Concistency with CH of a conbinatorial principle that implies there are no Suslin trees and all (ω_1, ω_1)-gaps are Hausdorff

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Abstract

A combinatorial statement concerning nice sequences on ω_1 is introduced and proved to be consistent with the continuum hypothesis. This statement implies that there are no Souslin trees, all (ω_1, ω_1)-gaps are Hausdorff, and every coherent sequence on ω_1 can be uniformized.

1 The principle

Our aim is to introduce a combinatorial statement, to prove its consistency with CH, and to give three examples of its use. In order to formulate this principle, let us first make a definition:

1. Let $I \subseteq \omega_1$ be an unbounded set of indices. A doubly indexed sequence $A =< A^n_\alpha | \alpha \in I, n \in \omega >$ is "nice" (over I) iff:

   (a) $A^n_\alpha \subseteq A^{n+1}_\alpha \subseteq \alpha$ for all $\alpha \in I$ and $n \in \omega$, and $\bigcup\{A^n_\alpha | n \in \omega\} = \alpha$.

(b) For every $\alpha < \beta$ in I and $n \in \omega$, there is $k \in \omega$ such that

$$A_\alpha^k \supseteq A_\beta^n \cap \alpha.$$

2. Given a nice sequence A as above, we say that $X \subseteq \omega_1$ has the *finite intersection property* with respect to A iff

$$\forall \beta \in I \; \forall n \in \omega \; X \cap A_\beta^n \text{ is } finite.$$

If, on the other hand, $X \subseteq \omega_1$ is such that

$$\forall \beta \in I \exists n \in \omega \;\; X \cap \beta \subseteq A_\beta^n,$$

then we say that X is *in* $\overline{A}$.

The combinatorial principle that will interest us in this paper is the following statement:

($\star$) For every nice sequence on ω_1, $\overline{A}$, either $\omega_1 = \bigcup_{i \in \omega} X_i$, where each X_i is in $\overline{A}$,
or
$\exists X \in [\omega_1]^{\aleph_1}$ with the finite intersection property with respect to $\overline{A}$.

($[\omega_1]^{\aleph_1}$ is the collection of uncounatble subsets of ω_1.)

2 Consequences of $\star$

We will derive three consequences of ($\star$), and in the following section the consistency of ($\star$) with CH will be proved.

2.1 No souslin trees

Theorem 2.1 *If ($\star$) holds, then every ω_1-tree is either a union of countably many chains, or else it contains an uncountable antichain.*

Let T be an ω_1-tree (height is ω_1, and every level, T_α, is countable). We may assume that every node in T has extensions to any higher level without restricting the theorem. For simplicity, assume that the universe of T is ω_1, so that $T \restriction \alpha = \alpha$ for a closed unbounded set $I \subseteq \omega_1$. Let $\{b_\alpha^n | n \in \omega\}$ be an enumeration of each T_α.

Define a nice sequence by setting, for $\alpha \in I$,

$$A^n_\alpha = \{x \in T \restriction \alpha \mid x <_T b^m_\alpha \text{ forsome } m \leq n\}$$

The relation $x <_T b^m_\alpha$ is of course the tree ordering, and A^n_α is thus the union of the first n branches of $T \restriction \alpha$, determined by nodes of T_α. It is immediate to check that $A = \langle A^n_\alpha | \alpha \in I, n \in \omega\rangle$ is a nice sequence, and so by $\star$ either ω_1 is a union of countably many sets in A (in which case T will be shown to be a union of countably many branches) or there is an uncounatble $X \subseteq T$ with the finite intersection property (and then an uncounatble antichain is derived).

Indeed, suppose that $X \subseteq T$ is uncountable and *in* A.

$$\forall \alpha \in I \, \exists n \in \omega \; X \cap \alpha \subseteq A^n_\alpha.$$

Then, for every α, $X \cap T \restriction \alpha$ is contained in a finite number of branches, and there is even a fixed n such that every $X \cap T \restriction \alpha$ is contained in n branches. Then X itself is contained in n branches.

Now if $X \subseteq T$ is uncounatble and with the finite intersection property, then, for every $t \in T, \{x \in X | x <_T t\}$ is finite. Let m be such that for uncountably many points $t \in X, \{x \in X | x <_T t\}$ has size m. Any two such points in X are incompatible, and hence T contains an uncountable antichain.

2.2 Hausdorff gaps

The partial order, $\subseteq^*$, of eventual inclusion is defined on $P(\omega)$ by setting $a \subseteq^* b$ iff $a \setminus b$ is finite. We write $a \subset^* b$ if $a \subseteq^* b$ and $b \setminus a$ is infinite. When $a \subseteq^* b$, let $e(a, b)$ be the least n such that $a \setminus b \subseteq n$.

