so that for every $u\neq v$ in A, $\{y_u^1, y_u^2\}\cap\{y_v^1, y_v^2\}=\emptyset$. By the second countability of Y and the openness of g, there is an uncountable $B\subseteq A$ such that for every distinct $u, v\in B$, $g(y^1, y^2)=1-l$. Since X does not contain uncountable 0-colored sets there are $u, v\in B$ such that $f(u,v)=1$. This contradicts the assumption about F, since $f(u,v)=1$, $y_u^1\neq y_u^2$ for $\langle u, y_u^1\rangle$, $\langle v, y_v^2\rangle\in F$ but $g(y_u^1, y_v^2)\neq l$.

The agrument why $|\{y \mid |F^{-1}(y)|>\aleph_0\}|\leqslant\aleph_0$ is similar. □

We now return to the proof of Lemma 4.4. For F as above let $D(F)=\{x \mid |F(x)|\leqslant\aleph_0\}$ and $R(F)=\{y \mid |F^{-1}(y)|\leqslant\aleph_0\}$. Let $\{F_i \mid i<\aleph_1\}$ be an enumeration of all closed subsets $F\subseteq X\times Y$ which satisfy (*): there is $l=l_F\in\{0,1\}$ such that for every $x_1, x_2\in X$ and $y_1, y_2\in Y$: if $f(x_1,x_2)=1$, $y_1\neq y_2$ and $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle\in F$, then $g(y_1,y_2)=l$.

We define by induction on $i<\aleph_1$, $x_i\in X$ and $\beta_i<\aleph_1$ with the purpose that X'

will be $\{x_i \mid i<\aleph_1\}$ and C will be $\{\beta_i \mid i<\aleph_1\}$. Suppose x_j, β_j have been defined for every $j<i$. If i is a limit ordinal let $\beta_i=\sup(\{\beta_j \mid j<i\})$, otherwise let β_i be an ordinal greater than any ordinal in the set $\{\gamma \mid y_\gamma \in \bigcup\{F_k(x_j) \mid k, j<i$ and $x_j \in D(F_k)\}\} \cup \{\beta_j \mid j<i\}$. Let $x_i \in X-\{x_j \mid j<i\}-\bigcup\{F_k^{-1}(y_{\beta_j}) \mid k<i,\ j\leq i$ and $y_{\beta_j} \in R(F_k)\}$.

Let $C=\{\beta_i \mid i<\aleph_1\}$ and $X'=\{x_i \mid i<\aleph_1\}$. Clearly X' is uncountable and C is a club. Let $Y'=\{y_\beta \mid \beta \in C\}$. Suppose by contradiction there is an uncountable 1-1 $h \subseteq X' \times Y'$ and $l \in \{0,1\}$ such that for every $x_1, x_2 \in \mathrm{Dom}(h)$: if $f(x_1,x_2)=1$, then $g(h(x_1),h(x_2))=l$. Let $F \subseteq X \times Y$ be the closure of h in $X \times Y$. Since $f^{-1}(1)$ is open and $g^{-1}(l)$ is closed, F satisfies $(*)$, hence for some i, $F=F_i$. Let $\langle x_j, y_{\beta_k}\rangle \in h$ be such that $i<j$, β_k, $x_j \in D(F)$ and $y_{\beta_k} \in R(F)$. If $j \geq k$, then we picked x_j not in $F^{-1}(y_{\beta_k})$, a contradiction. If $j<k$ and k is a successor, then we picked β_k such that $y_{\beta_k} \notin F_i(x_j)$, a contradiction. Suppose $j<k$ and k is a limit ordinal. Let $j<k_1<k$ be a successor ordinal. Then $\beta_k>\beta_{k_1}$ and β_{k_1} is greater than any element of the set $\{\gamma \mid y_\gamma \in F_i(x_j)\}$, hence $\beta_k \notin \{\gamma \mid y_\gamma \in F_i(x_j)\}$, a contradiction. This concludes the proof of the lemma. □

Proof of Lemma 4.5. Lemma 4.5 is very similar to the first part in the proof of Lemma 3.3. We leave it to the reader to translate the first part of the proof of 3.3 to a proof of this lemma.

