Hausdorff's Theorem for posets that satisfy the finite antichain property

Uri Abraham
Department of Mathematics and Computer Science
Ben-Gurion University, Be'er Sheva, Israel
Robert Bonnet
Laboratoire de Mathématiques
Université de Savoie, France

September 14, 1998

Abstract

Hausdorff characterized the class of scattered linear orderings as the least family of linear orderings that includes the ordinals and is closed under ordinal summations and inversions. We formulate and prove a corresponding characterization of the class of scattered partial orderings that satisfy the finite antichain condition (FAC).

Consider the least class of partial orderings containing the class of well-founded orderings that satisfy the FAC and is closed under the following operations: (1) inversion, (2) lexicographic sum, and (3) augmentation (where $\langle P, \preceq\rangle$ augments $\langle P, \leq\rangle$ iff $x \preceq y$ whenever $x \leq y$). We show that this closure consists of all scattered posets satisfying the finite antichain condition.

Our investigation also shed some light on the natural (Hessenberg) sum of ordinals and the related product and exponentiation operations.

1980 *Mathematical Subject Classification* (1985 *Revision*). Primary 06B30, 54E45, 54E12. Secondary 06B05.

Key words and phrases. Ordinals, Partial orderings, Scattered partial orderings, Hausdorff's theorem.

1 Introduction

The title of our paper refers to the well known analysis of the scattered linear orderings made by Hausdorff [5]: Let $\mathcal{H}^{\text{linear}}$ be the closure of the class of well–ordered sets under inversions and well-ordered summations, then $\mathcal{H}^{\text{linear}}$ consists exactly of all linear scattered orderings (all definitions and needed preliminaries will be given below). We relax the requirement on linearity and obtain a corresponding result concerning the class of partially ordered sets that satisfy the finite antichain condition (FAC— every set of pairwise incomparable points is finite). Precisely, we shall prove the following theorem in Section 3.

> Define $\mathcal{H}$ as the closure of the class of all FAC well-founded posets under lexicographical sums, inverses, and augmentations of the orderings. Then $\mathcal{H}$ is exactly the class of all scattered FAC posets.

In this formulation, the theorem is not an extension of Hausdorff's theorem because it (necessarily) allows augmentations, but we provide a finer investigation of the ranks of the posets involved and then obtain the theorem of Hausdorff as a particular case.

The proof is very elementary and should be accessible to readers with only a basic background in set theory. We therefore give all necessary definitions and some known results in the sequel of this introduction.

On posets

We recall here some definitions of elementary notions: posets, scattered posets, embedding, augmentation, etc. We recall some elementary facts. For example that an augmentation of a well quasi order (wqo) is wqo, and an augmentation of a scattered wqo is scattered.

If P is a partially ordered set (poset) then P, or more formally $|P|$, denotes the universe of the ordering, and $\leq^P$ (or $\leq$ when no confusion can arise) denotes the ordering itself. We write $a < b$ for $a \leq b$ and $a \neq b$.

We say that a and b are *incomparable* ($a \perp b$) when $a \not\leq b$ and $b \not\leq a$. An *antichain* is a set of pairwise incomparable elements. The Finite Antichain Condition (FAC) for a poset P is the property that any antichain in P is finite.

If P is a poset, then P^* denotes it inverse ordering, that is $|P| = |P^*|$ and for every p and q, $p \leq^{P^*} q$ iff $p \geq^P q$.

For any poset P and $x \in P$ we define

$$(\leq x) = \{y \in P \mid y \leq x\},$$
$$(\perp x) = \{y \in P \mid y \perp x\}.$$

$(< x)$, $(\geq x)$, and $(> x)$ are defined similarly. We may write $(\leq x)^P$, $(< x)^P$, etc. for emphasise.

1. We say that P is a *subposet* of Q if the universe of P is a subset of the universe of Q and whenever $p_1 <^P p_2$ then $p_1 <^Q p_2$.

