Partition Properties of ω_1 Compatible with CH

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Abstract. A combinatorial statement concerning ideals of countable subsets of ω_1 is introduced and proved to be consistent with the Continuum Hypothesis. This statement implies the Souslin Hypothesis, that all $(\omega_1, {\omega_1}^*)$-gaps are Hausdorff, and that every coherent sequence on ω_1 either almost includes or is orthogonal to some uncountable subset of ω_1.

Many combinatorial problems about ω_1 can be reduced to questions of the following sort. (Q) Given a family $\mathcal{I}$ of subsets of ω_1, when can we find uncountable $A \subseteq \omega_1$ such that $A \cap I$ is finite for all $I \in \mathcal{I}$, or dually (Q*), when can we find uncountable $B \subseteq \omega_1$ such that every countable subset of B is included is some element of $\mathcal{I}$? Clearly, there is no loss of generality assuming that $\mathcal{I}$ is an ideal of subsets of ω_1 in order to answer the version (Q) of the question, while for its dual form (Q*) the assumption sometimes makes a difference. It is clear that the answer to (Q) is negative if we can decompose ω_1 into countably many sets $\{S_n\}$ with the property that $[S_n]^\omega \subseteq \mathcal{I}$ for all n. Therefore it is quite natural to consider the following possible answer to (Q):

(A) For every nonprincipal $\aleph_1$-generated ideal $\mathcal{I}$ on ω_1, either

(1) there is uncountable $A \subseteq \omega_1$ such that $A \cap I$ is finite for all $I \in \mathcal{I}$, or, else

(2) ω_1 can be decomposed into countably many sets S_n $(n < \omega)$ such that $[S_n]^\omega \subseteq \mathcal{I}$ for all n.

The corresponding answer to the dual question (Q*) would therefore have the following form.

(A*) For every nonprincipal $\aleph_1$-generated ideal $\mathcal{I}$ on ω_1, either

(1) there is uncountable $A \subseteq \omega_1$ such that $[A]^\omega \subseteq \mathcal{I}$, or else

(2) ω_1 can be decomposed into countably many sets S_n $(n \in \omega)$ such that $S_n \cap I$ is finite for all $n < \omega$ and $I \in \mathcal{I}$.

The statements (A) and (A*) are rather strong combinatorial principles about ω_1 which decide many problems about the uncountable. They are both consistent with $\mathrm{MA}_{\aleph_1}$ (in fact, consequences of PFA) and they both easily imply the negation of CH (see [16], [17]). Note that under $\mathrm{MA}_{\aleph_1}$, the statement (A) is stronger than its dual form (A*) since the poset of all finite approximations to the decomposition (A*)(2) is *ccc* if we assume (A) and the negation of (A*)(1). Since we are interested here in partition properties of ω_1 consistent with CH it seems natural to try to weaken the statement (A*). First of all note that (A) and (A*) are really statements about ideals consisting only of countable subsets of ω_1. So let this be from now on the assumption about $\mathcal{I}$ which we implicitly assume. Moreover, we implicitly assume that $\mathcal{I}$ contains all finite subsets of ω_1. Such an ideal $\mathcal{I}$ is called a *P-ideal* if it is σ-directed under the relation $\subseteq^*$ of the inclusion modulo the ideal of finite sets, i.e., for every sequence I_n $(n<\omega)$ of elements of $\mathcal{I}$ there exist $J\in\mathcal{I}$ such that $I_n\subseteq^* J$ for all $n<\omega$. Call a set $A\subset\omega_1$ *orthogonal* to $\mathcal{I}$ if $A\cap I$ is finite for all $I\in\mathcal{I}$. The following weakening of (A*) is the object of our study in this paper.

(*) For every P-ideal $\mathcal{I}$ on ω_1, either

(1) there is uncountable $A\subseteq\omega_1$ such that $[A]^\omega\subseteq\mathcal{I}$, or else

(2) ω_1 can be decomposed into countably many sets orthogonal to $\mathcal{I}$.

It is interesting that in statements (A) and (A*) above ω_1 can be replaced by any set S without making the statements inconsistent, but the assumption that the ideal $\mathcal{I}$ is $\aleph_1$-generated is essential in (A) and (A*). On the other hand, if we assume $\mathcal{I}$ is a P-ideal (on an arbitrary set S rather than only on ω_1) then the assumption that the ideal be $\aleph_1$-generated is not needed even if we are working in the context of PFA (see [16; §8]). The purpose of this paper is to show that (*) is consistent with CH and that it is still strong enough to decide many problems.

§2. Consequences of (*)

It turns out that many of the standard consequences of (A*) are also consequences of its weaker version (*) as we shall now see.

2.1. Souslin Hypothesis

Theorem 2.1. *If (*) holds then there are no Souslin trees.*

Let T be an ω_1-tree, i.e., a tree of height ω_1, countable levels and with the property that every node of T has successors in any level of T bigger than its own. Let $\mathcal{I}_1$ be the set of all countable $a \subseteq T$ such that every node of T has only finitely many predecessors in a.

Claim 1. *$\mathcal{I}_1$ is a P-ideal.*

Proof: Given $\{I_n\} \subseteq \mathcal{I}_1$ let T_α be the level of T such that $I_n \subseteq T \restriction \alpha$ for all n. Let t_i $(i < \omega)$ be an enumeration of T_α. Let I_ω be the union of the I_n's and for $s \in I_\omega$ let $n_s = min\{n : s \in I_n\}$. Finally, let

$$J = \{s \in I_\omega : s \not<_T t_i \text{ for all } i \leq n_s\}.$$

Then it is easily checked that $J \in \mathcal{I}_1$ and that $I_n \subseteq^* J$ for all n.