We next have to deal with the following case. Suppose P_α, C_α have been defined, and the αth task is as follows: we are given a P_α-name of a c.c.c. forcing set R_α of power $\leq\aleph_1$, and we have either to add a generic filter to R_α or to destroy the c.c.c.-ness of R_α. If $\Vdash_{R_\alpha}$ "Y_α does not contain uncountable homogeneous sets", then $P_{\alpha+1}=P_\alpha * R_\alpha$ and $C_{\alpha+1}=C_\alpha$. We deal with the case when there is $r \in R_\alpha$ such that $r \Vdash_{R_\alpha}$ "Y_α contains an uncountable homogeneous set". In this case we will construct a c.c.c. forcing set Q_α such that $\Vdash_{Q_\alpha}$ "R_α is not c.c.c., and Y_α does not contain an uncountable homogeneous set".

Lemma 4.6. *Let $\langle Y, g\rangle$ be an OC pair that does not contain uncountable homogeneous subsets, let R be a c.c.c. forcing set and $r \in R$ be such that $r \Vdash_R$ "Y contains an uncountable homogeneous set". Then there is a c.c.c. forcing set Q of power $\aleph_1$ such that $\Vdash_Q$ "R is not c.c.c. and Y does not contain uncountable homogeneous subsets".*

~~**Remark.** The proof resembles Theorem 2.4 in [1].~~

Proof. Let M be a model with universe $\aleph_1$ encompassing the space Y, the function g and enough set theory. Let $C_M=\{\alpha \mid M \restriction \alpha \prec M\}$. Let $\tilde{B}$ be an R-name, $r \in R$ and $l \in \{0,1\}$ be such that $r \Vdash_R$ "$\tilde{B}$ is an uncountable l-colored subset of Y". W.l.o.g. $r=0_R$ and $l=1$. We choose a sequence $\{\langle r_\alpha, y_\alpha^1, y_\alpha^2\rangle \mid \alpha<\aleph_1\}$ such that: for every α, $r_\alpha \Vdash_R$ "$y_\alpha^1, y_\alpha^2 \in \tilde{B}$"; for every $\alpha<\beta<\aleph_1$, $y_\alpha^1<y_\alpha^2<y_\beta^1$, and

$\{y^1_\alpha, y^2_\alpha, y^1_\beta, y^2_\beta\}$ is C_M-separated. W.l.o.g. there are basic open sets U_1, U_2 of Y such that for every $\alpha<\aleph_1$ and $i\in\{1,2\}$, $y^i_\alpha\in U_i$ and $g(U_1\times U_2)=\{1\}$.

We define Q as follows. $Q=\{\sigma\in P_{\aleph_0}(\aleph_1)\mid$ for every distinct $\alpha,\beta\in\sigma$ there is $i\in\{1,2\}$ such that $g(y^i_\alpha, y^i_\beta)=0\}$. Note that the last clause in the definition of Q is just the 'explicit contradiction' clause, hence if $\alpha,\beta\in\sigma\in Q$ are distinct, then r_α and r_β are incompatible in R. The partial ordering in Q is of course: $\sigma\leqslant\tau$ if $\sigma\subseteq\tau$.

The proof that Q is c.c.c. resembles the analogous proof in Section 3. The argument why $\Vdash_Q$ "R is not c.c.c." is also as in Section 3.