2. If P is a subposet of Q such that for any $p_1, p_2 \in |P|$ $p_1 <^P p_2$ iff $p_1 <^Q p_2$, then P is said to be the *restriction* of Q to $|P|$, and we write $P = Q \lceil |P|$.

3. If P is a subposet of Q but they both have the same universe, then we say that Q *augments* P. We denote with $\text{aug}(P)$ the set of all possible augmentations of P. Clearly every augmentation of a FAC poset satisfies the FAC as well.

4. A poset is *well-founded* if it has no infinite decreasing sequence. P is well-founded iff there is a rank function for P; that is an ordinal valued function $r : P \to Ord$ such that $r(p) = \sup\{r(x) + 1 \mid x <^P p\}$ for every p in P.

5. A transitive and reflexive relation is called a quasi-ordering. If every antichain is finite and every decreasing sequence is finite then the quasi-ordering is said to be a well-quasi ordering. This is a very familiar term and a useful notion, but since we prefer to use partially ordered sets (rather than quasi-orderings) we will not use it and thus our rather long (but self explanatory) *FAC well-founded poset* terminology.

6. Let P and Q be two posets; an *embedding* of P into Q is a one-to-one function f from the universe of P into the universe of Q such that for every $a, b \in P$, $a <^P b$ iff $f(a) <^Q f(b)$.

   We say that P is *scattered* if there is no embedding of $\mathbb{Q}$ (the chain of rational numbers) into P.

7. Given two posets P and Q we define the *sum* $P+Q$ as the poset obtained by putting every member of P below every member of (a disjoint copy of) Q. The *perpendicular sum* $P \perp Q$ is defined by taking disjoint copies of P and of Q and making members of P incomparable to members of Q. The Cartesian product $P \times Q$ is the partial ordering obtained by defining $(a,b) \leq (a',b')$ iff $a \leq^P a'$ and $b \leq^Q b'$.

8. The *lexicographical ordering* $P \cdot Q$ is obtained, roughly speaking, by replacing each point in Q with a copy of P. This operation corresponds, more generally, to that of *lexicographical sum* $\sum_{q \in Q} P_q$ obtained by replacing each point q in Q with a copy P_q of P. More formally, this lexicographic sum is the the set $\bigcup_{q \in |Q|} |P| \times \{q\}$ endowed with the ordering $\langle p,q \rangle \leq \langle p',q' \rangle$ if and only if $q <^Q q'$ or, $q=q'$ and $p \leq^{P_q} p'$.

9. If X is a set of ordinals, then $\mathrm{Sup}(X)$ denotes the least ordinal that is strictly greater than all ordinals in X, while $\sup(X)$ is the least ordinal that is $\geq$ than all ordinals in X. That is $\mathrm{Sup}(X) = \sup\{\alpha+1 \mid \alpha \in X\}$.

An initial segment I of a poset P is a subset of $|P|$ such that for every $p \in I$, $(\leq p) \subseteq I$. We denote by $\mathcal{I}(P)$ the set of all initial segment of P.

Lemma 1.1 *Let P be a poset, then the following are equivalent.*

1. *P is a FAC well-founded poset.*

2. *For every infinite sequence $\langle p_i \mid i \in \omega \rangle$ of points from P there are $i<j$ with $p_i \leq^P p_j$.*

3. *Any augmentation of P is well-founded.*

4. *The ordered set $\langle \mathcal{I}(P), \subseteq \rangle$ is well–founded.*

We will not prove the lemma (see Chapter 4 of Fraïssé [4]) but only comment that Ramsey's theorem can be used.

Lemma 1.2 (Bonnet et Pouzet 1969) *The following are equivalent for any poset P.*

1. *P is scattered and satisfies the FAC.*

2. Every augmentation of P is scattered.

Proof. (See [2] and Fraïssé [4, chapter 7].) We prove only one direction, namely that 1 implies 2. Suppose that $P = \langle |P|, < \rangle$ is scattered and satisfies FAC, but $P' = \langle |P|, <' \rangle$ is a non–scattered augmentation of P. We first prove the following claim:

Suppose that $S \subseteq |P|$ is such that the restriction $P' \lceil S$ is isomorphic to $\mathbb{Q}$. Then there is $q \in S$ such that the restriction of P' to $(\perp q)^P \cap S$ contains a copy of $\mathbb{Q}$.