For an uncountable set of indices $I \subseteq \omega_1$, a double sequence $(\overline{a}, \overline{b}) = \langle a_\zeta, b_\zeta | \zeta \in I\rangle$ of subsets of ω is a *pregap* iff

$$a_\zeta \subseteq^* a_\xi \subseteq^* b_\xi \subseteq^* b_\zeta \text{ for every } \zeta < \xi \text{ in } I.$$

The pregap is a *gap* iff there is no x such that $a_\xi \subseteq^* x \subseteq^* b_\xi$ for every $\xi \in I$. Following Scheepers [Sch], we say that the pregap $(\overline{a}, \overline{b})$ is a *Hausdorff* pregap (or gap) iff for every $\alpha \in A$ and $n \in \omega$

$$\{\zeta < \alpha | \; e(a_\zeta, b_\alpha) \leq n\} \text{ is finite.}$$

Every (ω_1, ω_1) Hausdorff pregap is a gap, and it is by constructing Hausdorff gaps that (ω_1, ω_1)-gaps are shown to exist (in ZFC). If $\diamondsuit_{\omega_1}$ holds, then there are gaps that contain no Hausdorff subgaps.

Theorem 2.2 *If $(\star)$ holds, then every (ω_1, ω_1)-gap contains a Hausdorff subgap.*

For any pregap $(\overline{a}, \overline{b})$ we will define a nice sequence: For every α and n let

$$A^n_\alpha = \{\zeta < \alpha | \ e(a_\zeta, b_\alpha) \leq n\}.$$

That is, A^n_α is the set of those $\zeta < \alpha$ for which $a_\zeta \setminus b_\alpha \subseteq n$. So that a pregap is Hausdorff iff the sets A^n_α are all finite.

It is easy to see that, for any pregap, the A^n_α sets form a nice sequence, A, and so by $(\star)$ there is an uncountable X which is in A, or else an uncountable X with the finite intersection property. We will prove that the first possibility implies that $(\overline{a}, \overline{b})$ is not a gap, and show that the second implies that $(\overline{a}, \overline{b})$ contains a Hausdorf subgap.

Begin with the first alternative and suppose that for every $\alpha \in \omega_1$, for some $n(\alpha) \in \omega$ $X \bigcap \alpha \subseteq A^{n(\alpha)}_\alpha$. Find then an uncountable $I \subseteq \omega_1$ and a fixed n such that $n(\alpha) = n$ for all $\alpha \in I$. Then set $a = \bigcup \{a_\zeta \setminus n \mid \zeta \in X\}$. Obviously, $a_\zeta \subseteq^* a$ for *all* $\zeta \in \omega_1$. But also $a \subseteq^* b_\alpha$ for all $\alpha \in \omega_1$. To see this, pick a countable $X_0 \subseteq X$ such that $a = \bigcup \{a_\zeta \setminus n \mid \zeta \in X_0\}$. Then, for every $\alpha \in I$ such that $\alpha \geq \max X_0$, $\ a \subseteq b_\alpha$. Thus $a \subseteq^* b_\alpha$ for all $\alpha \in \omega_1$.

Thus, the first possibility implies that $(\overline{a}, \overline{b})$ is not a gap. Hencefor we assume that the second alternative holds, that is, to say, there is an uncountable $X \subseteq \omega_1$ such that $X \bigcap A^n_\alpha$ is finite for every α and n. This means exactly that the restriction of $(\overline{a}, \overline{b})$ to X is Hausdorff.

In a conversation, S. Shelah pointed to a stronger property, that of a Hausdorff* gap. We define $e^*(a, b) = |a \setminus b|$. Thus $e^*(a, b) \leq e(a, b)$. A pregap is defined to be Hausdorff* iff for very α and n

$$\{\zeta < \alpha \mid e^*(a_\zeta, b_\alpha) \leq n\} \text{ is finite.}$$

The proof that $(\star)$ implies that every gap is Hausdorff* is very similar, with e^* replacing e. The definition of the sequence A^n_α and the proof that it is nice is as before, but the proof that there is no uncountable $X \subseteq \omega_1$ in A, if A is a gap, is somewhat different: So assume that there is an uncountable $X \subseteq \omega_1$ in A. Then for some $n \in \omega$, for some uncountable $I \subseteq \omega_1$,

$$\forall \alpha \in I,\ X \bigcap \alpha \subseteq A^n_\alpha. \tag{1}$$

That is, for $\zeta \in X$ and $\alpha \in I$, $\zeta < \alpha$ implies $|a_\xi \setminus b_\alpha| \leq n$. We may assume that n is minimal with respect to that property, and so there is α_0 such that for every $\alpha \in I \setminus \alpha_0$ there is $\mu \in X \bigcap \alpha$ with $|a_\mu \setminus b_\alpha| = n$.