Let us show that $\Vdash_Q$ "Y does not contain uncountable homogeneous subsets". Suppose by contradiction that the above is not true. Then there is a sequence $\{\langle q_\beta, y_\beta\rangle\mid\beta<\aleph_1\}$ and $l\in\{0,1\}$ such that for every $\alpha<\beta<\aleph_1$, $q_\alpha\in Q$, $y_\alpha\neq y_\beta$ and if $q_\alpha\cup q_\beta\in Q$, then $g(y_\alpha, y_\beta)=l$. As usual we uniformize the sequence $\{\langle q_\beta, y_\beta\rangle\mid\beta<\aleph_1\}$ as much as possible, hence we may assume that $q_\beta=\{\alpha_0,\ldots,\alpha_{k-1},\alpha^\beta_k,\ldots,\alpha^\beta_{k-1}\}$ where for every $\beta<\gamma$, $\alpha_0<\cdots<\alpha_{k-1}<\alpha^\beta_k<\cdots<\alpha^\beta_{n-1}<\alpha^\gamma_k$. Let us denote $y^i_{\alpha^\beta_j}$ by $y(\beta,j,i)$, and $y^i_{\alpha_j}$ by $y(j,i)$. Let $\mathbf{y}=\langle y(0,0), y(0,1),\ldots,y(k-1,1)\rangle$, $\mathbf{y}(\beta,j)=\langle y(\beta,j,0), y(\beta,k,1)\rangle$, $\mathbf{y}(\beta)=\mathbf{y}^\frown\mathbf{y}(\beta,k)^\frown\cdots^\frown\mathbf{y}(\beta,n-1)$ and $\mathbf{z}(\beta)=\mathbf{y}(\beta)^\frown\langle y_\beta\rangle$. Recall that if we take two pairs $\mathbf{y}(\beta,i)$ and $\mathbf{y}(\beta,j)$ where $i\neq j$, then either their first or their second coordinates have color 0, i.e. either $g(y(\beta,i,0), y(\beta,j,0))=0$ or $g(y(\beta,i,1), y(\beta,j,1))=0$. Hence by more uniformization we can assume that there are m_{ij}'s for $k\leqslant i<j\leqslant n-1$ such that for every β, i and j, $g(y(\beta,i,m_{ij}), y(\beta,j,m_{ij}))=0$, and that there are basic sets U^m_i, $i=k,\ldots,n-1$, $m=0,1$, such that for every β, $y(\beta,i,m)\in U^m_i$ and for every $k\leqslant i<j\leqslant n-1$, $g(U^{m_{ij}}_i\times U^{m_{ij}}_j)=0$. Let F be the closure of $\{\mathbf{z}(\beta)\mid\beta<\aleph_1\}$ in Y^{n+1}, let $\gamma_0\in C_M$ be such that F is definable in M from a parameter belonging to γ_0. Let β be such that all the elements of $\mathbf{z}(\beta)$ except the first k of them do not belong to γ_0. We duplicate $\mathbf{z}(\beta)$. Note that $\mathbf{y}(\beta)$ is separated. Let E be the C_m-slice to which y_β belongs. Hence there is at most one element of $\mathbf{y}(\beta)$ which belongs to E. To simplify the notation let us assume that this element is $y(\beta,k,0)$. Hence by the duplication argument, and since we know that Y does not contain homogeneous uncountable subsets, we can find $\mathbf{z}^i=\mathbf{y}^\frown\langle y^i(k,0), y^i(k,1),\ldots,y^i(n-1,0), y^i(n-1,1), y^i\rangle\in F$, $i=1,2$, such that $g(y^1,y^2)\neq l$, and $g(y^1(t,j), y^2(t,j))=0$ for $\langle t,j\rangle=\langle k,1\rangle$, $\langle k+1,0\rangle$, $\langle k+1,1\rangle,\ldots,\langle n-1,0\rangle,\langle n-1,1\rangle$. Since g is continuous we can find neighbourhoods V_1, V_2 of $\mathbf{z}^1$, $\mathbf{z}^2$ respectively such that the same equalities hold whenever we pick $\mathbf{z}_1\in V_1$ and $\mathbf{z}_2\in V_2$. Let $\mathbf{z}(\alpha)\in V_1$ and $\mathbf{z}(\beta)\in V_2$ it is easy to check that $q_\alpha\cup q_\beta\in Q$ but $g(y_\alpha, y_\gamma)\neq l$, a contradiction. □

This concludes the proof of Theorem 4.1.

The tail method will be used again in Section 9 and 10. The reader can check by himself that combining the club method, the explicit contradiction method and the tail method one can e.g., get the following consistency result that was mentioned in Section 2.

Theorem 4.7. MA+SOCA+$\exists A$ (*A is increasing and rigid*) *is consistent.*

5. Enlarging the continuum beyond $\aleph_2$

According to our presentation in the previous sections, we always had to assume CH in the ground model in order to apply the club method. Thus in the resulting models of set theory $2^{\aleph_0}$ had to be equal to $\aleph_2$.

The goal of this section is to find a weaker assumption under which the club method can work. Hence we will be able to prove that some of the axioms considered in the previous sections are consistent with $2^{\aleph_0} > \aleph_2$.