Indeed, as P is scattered and S is not an infinite antichain, there are $a <^P b$ in S that are consecutive elements of S, i.e. there is no $c \in S$ satisfying $a <^P c <^P b$. $a <' b$ and the open interval $(a, b)^{<'} \stackrel{def}{=} \{c \in |S| \mid a <' c <' b\}$ is order–isomorphic to $\mathbb{Q}$. Also for $c \in (a, b)^{<'}$, we have $c \perp a$ or $c \perp b$ in P. Let $D_a = \{c \in (a, b)^{<'} : c \perp a\}$ and $D_b = \{c \in (a, b)^{<'} : c \perp b\}$. $D_a \cup D_b$ is a partition of $(a, b)^{<'}$ into two sets and hence one of the sets is not scattered in $<'$ and contains a copy of $\mathbb{Q}$. This proves the claim.

By repeatedly applying the claim, one obviously gets an infinite antichain of P, which is impossible. $\square$

Note that the class of FAC scattered posets contains the class of FAC well–founded posets and is closed under lexicographical sums, inverses, and augmentations of the orderings (that follows from 1.2).

On ordinals

We recall some elementary facts concerning ordinal operations that we shall use and then proceed to investigate the Hessenberg based operations on the ordinals (which we believe to be new, despite their simplicity). Several textbooks in set theory can be consulted; for example Sierpiński [8], Fraïssé [4], or Lévy [7]).

The Cantor normal form of an ordinal α

$$\alpha = \omega^{\alpha(0)} m(0) + \cdots + \omega^{\alpha(i)} m(i) + \cdots + \omega^{\alpha(\ell)} m(\ell)$$

is determined by a non-empty decreasing sequence $\alpha(0) > \alpha(1) > \cdots > \alpha(\ell) \geq 0$ of ordinals, and a sequence of natural numbers $m(i) \geq 0$. (Except for $\alpha = 0$, it is possible to demand that each $m(i) > 0$, but there are occasions where it seems to be convenient to allow for $m(i) = 0$.)

Each $\omega^{\alpha(i)}$ with $m(i) > 0$ is called here a *block* of α, and the least block is called the *tip* of α. So, for example, if $\alpha = \omega^{\omega 2} + \omega^7 + \omega^3 5$, then $\mathrm{tip}(\alpha) = \omega^3$, and if $\alpha = \omega + 5$ then $\mathrm{tip}(\alpha) = 1$.

The ordering of two ordinals can be determined by their coefficient series: Suppose that α_1 and α_2 are two ordinals. By allowing $m(i) = 0$ we can assume that they have the same sequence of exponents

$$\alpha(0) > \cdots > \alpha(\ell) \geq 0.$$

Let $m_1(i)$ and $m_2(i)$ for $i \leq \ell$ be the respective coefficients of α_1 and α_2. Then $\alpha_1 < \alpha_2$ iff

$$\langle m_1(0), \ldots, m_1(\ell)\rangle \prec \langle m_2(0), \ldots, m_2(\ell)\rangle$$

where $\prec$ is the lexicographical ordering of sequences of natural numbers.

The natural sum of ordinals. The *natural sum* (or Hessenberg sum) of two ordinals is defined by adding the coefficients of their normal forms as though these were polynomials in ω. For example if $\alpha = \omega^{\omega+\omega}8 + \omega^7 3$ and $\beta = \omega^\omega + \omega^7 + \omega^2 + \omega^0 4$ then $\alpha \oplus \beta = \omega^{\omega+\omega}8 + \omega^\omega + \omega^7 4 + \omega^2 + 4$.