For every finite $d \subseteq \omega$ let $X_d = \{\zeta \in X |\ d \subseteq a_\zeta\}$. By throwing away countably many indices from X we may assume that every X_d is either empty or uncountable. Likewise, we may assume that if $d \cap b_\alpha = \emptyset$ for some α in I, then $d \cap b_\alpha = \emptyset$ for uncountably many such α's. Let $\alpha \in I$, $\alpha \geq \alpha_0$, be large enough so that, for every d, $X_d \neq \emptyset \Longrightarrow X_d \bigcap \alpha \neq \emptyset$. Pick $\mu \in X \bigcap \alpha$ such that $d = a_\mu \setminus b_\alpha$ has size n. Since $d \subseteq a_\mu, X_d$ is uncountable. Set $a = \bigcup\{a_\xi \setminus d \mid \xi \in X_d \bigcap \alpha\}$. We will show that a is a buffer for A.

We claim first that $a = \bigcup\{a_\xi \setminus d | \xi \in X_d\}$ (and this implies that $a_\xi \subseteq^* a$ for *every* $\xi \in \omega_1$). Indeed, if $\xi \in X_d$ and $m \in a_\xi \setminus d$, then $d' = d \cup \{m\} \subseteq a_\xi$ (and $X_{d'} \subseteq X_d$ is uncountable), and so $X_{d'} \bigcap \alpha \neq \emptyset$ which implies $m \in a$.

Since $d \cap b_\alpha = \emptyset$, there are uncountably many $\beta \in I$ such that $d \cap b_\beta = \emptyset$. We claim now that for any such β above α, $a \subseteq b_\beta$. This shows that $a \subseteq^* b_\mu$ for every $\mu < \omega_1$. This is so because if $m \in a \setminus b_\beta$, then $m \in a_\xi \setminus d$ for some $\xi \in X_d \bigcap \alpha$, and so $d \cup \{m\} \subseteq a_\xi \setminus b_\beta$, which comtradicts $|a_\zeta \setminus b_\beta| \leq n$.

2.3 Coherent ω_1-sequences

A sequence $E = \langle E_\alpha | \alpha \in \omega_1 \rangle$ with $E_\alpha \subseteq \alpha$ for every $\alpha \in \omega_1$ is said to be *coherent* iff $E_\alpha =^* E_\beta \bigcap \alpha$ whenever $\alpha < \beta$. ($x =^* y$ means that the symmetric difference is finite.) If $X \subseteq \omega_1$ is such that $X \bigcap \alpha \subseteq^* E_\alpha$ for every $\alpha \in \omega_1$, then we say that X *uniformizes* E (and then, if X is uncountable , we say that E is *uniformizable*). (By coherence, it suffice to require that $X \bigcap \alpha \subseteq^* E_\alpha$ for uncountably many α's, to conclude that X uniformizes E.)

If E is a coherent sequence, then the *complement* of E, defined by

$$C(E) = \langle \alpha \setminus E_\alpha | \alpha \in \omega_1 \rangle,$$

is coherent as well.

We will see now that $(\star)$ implies that every coherent sequence, or its complement, is uniformizable (by an uncountable set). In fact, $(\star)$ implies that, for every coherent sequence E, either there is a decomposition $\omega_1 = \bigcup_n X_n$ such that each X_n uniformizes E, or else $C(E)$ is uniformizable.

To see this, enumerate each $\alpha < \omega_1$ as an ω sequence, $\alpha = \{x_i | i \in \omega\}$, and define $A^n_\alpha = E_\alpha \bigcup \{x_i | i < n\}$. Then the sets A^n_α form a nice sequence, and by $(\star)$ there are two alternatives. Observe that if X is in A (i.e., for every α, for some n, $X \cap \alpha \subseteq A^n_\alpha$), if and only if X uniformizes E. While if X has the finite intersection property with respect to A, then $X \bigcap E_\alpha$ is finite and thus X uniformizes $C(E)$.

A. Dow was interested in a somewhat stronger statement, (which was proved to follow from PFA by S. Todorcevic):

> There is a uniform filter U on ω_1 (which means that if $X \in U$ then $X \subseteq \omega_1$ is uncountable) such that for every coherent sequence E, either E or $C(E)$ is uniformizable by a subset of U.

To get the consistency with CH of this statement we strengthen $(\star)$: the following will be called $(\star\star)$

> For every collection $\{F_i \mid i \in \omega_1\}$, of subsets of ω_1, and nice sequence A, either for some i, F_i is a countable union of sets that are in A, or there is an uncountable X that has the finite intersection property for A and is such that $X \bigcap F_i$ is uncountable for every $i \in \omega_1$.