Indeed CH was used in more than one way. In Theorem 1.5 we used CH in order to prove the following claim. If $\{F_i \mid i \in I\}$ is a family of closed subsets of X, and X is not the union of countably many F_i's, then there is an uncountable $X' \subseteq X$ which intersects every F_i in at most countably many points. In Section 3 we used CH in order to preassign colors, and in Lemma 4.4 we used CH in still another way.

It so happens that MA+OCA implies $2^{\aleph_0} = \aleph_2$. However, MA+SOCA+$2^{\aleph_0} > \aleph_2$ is consistent. We present the new method by means of an example. We will show that MA+SOCA+$(2^{\aleph_0} > \aleph_2)$ is consistent. The reader will be able to check that the consistency of MA+NWD2+$(2^{\aleph_0} > \aleph_2)$ can be proved by the same method. The proof that MA+OCA$\Rightarrow 2^{\aleph_0} = \aleph_2$ will be presented in Section 11. But some questions remain open, and we will mention them in Section 11.

In view of this section and Section 11, certainly MA+SOCA$\not\Rightarrow$OCA, hence we do not have to prove Lemma 4.4 in the absence of CH, however since the proof exemplifies what can be done without CH we take the liberty to present its short proof. This is done in Lemma 5.5.

Let $A \subseteq B$ denote that $|A - B| \leq \aleph_0$. Let M be a model in a countable language such that $|M| \geq \aleph_1$, and let D be a finite subset of $|M|$; we denote $C_{M,D} \stackrel{\text{def}}{=} \{\alpha \in \aleph_1 \mid \text{there exists } N < M \text{ such that } D \subseteq |N| \text{ and } \alpha = |N| \cap \aleph_1\}$. Clearly $C_{M,D}$ contains a club. A club C of $\aleph_1$ is called M-thin, if for every finite $D \subseteq |M|$, $C \subseteq C_{M,D}$.

Let us reconstruct the proof of Theorem 1.1. The central point in the proof was to construct for a given SOC f of a second countable space X, a forcing set $P_{X,f}$ which adds to X an uncountable 1-colored subset. To do this we constructed a model M which included all the relevant information about X and f, and defined $P_{X,f}$ to be the set of all finite, C_M-separated, 1-colored subsets of X. In the proof that $P_{X,f}$ was c.c.c., the only property of C_M that was used, was its M-thinness.

Let W be a universe in which $2^{\aleph_1} > \aleph_2$. In order to be able to repeat the construction of Theorem 1.1. starting with W as the ground model, we thus need that W will have the following property. If $P \in W$ is a c.c.c. forcing set of power $< 2^{\aleph_1}$, $G \subseteq P$ is a generic filter and $M \in W[G]$ is a model which is constructed for some $\langle X, f \rangle \in W[G]$, then $W[G]$ contains an M-thin club.

We will show that such W's can be constructed, and in fact, the W's that we construct will contain M-thin clubs for a wider set of M's rather than for just those M's that come from some $\langle X, f\rangle$. This fact will be important in other applications of the method.

We define the countable closure of a set A. For $i \leqslant \aleph_1$ we define by induction $A^{(i)}$: $A^{(0)} = A$; if δ is a limit ordinal, then $A^{(\delta)} = \bigcup_{i<\delta} A^{(i)}$; and $A^{(i+1)} = A^{(i)} \cup \{B \subseteq A^{(i)} \mid |B| \leqslant \aleph_0\}$. Let $A^c = A^{(\aleph_1)}$, A^c is called the countable closure of A. For a model M, let M^c be the following model. $|M^c| = |M|^c$, the relations in M^c are those of M, and in addition: the belonging relation on $|M|^c$, and a unary predicate which represents $|M|$ in M^c. A model of the form M^c, where $\|M\| < 2^{\aleph_1}$ and M has a countable language is called a low model.

Axiom A1. If P is a c.c.c. forcing set of power $<2^{\aleph_1}$, and if M is a P-name of a model, such that $\Vdash_P$ "M is a low model", then $\Vdash_P$ "there is an M-thin club".