Carruth [3] characterized $\alpha \oplus \beta$ as the maximal order type of $A \cup B$, where A and B are sets of ordinals of order type α and β. We indicate a proof:

Relying on the criterion given above for the ordering of ordinals, one can see that the two-place function $\alpha \oplus \beta$ is strictly increasing in both arguments. This can be used to prove that if A, B are sets of ordinals of order type α and β respectively, then $tp(A \cup B) \leq \alpha \oplus \beta$ (here $tp(X)$ is the order type of X). Indeed, define a function f by setting

$$f(\gamma) = tp(A \cap \gamma) \oplus tp(B \cap \gamma)$$

for every $\gamma \in A \cup B$. Then $f : A \cup B \to \alpha \oplus \beta$ is order preserving, and hence $tp(A \cup B) \leq \alpha \oplus \beta$.

On the other hand, $\alpha \oplus \beta$ (as a set of ordinals) is clearly the order type of the union $A \cup B$ of two disjoint sets of ordinals with $tp(A) = \alpha$ and $tp(B) = \beta$. This proves Carruth's characterization.

The natural presentation of $\alpha \oplus \beta$ as the union $A \cup B$ as above has the following property: If $\gamma < \alpha \oplus \beta$ then $\gamma = tp(A \cap \gamma) \oplus tp(B \cap \gamma)$. So

(1) If $\gamma < \alpha \oplus \beta$, then $\gamma = \alpha_0 \oplus \beta_0$ where $\alpha_0 \leq \alpha$ and $\beta_0 \leq \beta$

(and $\alpha_0 < \alpha$ or $\beta_0 < \beta$). (Do not be tempted to think that if $\alpha < \gamma < \alpha \oplus \beta$ then $\gamma = \alpha \oplus \beta_0$ for some $\beta_0 < \beta$.)

The following remarks are for completeness, since we shall not use them in this paper. As Fraïssé [4, Chapter 7.2] remarks, for any two ordinals α and β:

$$\sigma = \mathrm{Sup}_{\substack{\alpha' < \alpha \\ \beta' < \beta}} \alpha' \oplus \beta' < \alpha \oplus \beta \ .$$

The following however is easy to prove:

If α is limit and $\mathrm{tip}(\alpha) \leq \mathrm{tip}(\beta)$ then $\alpha \oplus \beta = \mathrm{Sup}_{\alpha' < \alpha} \alpha' \oplus \beta$.

It follows from this that, for limit ordinals α and β:

$$\alpha \oplus \beta = \max\{\mathrm{Sup}_{\alpha' < \alpha} \alpha' \oplus \beta,\ \mathrm{Sup}_{\beta' < \beta} \alpha \oplus \beta'\}.$$

The Hessenberg based product. The natural sum operation is used to define a product operation $\alpha \odot \beta$ on the ordinals by the following relations:

$$\begin{aligned}
&\alpha \odot 0 = 0 \\
&\alpha \odot (\beta + 1) = (\alpha \odot \beta) \oplus \alpha \\
&\alpha \odot \beta = \mathrm{Sup}\{\alpha \odot \gamma \mid \gamma < \beta\} \text{ for limit } \beta.
\end{aligned}$$

We say that $\alpha \odot \beta$ is the *Hessenberg based product* of α and β. As usual parentheses may be deleted. For instance, $\alpha \oplus (\beta \odot \gamma) \oplus \delta$ is denoted by $\alpha \oplus \beta \odot \gamma \oplus \delta$.

Operation $\odot$ is not the same as the Hessenberg definition $\alpha \otimes \beta$ which is obtained from the normal forms of α and β viewed as polynomials and multiplied accordingly. In particular, $\alpha \otimes \beta$ is commutative, but $\odot$ is not. For example,

$$\begin{aligned}
&\omega \otimes 2 = 2 \otimes \omega = \omega + \omega \\
&\omega \odot 2 = \omega \oplus \omega = \omega + \omega, \quad 2 \odot \omega = \mathrm{Sup}\{2 \odot n \mid n < \omega\} = \omega
\end{aligned}$$

The function $\alpha \odot \beta$ is strictly increasing in the right variable, continuous in the right variable, and non-decreasing in the left variable. $\alpha \otimes \beta$ is strictly increasing in both coordinates, but it is not continuous. We always have (prove by induction on β)

$$\alpha\beta \leq \alpha \odot \beta \leq \alpha \otimes \beta,$$

where $\alpha\beta$ is the usual ordinal multiplication.