The proof of the consistency with CH of $(\star)$, will actually give $(\star\star)$ and $2^{\aleph_1} = \aleph_2$. Assuming this, the construction of the ultrafilter U is done in ω_2 steps, dealing with each coherent sequence E in its turn. Suppose the sets F_ζ, $\zeta < \tau$, for $\tau < \omega_2$, are already decided to be in U (and so the intersection of any finite collection of F'_ζ's is uncountable). Suppose also that E is a coherent sequence that presents itself at this stage of the construction. If for two ordinals $\zeta_0, \zeta_1 < \tau$, F_{ζ_0} is a countable union of sets that uniformizes E, and F_{ζ_1} is a countable union of sets that uniformizes $C(E)$, then $F_{\zeta_0} \bigcap F_{\zeta_1}$ contains an uncountable set that unifirmizes both E and $C(E)$. But no infinite set can do that. Thus, for one of the sequences E or $C(E)$, no F_ζ is a countable union of sets that uniformize that sequence. Hence the alternative given by $(\star\star)$ is a set which can be added to the filter, and which has the finite intersection property with respect to A or to $C(A)$. Such a set uniformizes $C(A)$ or A.

The question about the consistency of *for every coherent sequence E there is an uncountable uniformization of either (E) or $C(E)$* was asked by F. Galvin in somewhat different terms. He proves that CH negates this statement, and S. Todorcevic proved its consistency with $MA + 2^{\aleph_0} = \aleph_2$. The notion of coherent sequence is studied by A. Dow who applies it to study the space of ultrafilters on ω_1.

3 Consistency of $(\star)$ with CH

In this section we use forcing to prove $con(ZFC) \Longrightarrow con(ZFC+CH+(\star))$. Starting with a universe V in which $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, we iterate ω_2 posets taking the countable support iteration. The iteration satisfies the $\aleph_2$ chain condition and adds no new reals. Dealing with all the nice sequences that are formed along the iteration ensures that $(\star)$ holds finally. Only the basic step of the iteration is described, since the mechanizm for dealing with all the possible nice sequences is standard (the $\aleph_2$-c.c. is used). We will show that for any nice sequence A either there is a decomposition $\omega_1 = \bigcup_{i\in\omega} X_i$ with each X_i in A, or else there is a poset $P = P(A)$ such that:

1. in V^P there is an uncountable $X \subseteq \omega_1$ with the finite intersection property with respect to A.

2. The poset P is of size $2^{\aleph_1}$ and (assuming CH) satisfies the $\aleph_2$-p.i. condition.

3. P is α-proper for every $\alpha < \omega_1$, and is $\aleph_1$-Dee-complete.

For completness, the definition of Dee-completness will be repeated below. We use Shelah's theorems ([] chapter) saying that (1) assuming CH, the iteration with countable support of proper forcing posets that satisfy the $\aleph_2$-p.i.c. satisfies the $\aleph_2$-c.c., and (2) the iteration of Dee-complete posets that are α-proper for every $\alpha < \omega_1$ adds no new reals.

Thus if we start with a universe V that satisfies $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ and iterate with countable support ω_2 posets of type $P(A)$ for all possible nice sequences A, then the resulting extension satisfies $(\star)$ (and even $(\star\star)$).

So let $A = \langle A^n_\alpha | \alpha \in \omega_1,\ n \in \omega\rangle$ be a nice sequence such that ω_1 is not a countable union of sets that are in A. Then the poset $P = P(A)$ defined below introduces a generic uncountable $X \subseteq \omega_1$ with the finite intersection property with respect to A. Conditions in P are pairs $p = \langle x, I\rangle$ (denoted $\langle x^p, I^p\rangle$) that satisfy:

1. $x : \delta \to \{0, 1\}$, where δ is a countable ordinal (denoted $\delta(p)$). (The set $\{\zeta | x(\zeta) = 1\}$ is denoted $[x]$, or $[p]$, and it will generically develope the uncountable X with the finite intersection property).

2. For every $\gamma \leq \delta$ and $n \in \omega$,

$$[x] \bigcap A^n_\gamma \text{ is finite.}$$

3. I is a countable set, its members are called "promises". Each promise $C \in I$ is a pair $C = (X, n)$ where $X \in [\omega_1]^{\aleph_1}$ and $n \in \omega$. (We say that the *promise is in* p if it is in I^p.)

The role of the promises will become apparent later, they limit the possible extension of a condition and thus avoid bad things.

The ordering $p \leq q$ (q extends p) is defined when

1. $x^p \subseteq x^q$, and $I^p \subseteq I^q$.