Proposition 5.1. *Let W be a universe of set theory which satisfies* A1, *then there is a c.c.c. forcing set Q of power $2^{\aleph_1}$ such that $W^Q \models$SOCA.*

To prove the above proposition one has to reexamine the proof of Theorem 1.1. and check the following fact. Let f be a SOC of X, and suppose X does not contain uncountable 0-colored sets. W.l.o.g. $X = \aleph_1$; let $M = \langle \aleph_1 \cup \mathcal{B}; \in, <, f\rangle$ where $\mathcal{B}$ is a countable base for X, $\in$ is the belonging relation between elements of X and elements of $\mathcal{B}$, and $<$ is the ordering relation on $\aleph_1$; and let C be an M^c-thin club. Then if P is the set of all finite C-separated 1-colored subsets of X, then P is c.c.c., and $\Vdash_P$ "X contains an uncountable 1-colored subset". We leave all the other details to the reader.

Our next goal is to construct a W in which $2^{\aleph_1} > \aleph_2$ and which satisfies A1. Let us explain how such a W is constructed. We start with a universe V which satisfies CH. Let λ be a regular cardinal in V such that $\lambda^{\aleph_1} = \lambda$.

We define a countable support iteration $\{\langle P_\alpha, \pi_\alpha\rangle \mid \alpha \leqslant \lambda\}$ in which each π_α is the name of a forcing set which adds a club C to $\aleph_1$, such that C is almost contained in every club which belongs to V^{P_α}. We will show that in V^{P_λ}, $2^{\aleph_1} = \lambda$ and A1 holds.

Let $P_{Cb} = \{\langle D, f\rangle \mid D$ is a closed and bounded subset of $\aleph_1$, F is a club in $\aleph_1$ and $D \subseteq F\}$. let $\langle D_1, F_1\rangle, \langle D_2, F_2\rangle \in P_{Cb}$, then $\langle D_1, F_1\rangle \leqslant \langle D_2, F_2\rangle$ if D_1 is an initial segment of D_2 and $F_2 \subseteq F_1$.

Proposition 5.2 (R. Jensen). (a) *P_{Cb} is ω-closed.*

(b) (CH) *P_{Cb} is $\aleph_2$-c.c.*

(c) (CH) *Let $\langle \{P_{Cb}(\alpha) \mid \alpha \leqslant \lambda\}, \{\pi_{Cb}(\alpha) \mid \alpha < \lambda\}\rangle$ be a countable support iteration; $P_{Cb}(0)$ is a trivial forcing set, and for every $\alpha < \lambda$, $\pi_{Cb}(\alpha)$ is a $P_{Cb}(\alpha)$-name of the forcing set P_{Cb} of the universe $V^{P_{Cb}(\alpha)}$. Then $P_{Cb}(\lambda)$ is ω-closed and $\aleph_s$-c.c., and if $\lambda^{\aleph_1} = \lambda$, then $\Vdash_{P_{Cb}(\lambda)} 2^{\aleph_1} = \lambda$.*

Proof. Well known.

Lemma 5.3. *Let $P, Q \in V$ be forcing sets such that P is c.c.c. in V and Q is ω-closed in V. Then V^P is closed under ω-sequences in $V^{P\times Q}$.*

Proof. It suffices to show that every ω-sequence of ordinals in $V^{P\times Q}$ belongs to V^P. Let τ be a $P\times Q$-name of an ω-sequence of ordinals. We show that for every $q_0 \in Q$ there is $q_1 \geqslant q_0$ with the following property:

(*) For every $p_0 \in P$ and $n \in \omega$ there is $p_1 \geqslant p$ and an ordinal α such that $\langle p_1, q_1\rangle \Vdash \tau(n) = \alpha$. Suppose by contradiction $q^0 \in Q$ and there is no $q_1 \geqslant q^0$ which satisfies (*). We define by induction on i a sequence $\{\langle q_i, p_i, n_i\rangle \mid i < \aleph_1\}$. Let $q_0 \geqslant q^0$, $p_0 \in P$, $n_0 \in \omega$ be such that there is α_0 such that $\langle p_0, q_0\rangle \Vdash \tau(n_0) = \alpha_0$. Suppose $\langle q_j, p_j, n_j\rangle$ has been defined for every $j < i$. Let $q^i \geqslant q_j$ for every $j < i$. Since $q^1 > q^0$, (*) does not hold for q^i, and hence there is $p^i \in P$ and $n_i \in \omega$ such that for every $p \geqslant p^i$ and an ordinal α, $\langle p, q^i\rangle \nVdash \tau(n_i) = \alpha$. Let $\langle p_i, q_i\rangle \geqslant \langle p^i, q^i\rangle$ and α_i be an ordinal such that $\langle p_i, q_i\rangle \Vdash \tau(n_i) = \alpha_i$. Let $i < j$ be such that $n_i = n_j$; we show that p_i and p_j are incompatible. Suppose by contradiction $r \geqslant p_i, p_j$. Hence $\langle r, q_i\rangle \Vdash \tau(n_i) = \alpha_i$. But $q_i \leqslant q^j$, hence $\langle r, q^j\rangle \Vdash \tau(n_i) = \alpha_i$. But $r \geqslant p_j \geqslant p^j$, hence there is $p \geqslant p^j$ and α such that $\langle p, q^j\rangle \Vdash \tau(n_j) = \alpha$. This contradicts the choice of p^j, q^j and n_j. Let n be such that $|\{i \mid n_i = n\}| = \aleph_1$, hence $\{p_i \mid n_i = n\}$ is an uncountable antichain in P, a contradiction.