Since every countable antichain of T is clearly a member of $\mathcal{I}_1$, no set orthogonal to $\mathcal{I}_1$ can contain an infinite antichain, and therefore, must be the union of countably many chains. This means that the alternative (*)(2) cannot happen if T is to have uncountably many splitting nodes. So we are left with the alternative (*)(1). Since every $A \subseteq T$ with the property $[A]^\omega \subseteq \mathcal{I}_1$, considered as a subtree of T has height $\leq \omega$, the alternative (*)(1) means that T has an uncountable antichain.

2.2. Hausdorff gaps

A *pregap* in $\mathcal{P}(\omega)/\text{fin}$ is a double sequence $\langle \overrightarrow{a}, \overrightarrow{b} \rangle = \langle a_\xi, b_\xi : \xi \in I \rangle$ of subsets of ω indexed by some uncountable $I \subseteq \omega_1$ such that

(a) $a_\xi \subseteq^* a_\eta \subseteq^* b_\eta \subseteq^* b_\xi$ for all $\xi < \eta$ in I.

A pregap $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ is a *gap* in $\mathcal{P}(\omega)/\text{fin}$ if there is no x such that $a_\xi \subseteq^* x \subseteq^* b_\xi$ for all $\xi \in I$. Hausdorff ([7], [8]) was the first to construct such an object which had the additional property that

(b) $\{\xi \in I \cap \alpha : a_\xi \backslash b_\alpha \subseteq n\}$ is finite for all $\alpha \in I$ and $n \in \omega$.

Gaps with this property are called Hausdorff's gaps. It is easily seen that gaps having this property of Hausdorff remain to be gaps in any ω_1-preserving extension of the universe. The indestructibility condition by itself has been later reformulated by Kunen [9] in the following more transparent form:

(c) there is uncountable $J \subseteq I$ such that $a'_\xi \not\subseteq b'_\eta$ or $a'_\eta \not\subseteq b'_\xi$ for every $\xi < \eta$ in J,

where for $\xi \in I$ we let $a'_\xi = a_\xi \cap b_\xi$ and $b'_\xi = b_\xi$. This reformulation suggests that (without loss of generality) from now on we consider only pregaps $\langle a_\xi, b_\xi : \xi \in I \rangle$ which have the additional property that

(d) $a_\xi \subseteq b_\xi$ for all $\xi \in I$.

Kunen [9] also showed that for every gap $\langle \vec{a}, \vec{b} \rangle$ the natural poset which forces an uncountable subset $I \subseteq \omega_1$ with property (c) is a *ccc* poset. Similarly, one can show that the natural poset of finite approximations to an uncountable subset $I \subseteq \omega_1$ for which the corresponding subgap satisfies the Hausdorff condition (b) is also *ccc*. Thus, under $MA_{\aleph_1}$ every gap $\langle \vec{a}, \vec{b} \rangle$ contains a Hausdorff subgap or a subgap with the property (c). [Note that every Hausdorff gap has the property (c).] These results of Kunen led to discoveries of many analogies between gaps and Aronszajn trees. For example, it has become clear that Souslin trees correspond to gaps which can be filled in a *ccc* forcing extension, or equivalently in any ω_1 preserving extension, or equivalently to gaps which have the following property (in addition to (d))

(e) For every uncountable $J \subseteq I$ there exist $\xi < \eta$ in J such that $a_\xi \subseteq b_\eta$ and $a_\eta \subseteq b_\xi$.

Roughly speaking, we have the following analogies

Aronszajn Trees	special	Souslin	...
$(\omega_1, \omega_1{}^*)$-gaps	Hausdorff	destructible	...

Note that, in particular, the existence of destructible gaps implies that the *ccc* property is not productive which should be compared with the fact that the ccc property of a Souslin tree is not productive. This analogy was further enhanced with the discovery that destructible $(\omega_1, \omega_1{}^*)$-gaps can be constructed with the aid of Jensen's $\diamond$ principle which was originally intended for the purpose of constructing various Souslin trees. Also, destructible gaps exist in any forcing extension of the universe V obtained by adding a single Cohen subset $\dot{c} \subseteq \omega$. To see this, take any ground model gap $\langle \vec{a}, \vec{b} \rangle$ with the property (d) and show that

$$\langle a_\xi \cap \dot{c}, b_\xi \cap \dot{c} : \xi \in I \rangle$$

is a destructible gap. Yet another route to destructible gaps is the observation that similarly to Souslin trees their existence is expressible using the Magidor-Malitz quantifier Q^2. Thus, one can use the Completeness Theorem for $L(Q^2)$ to prove the existence of destructible gaps from the principle $\diamond$ (see [10]). Of course, direct $\diamond$-construction of destructible gaps is quite routine. Having in mind Jensen's celebrated result that SH is independent from CH (see [3], [13]) one arrives at the natural question whether CH alone is sufficient to give us a destructible $(\omega_1, {\omega_1}^*)$-gap in $\mathcal{P}(\omega)/\text{fin}$. It should be noted that the interest in the question whether a given gap is destructible or not is not based only on the analogy between gaps and Aronszajn trees. Analysis of this kind of questions forms an essential part of several important consistency results involving the embeddability properties of the algebra $\mathcal{P}(\omega)/\text{fin}$ (see [9], [2], [16]). Thus, finding another way of making a given $(\omega_1, {\omega_1}^*)$-gap indestructible, and in particular, doing this without adding any new reals, might give us some new information about these embeddability properties of $\mathcal{P}(\omega)/\text{fin}$. So the following result adds some further motivation to the consistency of (*) with CH.