2. For every promise (X, n) in p there are uncountably many indices $\alpha \in X$ such that
$$([q] \setminus [p]) \bigcap A^n_\alpha = \emptyset.$$
We will introduce a notation at this point to allow some ease of expression for the last requirement. For $E \subseteq \omega_1$ and a promise (X, n), define
$$D(E, X, n) = \{\alpha \in X | E \bigcap A^n_\alpha = \emptyset\}.$$
So condition 2 says, in other words, that for every $(X, n) \in I^p$, $D([q] \setminus [p], X, n)$ is uncountable.

3. The following strengthening of (2) is required: For every $(X, n) \in I^p$, not only that $X_1 = D([q] \setminus [p],\ X, n)$ is uncountable , but the pair (X_1, n) is even in I^q.

The relation $p \leq q$ is transitive (but not necessarily separative). In order to show that the generic filter gives an *uncountable* set (with the finite intersection property) we need the following lemma.

Lemma 3.1 *If ω_1 is not a countable union of sets in A, then for every $p \in P$ and $\mu \in \omega_1$, there is $q \geq p$ such that $[q] \setminus \mu \neq \emptyset$.*

proof. Let $p = \langle x^p, I^p \rangle \in P$ and $\mu \in \omega_1$ be given. For any countable $\alpha \geq \mu$ try to form a condition $p_\alpha = p + \{\alpha\}$ by adding α to $[p]$ in the manner described below. Then, assuming that we always fail and $p_\alpha \notin P$ for all α's, ω_1 is shown to be a union of countably many sets that are in A. Well, put $p_\alpha = (x', I')$ where $dom(x') = \alpha + 1$ and $[x'] = [x] \bigcup \{\alpha\}$, and $I' = I^P \bigcup \{(X', n)|$ for some $(X, n) \in I^p, X' = D(\{\alpha\}, X, n)\}$. If $p_\alpha \in P$ then $p_\alpha \geq p$ is as required. The only possible reason for $p_\alpha \notin P$ is that for some $(X, n) \in I^p$, $D(\{\alpha\}, X, n)$ is countable. Since I^p is countable, this association of α with such $(X, n) \in I^p$ is a partion of $\omega_1 \setminus \mu$ into countably many pieces. If $Y \subseteq \omega_1$ is such a piece, associated with $(X, n) \in I^p$, then for every $\alpha \in Y$ there is $\chi(\alpha) < \omega_1$ such that for all $\xi \in X \setminus \chi(\alpha)$, $\alpha \in A^n_\xi$. This implies that Y is in A.

Lemma 3.2 *P is proper: If κ is a large enough cardinal and $M \prec H_\kappa$ is countable (H_κ is the collection of all sets of cardinality heriditarily less than κ), and $p_0 \in P \bigcap M$ is given, then there is an extension $q \geq p_0$ which is M-generic. (That is, for every $q_1 \geq q$ and $D \in M$ dense in P, q_1 is compatible with some condition in $D \bigcap M$.)*

Proof. Say $p_0 = \langle x_0, I_0 \rangle$, $\delta = M \bigcap \omega_1$, and let $\langle D_m | m \in \omega \rangle$ be an enumeration of all dense sets of P that are in M. We will define an increasing sequence of conditions $p_i \in P \bigcap M$, for $i \in \omega$, such that $p_{i+1} \in D_i$, and such that some q in P extends all the p_i's. Such a condition q is obviously M-generic, and in fact this shows that P adds no new reals. In defining this increasing sequence we set

$$p_i = \langle x_i, I_i cw \rangle, \ \ x_\omega = \bigcup_{i \in \omega} x_i,$$

and we must take care of the following two requirements:

1. $[x_\omega] \bigcap A^n_\delta$ is finite for every $n \in \omega$,

2. If $(X,n)\in I_i$ then there are $\aleph_1$ α's in X such that $([x_\omega]\setminus[x_i])\bigcap A^n_\alpha=\emptyset$.

If we succeed to construct such a sequence, then $q=(x_\omega,I)$ can be defined by setting

$$I=(\bigcup_{i\in\omega}I_i)\cup\bigcup\{D([x_\omega]\setminus[x_i],X,n)|\langle X,n\rangle\in I_i,i\in\omega\}.$$

Then (1) and (2) imply that $q\in P$ extends each p_i and is thus M-generic.

To ensure (1) and (2),we will define an increasing sequence of integers $\langle k(i)|i\in\omega\rangle$ (with $k(0)=0$) and require that

$$([p_{i+1}]\setminus[p_i])\bigcap A^{k(i)}_\delta=\emptyset.$$

This takes care of (1): Because then, since $i<j$ implies $k(i)\leq k(j)$, $[x_\omega]\bigcap A^i_\delta\subseteq[x_i]\bigcap A^{k(i)}_\delta$ is finite. Now we explain why, if the $k(i)$ sequence grows "fast", then (2) is assured.