For a $P\times Q$-name τ of an ω-sequence of ordinals let $D_\tau = \{q \in Q \mid q$ satisfies (*)$\}$. We have thus shown that for every τ as above D_τ is dense in Q.

Let $G \subseteq P\times Q$ be a generic filter, and let $a \in V[G]$ be an ω-sequence of ordinals. Let G_1, G_2 be the restrictions of G to P and Q respectively. We show that $a \in V[G_1]$. Let τ be a $P\times Q$-name of a. Let $q \in D_\tau \cap G_2$. For every $n \in \omega$, let $p_n \in G_1$ and α_n be such that $\langle p_n, q\rangle \Vdash \tau(n) = \alpha_n$. Hence $a = \langle \alpha_n \mid n \in \omega\rangle$ and clearly $a \in V[G_1]$. □

Let $\langle\{P_\alpha \mid \alpha \leqslant \lambda\}, \{\pi_\alpha \mid \alpha < \lambda\}\rangle$ be an iteration of length λ. We denote by $\hat{P}_\beta$ and P_β-name of the iteration which is formed from the sequence of names $\{\pi_\alpha \mid \beta \leqslant \alpha < \lambda\}$. Hence $P_\beta * \hat{P}_\beta \cong P$.

Lemma 5.4. *Let $V \vDash$ CH and let λ be a regular cardinal in V such that $\lambda^{\aleph_1} = \lambda$. Then $\Vdash_{P_{Cb}(\lambda)}$ A1.*

Proof. Let $Q_0 = P_{Cb}(\lambda)$. Let $G \subseteq Q_0$ be a generic filter and $W = V[G]$. Let $P \in W$ be a c.c.c. forcing set of power $< 2^{\aleph_1}$, let $H \subseteq P$ be generic and $U = W[H]$. Let $M \in U$ be a model in a countable language such that $\aleph_1 \subseteq |M|$ and $\|M\| < 2^{\aleph_1}$. Since Q_0 is $\aleph_2$-c.c., $|P| < 2^{\aleph_0}$ and $(2^{\aleph_1})^{(W)} = \lambda$ it follows that for some $\alpha < \lambda$, $P \in V[G \cap P_{Cb}(\alpha)]$. Similarly for some $\alpha \leqslant \beta < \lambda$, $M \in V[G \cap P_{Cb}(\beta)][H]$. Let $G_1 = G \cap P_{Cb}(\beta)$, $V_1 = V(G_1)$ and $Q_1 = V_{G_1}(\hat{P}_{Cb}(\beta))$, and let G_2 be the generic filter of Q_1 determined by G. Hence P is c.c.c. in V_1, Q_1 is ω-closed in V_1, $H \times G_2$ is $P \times Q_1$-generic and $V_1[H][G_2] = U$. By the previous lemma $V_1[H]$ is closed under ω-sequences in U, and thus since $M \in V_1[H]$, also $M^c \in V_1[H]$. Let D be the club of $\aleph_1$ which is added by the restriction of G to $\pi_{Cb}(\beta)$, hence D is almost contained in every club of V_1. However since H is c.c.c. in V_1, every club of $V_1[H]$ contains a club of V_1, and hence D is almost contained in every club of V_1, and hence D is almost contained in every club of $V_1[H]$, and obviously this implies that D is M^c-thin. □

Lemma 5.5. *Lemma 4.4 is true in a universe V^P where $V \models$ A1 and P is a c.c.c. forcing set of power $<2^{\aleph_1}$.*