Theorem 2.2. *If (*) holds, then every $(\omega_1, {\omega_1}^*)$-gap contains a Hausdorff subgap.*

Let $\langle a_\xi, b_\xi : \xi < \omega_1 \rangle$ be a given gap. Let $\mathcal{I}_2$ be the set of all countable $A \subseteq \omega_1$ such that

(f) $\{\xi \in A \cap \alpha : a_\xi \backslash b_\alpha \subseteq n\}$ is finite for all $\alpha < \omega_1$ (equivalently, for all $\alpha \leq \sup(A)$) and $n < \omega$.

Claim. *$\mathcal{I}_2$ is a P-ideal.*

Proof: Let $\{A_i\} \subseteq \mathcal{I}_2$ be a given sequence and let $\alpha_0 < \omega_1$ be large enough such that $A_n \subseteq \alpha_0$ for all n. Let α_j $(j < \omega)$ be an enumeration of $\alpha_0 + 1$. For i, j, $n < \omega$, set

$$F_{ij}(n) = \{\xi \in A_j \cap \alpha_j : a_\xi \backslash b_{\alpha_j} \subseteq n\}$$

By our assumption $F_{ij}(n)$'s are all finite sets, so if for $i < \omega$ we set

$$B_i = A_i \backslash \bigcup_{j \leq i} F_{ij}(i)$$

we get a set which is almost equal to A_j. So $B = \bigcup_{i<\omega} B_i$ almost-includes all A_i's. To see that $B \in \mathcal{I}_2$ it suffices to check (f) for $\alpha \leq \alpha_0$. But any such α is equal to some α_j so by the choice of B_i for $i \geq \max\{j, n\}$, the set

$$\{\xi \in B \cap \alpha : a_\xi \backslash b_\alpha \subseteq n\}$$

is disjoint from any such B_i. It follows that this set is covered by finitely many finite sets

$$\{\xi \in A_i \cap \alpha : a_\xi \backslash b_\alpha \subseteq n\} \qquad (i < \max\{j, n\}).$$

Since the alternative (*)(1) is clearly equivalent to the conclusion of Theorem 2.2, we are left to show that (*)(2) is impossible. In fact, we shall show that there is no uncountable

$I \subseteq \omega_1$ orthogonal to $\mathcal{I}_2$. For suppose there is such an $I \subseteq \omega_1$. We have already noted that there is a *ccc* forcing extension which contains uncountable $J \subseteq I$ such that $\langle a_\xi, b_\xi : \xi \in J \rangle$ is Hausdorff. Let J_0 be the first ω many ordinals of J and let $\alpha_0 = \sup(J_0) + 1$. Thus, J_0 is an infinite subset of $I \cap \alpha_0$ which has the property (f) for every $n < \omega$ and $\alpha \leq \alpha_0$, a fact which is clearly absolute between the universe and the forcing extension. So, I contains an infinite subset which is in $\mathcal{I}_2$, contradicting our initial assumption that I is orthogonal to $\mathcal{I}_2$.

In a conversation, S. Shelah suggested to us a formally stronger property than (b)

(g) $\{\xi < \alpha : |a_\xi \setminus b_\alpha| \leq n\}$ is finite for all $\alpha < \omega_1$ and $n < \omega$.

It should be clear that the above proof also shows that (*) implies that every gap contains a subgap with this stronger property (g). For this one needs also the fact that for a given gap $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ the natural poset for forcing a subgap $\langle a_\xi, b_\xi : \xi \in I \rangle$ of $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$ with property (g) is indeed a *ccc* poset. This is proved by essentially the same argument used to prove this for the formally weaker property (b).

2.3 Coherent ω_1-sequences

A sequence $A_\alpha \subseteq \alpha$ ($\alpha < \omega_1$) is said to be *coherent* if $A_\alpha =^* A_\beta \cap \alpha$ whenever $\alpha < \beta$. Note in this case the sequence $A_\alpha^c = \alpha \setminus A_\alpha$ ($\alpha < \omega$) of complements is also coherent. A coherent sequence $\{A_\alpha\}$ is *trivial* if there is a set $A \subseteq \omega_1$ such that $A_\alpha =^* A \cap \alpha$ for all $\alpha < \omega_1$. Thus, coherent sequences correspond to the notion of a *pregap* in the structure $[\omega_1]^\omega$ i.e., $\langle A_\alpha, \omega_1 \setminus A_\alpha^c : \alpha < \omega_1 \rangle$ forms a pregap in $\langle [\omega_1]^\omega, \subseteq^* \rangle$ while the notion of nontrivial coherent sequence corresponds to the notion of a gap in this structure. In fact, this notion of special form of gaps in $[\omega_1]^\omega$ is even more closely tied to the notion of an Aronszajn tree than the notion of an $(\omega_1, \omega_1{}^*)$-gap in $\mathcal{P}(\omega)/\text{fin}$ seen in the previous subsection. For example, note that if $\{A_\alpha\}$ is a nontrivial coherent sequence on ω_1, then the tree $T\{A_\alpha\}$ of all finite changes of members of the sequence $\chi_{A_\alpha} : \alpha \to 2$ ($\alpha < \omega_1$) of characteristic functions is an Aronszajn tree, the correspondence which reduces the nontriviality condition to the nonexistence of a cofinal branch of $T\{A_\alpha\}$. As expected, there is a nontrivial coherent sequence in ZFC with proof closely resembling the proof of the existence of an Aronszajn tree ([18]). In fact, $T\{A_\alpha\}$ can represent the whole spectrum of different Aronszajn trees starting from Souslin and ending with the class of special Aronszajn trees (see [15; §6]). Note that Hausdorff's conditions (b) and (g) discussed above reduce to the following requirement on a coherent sequence $\{A_\alpha\}$.

(h) $\{\xi < \alpha : |A_\xi \Delta (A_\alpha \cap \xi)| \leq n\}$ is finite for all $\alpha < \omega_1$ and $n < \omega$.

Similarly to the proof of Theorem 2.2 we can show the following fact.