For every promise $R=\langle X,n\rangle$ in M define $K(R)$ to be the least $k\in\omega$ such that there are uncountably many $\alpha\in X$ for which $A^n_\alpha\cap\delta\subseteq A^k_\delta$ (recall that, for every $\alpha\in X\setminus\delta$, for some k, $A^n_\alpha\bigcap\delta\subseteq A^k_\delta$).

Now assume some enumeration device that gives, for every $i\in\omega$ and promise $\langle X,n\rangle$ in p_i, some $j>i$ such that $\langle X,n\rangle$ is presented at the jth stage of the contraction, the stage at which p_{j+1} is defined.

Then $k(j)$ is defined to ensure (2) for $\langle X,n\rangle$. Specifically, we first pick out $R=\langle X_1,n\rangle$ defined by $X_1=D([p_j]\setminus[p_i],X,n)$ (as $p_i\leq p_j$, X_1 is uncountable , and $R\in I_j$). Then $K(R)$ has the property that the set $X_2\subseteq X_1$, of those $\alpha\in X_1$ for which $A^n_\alpha\bigcap\delta\subseteq A^{K(R)}_\delta$, is uncountable . We set $k(j)$ bigger than $\max\{K(R),k(j-1)\}$. This ensures that $([x_\omega]\setminus[x_j])\bigcap A^{K(R)}_\delta=\emptyset$, and hence that $([x_\omega]\setminus[x_j])\bigcap A^n_\alpha=\emptyset$ for every $\alpha\in X_2$.

The question is how to satisfy the two seemingly contradictory demands:

1. $p_{i+1}\in D_i\bigcap M$,

2. $([p_{i+1}]\setminus[p_i])\bigcap A^{k(i)}_\delta=\emptyset$?

For this we have the following main lemma.

Lemma 3.3 *Suppose $M\prec H_k$ is countable as above, $\delta=\omega_1\bigcap M$, $p_0\in P\bigcap M$, and $D\in M$ dense open in P. Given $k\in\omega$, there is an extension $p_1\geq p_0$, $p_1\in D\bigcap M$, such that $([p_1]\setminus[p_0])\bigcap A^k_\delta=\emptyset$. In fact, suppose that $A\subseteq\delta$ is such that for every $\delta'<\delta$ there is $n\in\omega$ such that $A\bigcap\delta'\subseteq A^n_{\delta'}$, then there is $p_1\geq p_0$ in $D\bigcap M$ such that $([p_1]\setminus[p_0])\bigcap A=\emptyset$.*

Proof. Suppose this is not the case, and, whenever $p_1 \geq p_0$ is in $D \bigcap M$, then $([p_1] \setminus [p_0]) \bigcap A \neq \emptyset$. For any ordinal δ' such that $\delta(p_0) \leq \delta' < \delta$, let n be such that $A \bigcap \delta' \subseteq A^n_{\delta'}$. Then $n =$ satisfies the following:

$$\text{For every } p_1 \geq p_0,\ p_1 \in D \bigcap N,\ \text{if } \delta(p_1) \leq \delta',\ \text{then } ([p_1] \setminus [p_0]) \bigcap A^n_{\delta'} \neq \emptyset. \tag{2}$$

Since (2) involves only parameters in M (A does not appear there), there is a function L in M such that, for every δ' with $\delta(p_0) \leq \delta' < \delta$, $n = L(\delta')$ satisfies (2).

Hence (in H_k), the function L is defined on all of ω_1 and it satisfies the following, for every δ' with $\delta(p_0) \leq \delta' < \omega_1$,

$$\text{For every } p_1 \geq p_0,\ \text{if } p_1 \in D \text{ and } \delta(p_1) \leq \delta',\ \text{then } ([p_1] \setminus [p_0]) \bigcap A^{L(\delta')}_{\delta'} \neq \emptyset. \tag{3}$$

For some fixed n_0, for an uncountable $X \subseteq \omega_1$, $L(\delta') = n_0$ for all $\delta' \in X$. Set $p^* = (x^{p_0}, I^{p_0} \bigcup \{\langle X, n_0 \rangle\})$. Then $p^* \in P$, and trivially $p^* \geq p_0$. There is some $p_1 \geq p^*$ such that $p_1 \in D$. By definition of extension (and as $x^{p_0} = x^{p^*}$), there are uncountably many $\alpha \in X$ such that $([p_1] \setminus [p_0]) \bigcap A^{n_0}_\alpha = \emptyset$. Pick any such $\alpha \in X$ with $\delta(p_1) \leq \alpha$. As $L(\alpha) = n_0$, (3) implies that $([p_1] \setminus [p_0]) \bigcap A^{n_0}_\alpha \neq \emptyset$, which is a contradiction, and P is thus proper.