Strict inequalities are possible, for instance with $\alpha = \omega+1$ and $\beta = \omega+2$, we have: $\alpha\beta = \omega^2 + \omega 2 + 1$, $\alpha\odot\alpha = \omega^2 + \omega + 1$, $\alpha\odot\beta = \omega^2 + \omega 2 + 2$, and $\alpha \otimes \beta = \omega^2 + \omega 3 + 2$.

We shall recall a well-known formula for the product of ordinals, and show that a similar result also holds for the Hessenberg based product. For this we decompose any ordinal β into its infinite parts and the finite (possibly empty) tip.

Lemma 1.3 *If $\beta = \beta_0 + k$ where β_0 is limit or 0 and $k \in \omega$, then for any α with Cantor normal form $\alpha = \omega^{\alpha(0)} m(0) + \cdots + \omega^{\alpha(\ell)} m(\ell)$ (where $m(0) > 0$),*

$$\alpha\beta = \omega^{\alpha(0)}\beta_0 + \alpha k = \omega^{\alpha(0)}\beta_0 \oplus \alpha k. \tag{2}$$

$$\alpha\odot\beta = \omega^{\alpha(0)}\beta_0 \oplus \alpha\odot k. \tag{3}$$

Hence if $\beta = \beta_0 + 0$ is a limit ordinal then $\alpha\beta = \alpha\odot\beta = \omega^{\alpha(0)}\beta$.

Proof. We only prove (3), but the proof for (2) is the same. The proof is by induction on β. If $\beta = 0$ then $\beta_0 = 0$, $k = 0$ and (3) is trivial. If $\beta = \beta' + 1$ is a successor ordinal, and $\beta = \beta_0 + k + 1$ where β_0 is the last limit ordinal (or zero) below β, then $\beta' = \beta_0 + k$ and, by the inductive assumption,

$$\begin{aligned} \alpha\odot\beta &= \alpha\odot\beta' \oplus \alpha = \omega^{\alpha(0)}\beta_0 \oplus \alpha\odot k \oplus \alpha \\ &= \omega^{\alpha(0)}\beta_0 \oplus \alpha\odot(k+1). \end{aligned}$$

Now if β is limit, then $\beta = \beta_0$, $k = 0$ and we shall prove that $\alpha\odot\beta = \omega^{\alpha(0)}\beta$. By definition

$$\alpha\odot\beta = \mathrm{Sup}_{\beta' < \beta}\alpha\odot\beta'.$$

There are two possibilities.

1. β is a limit of limit ordinals β_i, $i \in I$. Then

$$\alpha\odot\beta = \mathrm{Sup}_{i\in I}\ \alpha\odot\beta_i = \mathrm{Sup}_{i\in I}\ \omega^{\alpha(0)}\beta_i = \omega^{\alpha(0)}\beta$$

because (regular) ordinal multiplication is continuous on the right.

2. In the second possibility β has the form $\beta = \gamma + \omega$ where γ is limit or 0. Then

   (4) $$\alpha \odot \beta = \mathrm{Sup}_{n\in\omega}\ \alpha \odot (\gamma + n) = \mathrm{Sup}_{n\in\omega}\ \omega^{\alpha(0)}\gamma \oplus \alpha \odot n.$$

   Observe first that $\mathrm{Sup}_{n\in\omega}\alpha \odot n = \omega^{\alpha(0)}\omega$. So that if $\beta = \omega$ then $\gamma = 0$,

   $$\alpha \odot \beta = \mathrm{Sup}_{n\in\omega}\alpha \odot n = \omega^{\alpha(0)}\beta$$

   and we are done. Thus we assume $\gamma \geq \omega$.