Theorem 2.3. *If (*) holds then every nontrivial coherent sequence on ω_1 contains an uncountable subsequence with the property (h).*

It can be shown that if a coherent sequence $\{A_\alpha\}$ has the Hausdorff property (h) then the corresponding Aronszajn tree $T\{A\}$ is special. This gives still further evidence to the close relationship between Hausdorff gaps and Aronszajn trees. However, the main goal of this subsection is to discuss another set of problems related to coherent sequences. To state this, let us say that a set $X \subseteq \omega_1$ is *almost included* in a coherent sequence $\{A_\alpha\}$ if $X \cap \alpha \subseteq^* A_\alpha$ for all α. Let us also say that a set $X \subseteq \omega_1$ is *almost disjoint* from $\{A_\alpha\}$ (or is *orthogonal to* $\{A_\alpha\}$) if $X \cap A_\alpha$ is finite for all α. Thus, saying that there is no uncountable $X \subseteq \omega_1$ which is orthogonal to $\{A_\alpha\}$ or is almost included in $\{A_\alpha\}$ is stronger than just saying $\{A_\alpha\}$ is nontrivial. So one naturally comes to the following question apparently first considered by Galvin [5]:

(P) Is there a coherent sequence $\{A_\alpha\}$ such that no uncountable subset of ω_1 is almost included in, nor is orthogonal to, $\{A_\alpha\}$?

A positive answer to (P) is given in [5] but only under the assumption of Jensen's combinatorial principle $\diamond$ (see also [4]). This of course leads to two natural questions which ask whether such a coherent sequence can be constructed in ZFC or at least using CH ([5], [6]). The negative answer to the first question is given by Todorcevic [14], while the following fact together with the main result of this paper show that indeed CH is not sufficient for a positive answer to (P).

Theorem 2.4. *If (*) holds, then every coherent sequence on ω_1 either almost includes an uncountable subset of ω_1 or ω_1 can be decomposed into countably many sets orthogonal to it.*

To see this, note that the ideal $\mathcal{I}_3$ generated by a coherent sequence is clearly a P-ideal and that the alternatives (1) and (2) of (*) in this case are exactly the alternatives from the conclusion of Theorem 2.4. In fact, all results about coherent sequences in the PFA context given in [16; §8] can be proved to hold in the CH-model constructed in the last section of this paper. We refer the reader to [18], [1; §14], [4] and [16] for the topological meanings behind the notion of a coherent sequence which involves the space of subuniform ultrafilters on ω_1.

§3. Consistency of (*) with CH

Under CH any P-ideal $\mathcal{I} \subseteq [\omega_1]^{\leq\omega}$ is generated by an ω_1-tower A_α $(\alpha < \omega_1)$ of sets such that $A_\alpha \subseteq^* A_\beta$ whenever $\alpha < \beta$. Clearly, we may assume that the tower $\{A_\alpha\}$ is such that $A_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$. In these terms the statement (*) reduces to:

(*) For every $\subseteq^*$-tower A_α $(\alpha < \omega_1)$ of countable subsets of ω_1, either there is an uncountable set $X \subseteq \omega_1$ almost included in $\{A_\alpha\}$, i.e., every countable subset of X is almost included in some A_α, or else ω_1 can be decomposed in countably many sets orthogonal to $\{A_\alpha\}$.

In this section we use forcing to prove the consistency of (*) with CH. Starting with a universe V of GCH, we iterate ω_2 posets taking countable supports. The iteration will add no new reals and it will satisfy the $\aleph_2$ chain condition. So we will be able to deal with all $\subseteq^*$-towers A_α $(\alpha < \omega_1)$ which occur in the course of iteration. Since the mechanism for dealing with all possible towers $\{A_\alpha\}$ is standard, we concentrate on the basic step of the iteration. So given a tower A_α $(\alpha < \omega_1)$ with the property that ω_1 cannot be decomposed into countably many sets orthogonal to it, we produce a poset $\mathcal{P} = \mathcal{P}\{A_\alpha\}$ such that:

I. In $V^{\mathcal{P}}$ there is uncountable $X \subseteq \omega$ almost included in $\{A_\alpha\}$.

II. The poset $\mathcal{P}$ is of size $2^{\aleph_1}$ and satisfies the properness-isomorphism-condition.

III. $\mathcal{P}$ is α-proper for every countable ordinal α and is complete with respect to some simple σ-complete completeness system.

This will finish the consistency proof since by results of Shelah [13; Ch. V] posets with these properties can be iterated ω_2 times preserving the $\aleph_2$ chain condition and without adding reals. For the convenience of the reader we shall reproduce here all relevant definitions from [13; Ch. V].

The poset $\mathcal{P} = \mathcal{P}\{A_\alpha\}$ is the set of all pairs $p = \langle x_p, \mathcal{X}_p \rangle$ where

(i) x_p is a countable subset of ω_1.

(j) $\mathcal{X}_p$ is a countable collection of uncountable subsets of ω_1 called *promises*.

The ordering of $\mathcal{P}$ is defined by letting $q \leq p$ (q extends p) when

(k) $x_p \trianglelefteq x_q$ (x_q end-extends x_p).

(l) $\mathcal{X}_p \subseteq \mathcal{X}_q$.

(m) for every $X \in \mathcal{X}_p$ the set $\{\xi \in X : x_q \setminus x_p \subseteq A_\xi\}$ is uncountable and it belongs to $\mathcal{X}_q$.

The relation $q \leq p$ is transitive but not necessarily separative. Provided we show $\mathcal{P}$ preserves ω_1, the following extension lemma shows that $\mathcal{P}$ introduces an uncountable subset of ω_1 which is almost included in $\{A_\alpha\}$.