We can modify the proof to obtain that P is α-proper for every $\alpha < \omega_1$. This means the following:

> suppose $M_i \prec H_k$, for $i < \alpha < \omega_1$ are countable elementary submodels with $\langle M_i | i \leq i_0 \rangle \in M_{i_0+1}$ for every $i_0 < \alpha$, and $M_\delta = \bigcup_{i<\delta} M_i$ for every limit $\delta \leq \alpha$. (Such a sequence will be called an α-tower.) Let $p_0 \in P \bigcap M_0$ be given. Then there is $q \geq p_0$ which is M_i-generic for all $i < \alpha$.

Observe that if δ is a limit and q is M_i-generic for every $i < \delta$, then q is M_δ-generic. The proof that the poset P that we constructed is α-proper is by induction on α, and the following stronger property is the inductive statement:

> *If $\langle M_i | i \leq \alpha \rangle$ is a tower of countable elementary submodels as above, of length $\alpha < \omega_1$, and $p_0 \in P \bigcap M_0$, then for every $k \in \omega$ there is $q \geq p_0$ which is M_i-generic for every $i \leq \alpha$ and such that $([q]) \setminus [p_0]) \bigcap A^k_\delta = \emptyset$ (where $\delta = M_\alpha \bigcap \omega_1$.)*

We have just seen the case $\alpha = 1$, and the successor case is done similary, using the inductive assumption. If α is a limit ordinal, pick an increasing sequence $\langle \alpha_i | i \in \omega \rangle$ with $\sup\{\alpha_i \mid i \in \omega\} = \alpha$. Set $\delta_i = M_{\alpha_i} \bigcap \omega_1$. We will define an increasing sequence p_i, $i \in \omega$, where $p_{i+1} \in M_{\alpha_i+1}$ is M_μ-generic for every $\mu \leq \alpha_i$, and $\delta(p_{i+1}) = \delta_i$. Thus, if the sequence $\langle p_i \mid i \in \omega \rangle$ has a supremum q in p, then q is M_i-generic for every $\mu \leq \alpha$. Just as in the proof of properness, we define for this aim an increasing sequence of integers $\langle k(i) | i \in \omega \rangle$ and demand that

$$([p_{i+1}] \setminus [p_i]) \bigcap A_\delta^{k(i)} = \emptyset.$$

This p_{i+1} can be obtained by aplying the inductive assumption to p_i and to $A^k_{\delta_i}$ chosen so that $A_\delta^{k(i)} \bigcap \delta_i \subseteq A^k_{\delta_i}$.

We are going to prove now that P is simply $\aleph_1$-Dee-complete, a property which, together with α-properness ensures that the countable support iteration adds no new reals. For completeness we shall review the relevant definitions that lead to this property (introduced in Shelah []).

If N is a countable $\in$-structure, P, $p_0 \in N$ and $N \models$ "P is a poset and $p_0 \in P''$, then $Gen(N; p, p_0)$ denotes the set of all N-generic filters over P that contain p_0. That is $G \in \ Gen(N; P, p_0)$ iff $G \subseteq P \bigcap N$, $\ p_0 \in G$, and, for every $D \in N$ dense open in P (in N), $G \bigcap D \neq \emptyset$.

We are going to use second-order formules; these have two types of variables, class variables and sets (elements) variables. If $\psi(X_1, X_2; y_1, y_2, y_3)$ is a second-order formula (in the $\in$ language) the semicolumn seperates the class variables X_1, X_2 from the set variables y_1, y_2, y_3.

A (simple) Dee-system for a poset P is given by a second-order formula ψ as above, and a parameter $a \in H_k$ such that whenever $N \prec H_k$ is countable, with $a, P, p_0 \in N$, and $p_0 \in P$, then for every $X \subseteq N$, if $G \subseteq N$ is such that

$$N \models \psi(G, X; P, p_0, a),$$

then $G \in \ Gen(N; P, p_0)$. Intuitively, then, a Dee-system assigns to every class $X \subseteq N$ a collection of N-generic filters "constructed with respect to X", namely, the collection of all those G's such that N satisfies the formula ψ.

We say that the Dee-system as above is $\aleph_1$-complete (or countably complete) iff for every countable $N \prec H_k$ and $p_0 \in P$ as above, and for every countable collection $\{X_i \mid i \in \omega\}$, *there is* $G \subseteq N$ such that $N \models \psi(G, X_i; P, p_0, a)$, for every $i \in \omega$.

Finally, we say that the poset P is *complete* with respect to the (simple) Dee-system given by ψ (and a) iff for every countable $N \prec H_k$ as above, there is $X \subseteq N$ such that if $N \models \psi(G, X, P, p_0, a)$, then there is $q \in P$, $q \geq p_0$ such that $\forall p \in G(p \leq q)$. We say that P is (simply) $\aleph_1$-Dee-complete iff P is complete with respect to some countably complete simple Dee-system.