Proof. Let $\langle X, f\rangle$, $\langle Y, g\rangle$, $\{y_\beta \mid \beta < \aleph_1\}$ be as in 4.4, and let M be a model which encodes X, f, Y, g and $\{y_\beta \mid \beta < \aleph_1\}$. Let C be M^c-thin, and let $\{E_\alpha \mid \alpha < \aleph_1\}$ be an enumeration of the C-slices in an increasing order. Let D be a club in $\aleph_1$ such that $\{\alpha \mid E_\alpha \cap \{y_\gamma \mid \gamma \in D\} = \emptyset\}$ is uncountable, and let $X' \subseteq X$ be an uncountable set such that for every $\alpha < \aleph_1$ if $X' \cap E_\alpha \neq \emptyset$, then $\{y_\gamma \mid \gamma \in D\} \cap E_\alpha = \emptyset$. We show that D and X' are as required. Let $Y' = \{y_\gamma \mid \gamma \in D\}$, and suppose by contradiction for some $h \subseteq X' \times Y'$ and $l \in \{0, 1\}$, h is uncountable and 1-1 and whenever $x_1, x_2 \in X'$ and $f(x_1, x_2) = 1$, then $g(h(x_1, h(x_2)) = l$. Let $F = \mathrm{cl}(h)$. Then F is definable in M^c. Using the notation of 4.4, F satisfies (*). Hence by the proof of 4.4 for all but countably many elements $x \in X'$, $F(x)$ lies in the same C-slice that x does. This contradicts the choice of X' and Y'. □

6. MA, OCA and the embeddability relation on $\aleph_1$-dense real order types

Let $A \subseteq \mathbb{R}$. A is $\aleph_1$-dense if it has no first and no last element, and if between any two members of A there are exactly $\aleph_1$ members of A. If $A, B \subseteq \mathbb{R}$, let $A^* = \{-a \mid a \in A\}$, let $A \cong B$ mean that the structures $\langle A, <\rangle$ and $\langle B, <\rangle$ are isomorphic, let $A \leqslant B$ mean that $\langle A, <\rangle$ is embeddable in $\langle B, <\rangle$, and let $A \perp B$ mean that for no $\aleph_1$-dense $C \subseteq \mathbb{R}: C \leqslant A$ and $C \leqslant B$. $f: A \to B$ is *order preserving* (OP), if for every $a_1, a_2 \in A$, $a_1 < a_2 \Rightarrow f(a_1) < f(a_2)$; it is *order reversing* (OR), if for every $a_1, a_2 \in A$, $a_1 < a_2 \Rightarrow f(a_2) < f(a_1)$; f is *monotonic* if F is either OP or OR. Let $K = \{A \subseteq \mathbb{R} \mid A$ is $\aleph_1$-dense$\}$. $A \subseteq \mathbb{R}$ is *homogeneous*, if for every $a, b \in A$ there is an automorphism f of $\langle A, <\rangle$ such that $f(a) = b$. Let $K^H = \{A \in K \mid A$ is homogeneous$\}$. It follows easily from ZFC that for every $A \in K$ there is $B \in K^H$ such that $A \subseteq B$. Let $\{A_i \mid i < \alpha\} \subseteq K$, and let $B \in K$. We say that B is a *shuffle* of $\{A_i \mid i < \alpha\}$ if there are A_i^1 such that $A_i^1 \cong A_i$, $B = \bigcup_{i<\alpha} A_i^1$ and for every $i < \alpha$ and $b_1, b_2 \in B$ such that $b_1 < b_2$ there is a $a \in A_i^1$ such that $b_1 < a < b_2$.

Let $A \in K$; $B \in K$ is a *mixing* of A if for every rational interval I there is A_I such that $A_I \subseteq I$, $A_I \cong A$ and $B = \bigcup \{A_I \mid I$ is a rational interval$\}$.

Baumgartner [2] proved that it is consistent that all members of K are isomorphic. Shelah [1] invented the club method and used it to show that $\mathrm{MA} + \aleph_1 < 2^{\aleph_0}$ does not imply Baumgartner's axiom (BA). He constructed a universe in which MA holds but $\mathbb{R}$ contains an entangled set. An entangled set A

has the property that there is no uncountable monotonic function g with no fixed points such that Dom(g), Rng(g) $\subseteq A$. Thus A is rigid in a very strong sense. The consistency of MA$+\aleph_1<2^{\aleph_0}$ with the existence of an increasing set was proved by Avraham in [1]. An increasing set is an analogue of an entangled set when monotonic functions are replaced by OR functions.