Lemma 3.1. *For every $p \in \mathcal{P}$ and $\gamma < \omega_1$ there is an extension q of p such that $x_q \setminus \gamma \neq \emptyset$.*

Proof: Otherwise, for every large enough $\beta < \omega_1$ there is a promise $X \in \mathcal{X}_p$ such that $\beta \notin A_\xi$ for all but countably many ξ in X. Let X_n $(n < \omega)$ be an enumeration of $\mathcal{X}_p$ and let B_n be the set of all $\beta < \omega_1$ such that $\beta \notin A_\xi$ for all but countably many $\xi \in X_n$. Then each B_n is orthogonal to $\{A_\alpha\}$ and they cover a cocountable subset of ω_1. This of course contradicts our initial assumption about the tower $\{A_\alpha\}$.

Lemma 3.2. *$\mathcal{P}$ is proper.*

Proof: Let θ be a given large enough regular cardinal and let M be a countable elementary submodel of H_θ containing $\mathcal{P}$ and a condition p_0 of $\mathcal{P}$. We need to find an M-generic condition q extending p_0. Let $\delta = M \cap \omega_1$ and let $\mathcal{D}_n$ $(n < \omega)$ be an enumeration of all dense-open subsets of $\mathcal{P}$ that are in M. Starting from p_0 we shall define a sequence $p_n = \langle x_n, \mathcal{X}_n \rangle$ $(n < \omega)$ of stronger and stronger conditions from $\mathcal{P} \cap M$ such that $p_{n+1} \in \mathcal{D}_n$ for all n. This will be done in such a way that there will be a condition q of $\mathcal{P}$ extending all the p_n's. Such a condition q will obviously be M-generic. Having in mind an even stronger completeness property of $\mathcal{P}$ to be proved later, we choose an arbitrary $B \subseteq \delta$ such that $A_\alpha \subseteq^* B$ for all $\alpha < \delta$ and we try to construct the sequence of p_n's such that the following two conditions are satisfied when we let x_ω be the union of the x_n's:

(n) $x_\omega \setminus x_0 \subseteq B$

(o) for every $n < \omega$ and $X \in \mathcal{X}_n$ the set $Z(X, n) = \{\xi \in X : x_\omega \setminus x_n \subseteq A_\xi\}$ is uncountable.

If we succeed in constructing such a sequence, then $q = \langle x_q, \mathcal{X}_q \rangle$ can be defined by letting

$$x_q = x_\omega \quad \text{and} \quad \mathcal{X}_q = \{Z(X, n) : n < \omega, X \in \mathcal{X}_n\} \cup \bigcup_{n<\omega} \mathcal{X}_n.$$

It is clear that such a condition q will be as required. To ensure (o), we shall use some enumeration device which will give us for every $n < \omega$ and $X \in \mathcal{X}_{p_n}$ an integer $m > n$ such that X will be taken care of while defining p_i for $i > m$. We shall first let

$$X_1 = \{\xi \in X : x_m \setminus x_n \subseteq A_\xi\}.$$

Since $p_m \leq p_n$ this set is uncountable and is a member of $\mathcal{X}_m$. Since $A_\delta \subseteq^* A_\xi$ for all but countably many $\xi \in X_1$ there is an uncountable $X_2 \subseteq X_1$ (now, not in M) and a finite set $F \subseteq \delta$ such that $A_\delta \subseteq A_\xi \cup F$ for all $\xi \in X_2$. So, if we are able to find an extension p_{m+1} of p_m in $\mathcal{D}_m \cap M$ such that $x_{m+1} \backslash x_m$ is included in $(A_\delta \cap B) \backslash F$ and, moreover, keep $x_{i+1} \backslash x_i$ disjoint from F at any later stage $i \geq m$, this will ensure that $Z(X, n)$ includes the uncountable set X_2. So, we are left to satisfy the following two demands for a given finite set $F \subseteq \delta$:

(p) $p_{m+1} \in \mathcal{D}_m \cap M$

(q) $x_{m+1} \backslash x_m \subseteq (A_\delta \cap B) \backslash F$.

For this we have the following Lemma.

Lemma 3.3. *Suppose $M \prec H_\theta$ is countable and it contains $\mathcal{P}$. Let $\delta = M \cap \omega_1$, $p_0 \in \mathcal{P} \cap M$ and let $\mathcal{D} \in M$ be a dense open subset of $\mathcal{P}$. Let $A \subseteq \delta$ be such that $A_\alpha \subseteq^* A$ for all $\alpha < \delta$. Then there is an extension p of p_0 in $\mathcal{D} \cap M$ such that $x_p \backslash x_{p_0} \subseteq A$.*

Proof: Suppose such a p cannot be found. Let X be the set of all $\gamma < \omega_1$ for which there is a finite set $F \subseteq \gamma$ such that there is no $p \in \mathcal{D}$ extending p_0 such that $x_p \backslash x_{p_0} \subseteq A_\gamma \backslash F$. Note that every ordinal $\gamma < \delta$ is a member of X since for such γ the finite set $A_\gamma \backslash A$ works. Since X is clearly an element of M this means that X is actually equal to ω_1. By the Pressing Down Lemma there is a stationary set $S \subseteq \omega_1$ in M and a finite set $F \subseteq \delta$ such that F witnesses $\gamma \in X$ for *all* $\gamma \in S$. Then $p_1 = \langle x_{p_0}, \mathcal{X}_{p_0} \cup \{S\} \rangle$ is a member of $\mathcal{P} \cap M$ which clearly extends p_0, so by the extension Lemma 3.1. there is $\beta_0 \in (\max(F), \delta)$ such that

$$p_2 = \langle x_{p_1} \cup \{\beta_0\}, \mathcal{X}_{p_1} \cup \{\{\xi \in X : \beta_0 \in A_\xi\} : X \in \mathcal{X}_{p_1}\} \rangle$$

is a member of $\mathcal{P} \cap M$ extending p_1. Since $\mathcal{D} \in M$ is a dense subset of $\mathcal{P}$, we can find $p \in \mathcal{D} \cap M$ extending p_2 and therefore p_1. It follows that

$$S_1 = \{\gamma \in S : x_p \backslash x_{p_0} \subseteq A_\gamma\}$$

is uncountable. Since $\min(x_p \backslash x_{p_0}) = \beta_0$ is above $\max(F)$, this means that for $\gamma \in S_1$ we actually have the stronger inclusion

$$x_p \backslash x_{p_0} \subseteq A_\gamma \backslash F$$

contradicting the fact that for any such γ, the set F witnesses that γ belongs to X.