Now we return to the poset P defined above, and prove that it is $\aleph_1$-Dee-complete. So let $A = \langle A^n_\alpha | \alpha \in \omega_1,\ n \in \omega \rangle$ be a nice sequence (it will be the parameter a), let P be the poset as above, constructed under the assumption that it is not the case that ω_1 is a countable union of sets that are in A. Let $N \prec H_k$ be countable, containing $A, P, p_0 \in P$. Let $\delta = \omega_1 \bigcap N$. We say that a sequence $\langle A^i | i \in \omega \rangle$ is "appropriate for N" iff

1. $\bigcup_{i \in \omega} A^i = \delta$, $A^i \subseteq A^{i+1}$, and
2. for every $i \in \omega$ and $\alpha < \delta$, for some $k \in \omega$, $A^i \bigcap \alpha \subseteq A^k_\alpha$.

Now, let K be a function that assigns to every promise $\langle X, n \rangle$ in N a number $K(X, n) \in \omega$. In some standard way, code $< A^i | i \in \omega >$ and K as some class $X \subseteq N$. Let $\psi(G, X, P, p_0, a)$ be the statement that G is obtained by constructing an increasing sequence of conditions $p_n \in P \bigcap N$ (starting with some extension of the given p_0) as we did above in proving Lemma 3.2, but with the sequence $\langle A^i | i \in \omega \rangle$, decoded from X, replacing the "real" sequence $< A^i_\delta | i \in \omega >$, and the function K, decoded from X, replacing the "real" function.

One must check, in order to prove the $\aleph_1$-completness, that for any countable collection, $\{X_j \mid j \in \omega\}$, of parameters, there is a single G that satisfies $\psi(G, X_j; P, p_0)$ for all X_j.

The proof is essentially as before, observing that if $A \subseteq \delta$ is a finite unions of sets A^i_j, then A satisfies the requirement of the main lemma, namely that for every $\delta' < \delta$, for some $n \in \omega$, $A \bigcap \delta' \subseteq A^n_{\delta'}$.

Thus the increasing sequence $< p_n | n \in \omega >$ can be constructed in such a way that for every j the sequence $< p_n | n \geq j >$ is constructed in accordance with X_j.

Finally, it remains to prove the $\aleph_2$-chain condition for the iteration of our posets, in order to finish the consistency proof. A stronger property, the $\aleph_2$-p.i.c holds, which suffices to ensure that if CH holds, than the countable support iteration of length ω_2 of proper and $\aleph_2$- p.i.c posets satisfies the $\aleph_2$-c.c. This $\aleph_2$-p.i.c. follows from the fact that whenever $(x_1, I_1), (x_2, I_2) \in P$

are such that $x_1 = x_2$, then $(x_1, I_1 \cup I_2)$ extends both conditions. Thus, if CH holds, P is $\aleph_2$-linked: $P = \bigcup_{\alpha \in \omega_1} \ P_\alpha$ where any two conditions in P_α are compatible. This implies the $\aleph_2$-p.i.c.

4 On the possible values of $2^{\aleph_1}$.

If $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ hold in V, then the iteration devised to prove the consistency of (*) gives an $\aleph_2$-c.c. poset of size $\aleph_2$, and thus $2^{\aleph_1} = \aleph_2$ in the extension. In this section we indicate how to modify the proof to accomodate any value $2^{\aleph_1} = \kappa$ (when $cf(k) > \aleph_1$). (It is for this flexibility that the concept $\aleph_2$- p.i.c. was introduced). The scheme is to force with an iteration of length ω_2 of α-proper (for every $\alpha < \omega_1$), $\aleph_1$-Dee-complete, $\aleph_2$- p.i.c. posets, just as before, only that now each poset has size $2^{\aleph_1}$, which may be larger than $\aleph_2$. The first poset, P_1, just adds κ subsets to ω_1 (with countable conditions). $P_{\alpha+1}$ is defined in V^{p_α} to take care, simultaneously, of *all* nice sequences in V^{P_α}. Thus, $P_{\alpha+1} = P_\alpha * Q_\alpha$ where Q_α is in V^{p_α} the countable support *product* of all posets of type $P[A]$, for nice sequences $\overline{a}$'s. That is, $p \in Q_\alpha$ iff $p \in \Pi_{i \in I} \ P_i$, where $I = \{A | A \text{ is a nice sequence on } \omega_1\}$ and $P_A = P(A)$ is as defined above, and $p(i) \neq \emptyset$ only for countably many indices.

The main point to check is that Q_α is $\aleph_1$-Dee-complete .

5 References