   But $\omega^{\alpha(0)}\gamma \oplus \alpha \odot n = \omega^{\alpha(0)}\gamma + \alpha \odot n$, because $\mathrm{tip}(\omega^{\alpha(0)}\gamma) \geq \omega^{\alpha(0)}\omega > \alpha \odot n$. So, by the right-continuity property of ordinal addition, Equation 4 gives

   $$\alpha \odot \beta = \omega^{\alpha(0)}\gamma + \omega^{\alpha(0)}\omega = \omega^{\alpha(0)}(\gamma + \omega) = \omega^{\alpha(0)}\beta.$$

Corollary 1.4 *Distributivity holds on the right:* $\alpha \odot (\beta_1 \oplus \beta_2) = \alpha \odot \beta_1 \oplus \alpha \odot \beta_2$.

Proof. This can be proved by induction on $\gamma = \beta_1 \oplus \beta_2$. The case $\gamma = 0$ is trivial, and the case γ a successor is easy. So assume that γ is a limit ordinal, that is, that both β_1 and β_2 are limit ordinals. Then

$$\alpha \odot \gamma = \omega^{\alpha(0)}\gamma = \omega^{\alpha(0)}(\beta_1 \oplus \beta_2) = \omega^{\alpha(0)}\beta_1 \oplus \omega^{\alpha(0)}\beta_2 = \alpha \odot \beta_1 \oplus \alpha \odot \beta_2.$$

We have used the formula $\omega^{\delta}(\beta_1 \oplus \beta_2) = \omega^{\delta}\beta_1 \oplus \omega^{\delta}\beta_2$ which can be easily computed. □

Corollary 1.5 $\odot$ *is associative:* $(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma)$.

Proof. By induction on γ. The successor case is handled as follows:

$$(\alpha \odot \beta) \odot (\gamma + 1) = (\alpha \odot \beta) \odot \gamma \oplus \alpha \odot \beta.$$
$$\alpha \odot (\beta \odot (\gamma + 1)) = \alpha \odot (\beta \odot \gamma \oplus \beta) = \alpha \odot (\beta \odot \gamma) \oplus \alpha \odot \beta.$$

So we see that the two terms are equal by the induction hypothesis. (To get the second equality, we have used distributivity.)

The case where γ is a limit ordinal is as follows.

$$\begin{aligned}(\alpha \odot \beta) \odot \gamma &= \mathrm{Sup}_{\gamma' < \gamma}(\alpha \odot \beta) \odot \gamma' = \\ &= \mathrm{Sup}_{\gamma' < \gamma}\alpha \odot (\beta \odot \gamma') = \alpha \odot (\beta \odot \gamma)\end{aligned}$$

We have used here the fact that $\mathrm{Sup}_{\gamma'<\gamma}(\beta\odot\gamma') = \beta\odot\gamma$ by definition, and the continuity property on the right, which says that if $\mathrm{Sup}_{g\in G}\beta_g = \beta$ is a limit ordinal, then $\mathrm{Sup}_{g\in G}\, \alpha\odot\beta_g = \alpha\odot\beta$.

HESSENBERG BASED EXPONENTIATION. In the following step $\odot$ is used to define the Hessenberg based exponentiation, which we denote with the superscript H (as in $\alpha^{H\beta}$). Whenever it is perfectly clear that we deal with the Hessenberg based exponentiation, we shall omit H:

$$\alpha^0 = 1 \text{ for } \alpha > 0 \text{ (but } 0^\beta = 0)$$
$$\alpha^{\beta+1} = \alpha^\beta\odot\alpha$$
$$\alpha^\beta = \mathrm{Sup}\{\alpha^\gamma \mid \gamma < \beta\} \text{ for limit } \beta.$$

Lemma 1.6 *The usual rules hold for the Hessenberg based exponentiation:*

$$\alpha^{\beta+\gamma} = \alpha^\beta\odot\alpha^\gamma, \tag{5}$$
$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}. \tag{6}$$

Proof. All exponentiations are Hessenberg based exponentiations. We first show (5). The proof is by induction on γ. The case $\gamma = 0$ is trivial. Look at the following for $\gamma + 1$. By definition of exponentiation and by the inductive assumption we have: $\alpha^{\beta+\gamma+1} = \alpha^{\beta+\gamma}\odot\alpha = (\alpha^\beta\odot\alpha^\gamma)\odot\alpha = \alpha^\beta\odot(\alpha^\gamma\odot\alpha) = \alpha^\beta\odot\alpha^{\gamma+1}$.