The proof of Lemma 3.2 can be modified to give us the following stronger result.

Lemma 3.4. *P is α-proper for all $\alpha < \omega_1$.*

Proof: This means that for every continuous $\in$-chain M_ξ $(\xi < \alpha)$ of countable elementary submodels of some large enough structure H_θ every condition $p_0 \in \mathcal{P} \cap M_0$ can be extended to a condition q which is M_ξ-generic for all $\xi < \alpha$. This is done by induction on α with the following inductive hypothesis.

(r) For every continuous $\in$-chain M_ξ $(\xi \leq \alpha)$ of countable elementary submodels of H_θ, every $p_0 \in \mathcal{P} \cap M_0$ and every finite set $F \subseteq \delta$, where $\delta = M_\alpha \cap \omega_1$, there is $q \in \mathcal{P}$ extending p_0 which is M_ξ-generic for all $\xi < \alpha$ and which has the property that $x_q \backslash x_{p_0} \subseteq A_\delta \backslash F$.

We have just seen this when $\alpha = 1$ during the course of the proof of Lemma 3.2. The successor step is done similarly using the inductive hypothesis. So let us assume α is a limit ordinal and fix an increasing sequence α_i $(i < \omega)$ cofinal with α. Let $\delta_i = M_{\alpha_i} \cap \omega_1$ and $F_i = A_{\delta_i} \backslash A_\delta$. We will define a sequence p_i $(i < \omega)$ of stronger and stronger conditions such that $p_{i+1} \in M_{\alpha_i+1} \cap \mathcal{P}$ is M_ξ-generic for all $\xi \leq \alpha_i$ and such that

$$x_{p_{i+1}} \backslash x_{p_i} \subseteq A_{\delta_i} \backslash (F_i \cup F \cup E_i),$$

where E_i is an additional finite set given by some book-keeping device as in the proof of Lemma 3.2 in order to ensure that the sequence $\{p_i\}$ will have an extension q. The condition p_{i+1} is obtained by working in M_{α_i+1} and applying the inductive hypothesis (r) for $\alpha = \alpha_i$ to the condition $p_i \in \mathcal{P} \cap M_{\alpha_{i-1}+1}$, the finite set $F \cup F_i \cup E_i$, and the chain M_ξ $(\alpha_{i-1} \leq \xi \leq \alpha_i)$ of countable elementary submodels of H_θ $(\alpha_{-1} = 0)$.

To prove the completeness property of $\mathcal{P}$ which together with α-properness ensures that the countable support iteration of such posets does not add reals we need to review the relevant definitions from [13; Ch. V]. For a countable elementary submodel M of some large enough structure H_θ, a poset $\mathcal{P} \in M$ and $p_0 \in \mathcal{P} \cap M$ let $\mathrm{Gen}(M, \mathcal{P}, p_0)$ denote the set of all M-generic filters $G \subseteq \mathcal{P} \cap M$ containing p_0, i.e., filters G of $\mathcal{P} \cap M$ such that $G \cap \mathcal{D} \neq \emptyset$ for every $\mathcal{D} \in M$ which is dense open in $\mathcal{P}$. A *completeness system* is a function which to every triple $(M, \mathcal{P}, p_0)$ associates a filter of nonempty subsets of $\mathrm{Gen}(M, \mathcal{P}, p_0)$. Such a system is called *simple* if it can be described by a single second-order formula $\psi(Y_1, Y_2; y_1, y_2, y_3)$ in the $\in$-language so that for some fixed parameter $a \in H_\theta$, the filter associated to a given triple $\langle M, \mathcal{P}, p_0 \rangle$ such that $a \in M$ is generated by the sets of the form

$$\mathcal{G}_X^\psi = \{G \in \mathrm{Gen}(M, \mathcal{P}, p_0) : M \models \psi(G, X; \mathcal{P}, p_0, a)\},$$

where X is an arbitrary subset of M. We say that the simple completeness system given by a formula $\psi(Y_1, Y_2; y_1, y_2, y_3)$ and a parameter $a \in H_\theta$ is *σ-complete* if all filters are in some sense σ-complete, or, more precisely, if for every $a, \mathcal{P}, p_0 \in M \prec H_\theta$ and every sequence X_n $(n < \omega)$ of subsets of M there is a $G \subseteq M$ such that $M \models \psi(G, X_n; \mathcal{P}, p_0, a)$ for all $n < \omega$. Finally, we say that the poset $\mathcal{P}$ is *complete* with respect to the simple completeness system given by a formula $\psi(Y_1, Y_2; y_1, y_2, y_3)$ and a parameter $a \in H_\theta$ iff for every countable $M \prec H_\theta$, $a, \mathcal{P}, p_0 \in M$, there is an $X \subseteq M$ such that every element G of $\mathcal{G}^\psi_X$ has an extension in $\mathcal{P}$.