It was natural to ask how much freedom do we have in determining the structure of the category whose members are the elements of K and whose morphisms are the monotonic functions. In this section we start investigating such questions under the assumption of MA$+\aleph_1<2^{\aleph_0}$.

We shall first see that MA$_{\aleph_1}$ already implies many properties of K. Next we shall see that MA$_{\aleph_1}$+OCA determines K quite completely, namely if we conjunct MA$_{\aleph_1}$+OCA with the existence of an increasing set, then K^H consist up to isomorphism of three elements and every element of K is built from these elements in a simple way. On the other hand MA$_{\aleph_1}$+OCA+$\neg$ISA implies Baumgartner's axiom.

Theorem 6.1. (MA$_{\aleph_1}$). (a) *If $A\in K^H$, $a_1, b_1, a_2, b_2 \in A$, and $a_1<b_1$ and $a_2<b_2$, then there is an automorphism f of $\langle A,<\rangle$ such that $f(a_1)=a_2$ and $f(b_1)=b_2$.*

(b) *Let $A, B\in K$ and let $\{g_i \mid i\in\omega\}$ be a family of OP functions such that for every $a\in A$ and $b_1<b_2$ in B there is $i\in\omega$ such that $g_i\in B\cap(b_1, b_2)$. Then $A\preccurlyeq B$.*

(c) *Let A, B and $\{g_i \mid i\in\omega\}$ be as in* (b), *and suppose in addition that for every $b\in B$ and $a_1<a_2$ in A there is $i\in\omega$ such that $g_i^{-1}(b)\in A\cap(a_1, a_2)$. Then $A\cong B$.*

(d) *If $A, B\in K^H$, $A\preccurlyeq B$ and $B\preccurlyeq A$, then $A\cong B$. (Hence $\preccurlyeq$ is a partial ordering of $K^H/\cong$.)*

(e) *If $A\in K^H$, then A is isomorphic to every non-empty open interval of A.*

(f) *Let $\{A_i \mid i<\alpha\leqslant\omega\}\subseteq K^H$. Then* (a) *all shuffles of $\{A_i \mid i<\alpha\}$ are isomorphic and they belong to K^H. In particular,if all the A_i's are isomorphic to some fixed A, then every shuffle of $\{A_i \mid i<\alpha\}$ is isomorphic to A; and* (b) *if B is a shuffle of $\{A_i \mid i<\alpha\}$ and $C\in K^H$ and for every $i<\alpha$, $A_i\preccurlyeq C$, then $B\preccurlyeq C$.*

(g) *If $A\in K$ and B_1, B_2 are mixings of A, then $B_1\cong B_2$ and $B_1\in K^H$, and if $C\in K^H$ and $A\preccurlyeq C$, then $B_1\preccurlyeq C$.*

(h) *If $A\in K$ and for every $B\in K$ $A\preccurlyeq B$, then $A\in K^H$.*

(i) *If for every $A, B\in K$ $A\preccurlyeq B$, then* BA *holds.*

(j) *If $|K^H/\cong|=1$, then* BA *holds.*

Proof. All parts of 6.1 follow easily from (b) and (c). We prove (c), (a) and (j), and leave the other parts to the reader.

(c) Let $P=\{f\in P_{\aleph_0}(A\times B) \mid f$ is OP, and for every $a\in\mathrm{Dom}(f)$ there is g_i such that $f(a)=g_i(a)\}$. $f\leqslant g$ if $f\subseteq g$. It is easy to see that P is c.c.c.. It is also easy to see that for every $a\in A$, $D_a\overset{\text{def}}{=}\{f\in P \mid a\in\mathrm{Dom}(f)\}$ is dense in P, and for every $b\in B$, $D^b\overset{\text{def}}{=}\{f\in P \mid b\in\mathrm{Rng}(f)\}$ is dense in P, hence if G is a filter in P which intersects all D_a's and D^b's, then $\bigcup\{f \mid f\in G\}$ is an isomorphism between A and B.