Lemma 3.5. *The poset $\mathcal{P} = \mathcal{P}\{A_\alpha\}$ defined above is complete with respect to some simple σ-complete completeness system.*

Proof: To see what the formula $\psi(Y_1, Y_2; y_1, y_2, y_3)$ should say in order that every member of $\mathcal{G}^\psi_X$, for suitably chosen X, is an extendable (and sufficiently generic) filter, let us fix some countable elementary submodel $M \prec H_\theta$ containing $\{A_\alpha\}$, $\mathcal{P}$ and some $p_0 \in \mathcal{P}$. Suppose X is a subset of M which codes in some standard way the following objects:

(1) an enumeration $\mathcal{D}_n$ $(n < \omega)$ of all dense open subsets $\mathcal{D}$ of $\mathcal{P}$ which are members of M,

(2) an enumeration X_n $(n < \omega)$ of all uncountable subsets of ω_1 which are members of M,

(3) a function $F : \omega \to [\delta]^{<\omega}$ where $\delta = M \cap \omega_1$, and

(4) a subset $B \subseteq \delta$ such that $A_\alpha \subseteq^* B$ for all $\alpha < \delta$.

We let the formula $\psi(G, X; \mathcal{P}, p_0, \{A_\alpha\})$ say that when we decode X to give us objects of (1), (2), (3) and (4), then G is a filter generated by a sequence p_n $(n < \omega)$ of stronger and stronger conditions which starts from p_0 such that for all but finitely many m the condition p_{m+1} belongs to the intersection of all $\mathcal{D}_i$ $(i \leq m)$ and $x_{p_{m+1}} \setminus x_{p_m}$ is a subset of B which is disjoint from any finite set $F \subseteq \delta$ of the following form: There exists $i \leq m$ such that X_i belongs to $\mathcal{X}_{p_j}$ for some $j \leq m$ and if k is the minimal such j, then $F = F(l)$ where l is the minimal integer such that

$$X_l = \{\xi \in X_i : x_{p_m} \setminus x_{p_k} \subseteq A_\xi\}.$$

This definition is of course based on the proof of Lemma 3.2 where the function $F : \omega \to [\delta]^{<\omega}$ is given by letting $F(i)$ be the minimal finite subset F of δ such that

$$\{\xi \in X_i : A_\delta \subseteq A_\xi \cup F\}$$

is uncountable. Thus any $X \subseteq M$ which codes this particular $F : \omega \to [\delta]^{<\omega}$ and whose set B of (4) is equal to A_δ will have the property that any G such that

$$M \models \psi(G, X; \mathcal{P}, p_0, \{A_\delta\})$$

will be an M-generic filter containing p_0 and having extensions in $\mathcal{P}$. In other words, such an $X \subseteq M$ will witness the fact that $\mathcal{P}$ is complete with respect to the completeness system determined by ψ and the parameter $\{A_\alpha\}$.

It remains to be shown that the completeness system is σ-complete. So let X^n $(n < \omega)$ be a given sequence of subsets of M. We need to find a filter G such that for all $n < \omega$,

$$M \models \psi(G, X^n; \mathcal{P}, p_0, \{A_0\}).$$

So G will be generated by a sequence p_n $(n < \omega)$ of stronger and stronger conditions which is constructed almost exactly as in the proof of Lemma 3.2 except that now going from p_m to p_{m+1} we take care about all of the finitely many requirements obtained by decoding the sets X^n $(n \leq m)$. For example, p_{m+1} will be a member of the intersection of all $\mathcal{D}_i^n$ $(n, i \leq m)$ extending p_m and having the property that $x_{p_{m+1}} \backslash x_{p_m}$ is a subset of $\bigcap_{n \leq m} B^n$ avoiding all of the finitely many finite sets $F \subseteq \delta$ which can be obtained from one of the sets X_i^n $(n, i \leq m)$ via the process described above, i.e., $F = F^n(l)$ for a suitably chosen l depending on some $i \leq m$. Note that the general form of Lemma 3.3 is exactly what is needed in order to show that such a condition p_{m+1} can always be found.

Finally, it remains to prove the $\aleph_2$ chain condition of the countable support iteration of our posets. For this it suffices to show the following (see [13; p. 262]).

Lemma 3.6. *The poset $\mathcal{P} = \mathcal{P}\{A_\alpha\}$ satisfies the properness isomorphism condition.*

Proof: This means that if we are *given* two countable elementary submodels $M_0, M_1 \prec H_\theta$ such that $\mathcal{P} \in M_0 \cap M_1$ and an isomorphism h between M_0 and M_1 which is equal to the identity on $M_0 \cap M_1$ *then* for every p in $\mathcal{P} \cap M_0$ there is an extension q of p and $h(p)$ which is M_0-generic and which forces that the mapping h sends $\dot{G} \cap M_0$ isomorphically onto $\dot{G} \cap M_1$, where $\dot{G}$ is the canonical name for the generic filter. The condition q is obtained as the extension of a sequence p_n $(n < \omega)$ of conditions constructed similarly as in the proof of Lemma 3.2 but working only inside the model M_0. Of course, we let $p_0 = p$ and construct p_{m+1} from p_m as in the proof of Lemma 3.2 but now at various stages taking care not only about promises appearing in some $\mathcal{X}_{p_i}$ but also about their h-images. Thus, at a given stage m where we have to take care about some $X \in \mathcal{X}_{p_i}$ $(i \leq m)$ we additionally take care about $h(X)$ by first setting

$$X_1 = \{\xi \in h(X) : x_{p_m} \backslash x_{p_i} \subseteq A_\xi\}$$

and noting that this is indeed an uncountable subset of $h(X)$. It follows that there is a finite set $F \subseteq \delta$ ($= M_0 \cap \omega_1 = M_1 \cap \omega_1$) such that

$$X_2 = \{\xi \in X_1 : A_\delta \subseteq A_\xi \cup F\}$$

is uncountable. This will be another finite set which we have to avoid while constructing p_n for $n > m$. When this is done, we set $x_\omega = \bigcup_n x_n$ and

$$q = \langle x_\omega, \bigcup_n (\mathcal{X}_n^* \cup (h''\mathcal{X}_n)^*)\rangle,$$

where for $n < \omega$,

$$\mathcal{X}_n^* = \mathcal{X}_n \cup \{\{\xi \in X : x_\omega \backslash x_n \subseteq A_\xi\} : X \in \mathcal{X}_n\}, \quad \text{and}$$
$$(h''\mathcal{X}_n)^* = h''\mathcal{X}_n \cup \{\{\xi \in h(X) : x_\omega \backslash x_n \subseteq A_\xi\} : X \in \mathcal{X}_n\}.$$

Then q is an M_0-generic condition which extends not only the p_n's but also their h-images $h(p_n)$'s and, therefore, it forces that $\dot{G} \cap M_0$ is generated by $\{p_n\}$ while $\dot{G} \cap M_1$ is generated by $\{h(p_n)\}$.

§4. Concluding remarks

One way of strengthening the principle (*) would be trying to approximate (A) and (A*) more closely. Such approximation should imply as many consequences of (A) and (A*) as possible, e.g., those presented in [16; §8] and [17] related to the Moore-Mrowka problem. That the methods of this paper might be relevant is most easily seen by considering the standard example from this area, the Ostaszewski space ([12]). This example is so closely tied to the $\diamond$-principle that many attempts to construct such a space using only CH have failed so far. For example, in [11; Problem 174] one finds a combinatorial principle about $[\omega_1]^\omega$ (due to S. Watson) which together with CH implies the existence of an Ostaszewski space, so the natural question was whether the principle is a consequence of CH or not. The result of this paper shows that the answer is negative, since it is easily seen that the ideal generated by 'small' sets of [11; Problem 174] is a P-ideal of countable subsets of ω_1 so the alternatives of (*) would contradict the four properties required in [11; Problem 174]. However, there might be another possible line to strengthen the principle (*). To see this, let us reformulate (*) in terms of integer-valued functions defined on (sets of) ordinals. For two such functions f and g, set $f \leq^0 g$ iff $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and the function

g is unbounded on any subset $X \subseteq \mathrm{dom}(f)$ on which the function f itself is unbounded. Call a family F of integer-valued functions *unbounded* if there is no g such that $f \leq^0 g$ for all $f \in \mathcal{F}$. A typical example of an unbounded family is a sequence $e_\alpha : \alpha \to \omega$ $(\alpha < \omega_1)$ of finite-to-one functions. Such a sequence is, of course, $\leq^0$-increasing, i.e., $e_\alpha \leq^0 e_\beta$ for $\alpha < \beta$. The following fact shows that any other $\leq^0$-increasing unbounded ω_1-sequence of functions will essentially have to look like this one.

Theorem 4.1 (CH). *The statement (*) is equivalent to the statement that for every $\leq^0$-increasing unbounded sequence $f_\alpha : \alpha \to \omega$ $(\alpha < \omega_1)$ there is uncountable $X \subseteq \omega_1$ such that $f_\alpha {}^{-} X$ is finite-to-one for all $\alpha < \omega_1$.*

To see the direct implication, let $\mathcal{I}_4$ be the set of all countable $X \subseteq \omega_1$ such that $f_\alpha {}^{-} X$ is finite-to-one for all $\alpha < \omega_1$. It is easily checked that $\mathcal{I}_4$ is a P-ideal and that the alternative (*)(1), applied to $\mathcal{I}_4$, gives us the uncountable $X \subseteq \omega_1$ such that $f_\alpha {}^{-} X$ is finite-to-one for all α. The alternative (*)(2), however, is equivalent to the boundedness of f_α $(\alpha < \omega_1)$ and is therefore impossible by our assumption about the sequence.

To see the reverse implication, let $\mathcal{I}$ be a given P-ideal of countable subsets of ω_1 and (using CH) fix a tower $\{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{I}$ which generates $\mathcal{I}$. Choose also a sequence $e_\alpha : \alpha \to \omega$ $(\alpha < \omega_1)$ of finite-to-one functions, and define a new sequence $f_\alpha : \alpha \to \omega$ $(\alpha < \omega_1)$ as follows:

$$f_\alpha(\xi) = \begin{cases} e_\alpha(\xi), & \text{if } \xi \in A_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\{f_\alpha\}$ is an $\leq^0$-increasing sequence of functions. If $g : \omega_1 \to \omega$ is an upper-bound of $\{f_\alpha\}$, the sets $g^{-1}(n)$ $(n < \omega)$ would form a decomposition of ω_1 into countably many pieces orthogonal to $\mathcal{I}$. So we may assume $\{f_\alpha\}$ is unbounded and, therefore, we can find uncountable $X \subseteq \omega_1$ such that $f_\alpha {}^{-} X$ is finite-to-one for all α. Going back to the definition of $\{f_\alpha\}$, it is clear that this means that $X \cap \alpha \subseteq^* A_\alpha$ for all α.

The ordering $\leq^0$ is one of the many orderings which can be considered in this context. For example, another natural ordering would be as follows:

$$f \leq^1 g \quad \text{iff} \quad \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \quad \text{and} \quad f(\alpha) \leq \max\{m, g(\alpha)\}$$

for a fixed $m < \omega$ and all $\alpha \in \mathrm{dom}(f)$. One may also consider larger families of functions, e.g., of cardinality $\aleph_2$ or bigger, and consider the problem when they contain an ω_1-subsequence which behaves like the prototype $e_\alpha : \alpha \to \omega$ $(\alpha < \omega_1)$ of finite-to-one functions. Note that this must necessarily involve something stronger than the assumption of the consistency of ZFC alone (see [15; (1.14)]).

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