Now, in case γ is a limit ordinal, we argue as follows. $\alpha^{\beta+\gamma} = \mathrm{Sup}_{\gamma'<\gamma}\alpha^{\beta+\gamma'} = \mathrm{Sup}_{\gamma'<\gamma}\alpha^\beta\odot\alpha^{\gamma'} = \alpha^\beta\odot\alpha^\gamma$. This last equality is derived from the definition of $\mu\odot\delta$ when δ is a limit ordinal, and from the fact that $\mathrm{Sup}_{\gamma'<\gamma}\alpha^{\gamma'} = \alpha^\gamma$. So (5) holds.

The case $\gamma = 0$ for (6) is trivial. Look at the following for (6) in the successor case $\gamma + 1$.

$$(\alpha^\beta)^{\gamma+1} = (\alpha^\beta)^\gamma\odot\alpha^\beta = \alpha^{\beta\gamma}\odot\alpha^\beta = \alpha^{\beta\gamma+\beta} = \alpha^{\beta(\gamma+1)}.$$

Here, we have used in the second equality the induction hypothesis, and (5) in the third equality.

Formula (6), in case γ is limit, follows from the following equalities.

$$(\alpha^\beta)^\gamma = \mathrm{Sup}_{\gamma'<\gamma}(\alpha^\beta)^{\gamma'} = \mathrm{Sup}_{\gamma'<\gamma}\alpha^{(\beta\gamma')} = \alpha^{\beta\gamma}.$$

□

For many ordinals μ, $\mu^{H\alpha} = \mu^\alpha$ holds for all α's. That is, the Hessenberg based exponentiation coincides with the regular one. For example:

Lemma 1.7 *For every α, $\omega^{H\alpha} = \omega^\alpha$.*

Proof. By induction on α. The cases $\alpha = 0, 1$ and α a limit ordinal are very easy. Now consider the successor case:
$$\omega^{H(\alpha+1)} = \omega^{H\alpha} \odot \omega = \omega^\alpha \odot \omega = \omega^\alpha \omega = \omega^{\alpha+1} . \qquad \square$$

For instance, note that $(\omega + 2)^{\omega+1} < (\omega + 2)^{H(\omega+1)}$.

The following lemma is used in the proof of Lemma 1.11

Lemma 1.8 *If $\gamma < \alpha \odot \beta$ then for some $k < \omega$*

$$\gamma = \alpha_1 \oplus \cdots \oplus \alpha_k \oplus \alpha \odot \beta_0$$

where $\alpha_i < \alpha$ for every i and $\beta_0 < \beta$, and in fact $\beta_0 + k \leq \beta$.

Moreover, if $\gamma' < \alpha_1 \oplus \cdots \oplus \alpha_k \oplus \alpha \odot \beta_0$, then for some $\ell \in \omega$:

$$\gamma' = \alpha'_1 \oplus \cdots \oplus \alpha'_k \oplus \alpha'_{k+1} \oplus \cdots \alpha'_{k+\ell} \oplus \alpha \odot \beta'_0$$

where $\alpha'_i \leq \alpha_i$ for each $1 \leq i \leq k$ and either

1. *$\ell = 0$ and $\beta'_0 = \beta_0$, or*

2. *$\ell > 0$ and then $\beta'_0 + \ell \leq \beta_0$.*

Proof. We prove the first statement by induction on β. In case β is 0 or is a limit ordinal the statement is obvious. So assume that $\gamma < \alpha \odot (\beta + 1) = \alpha \odot \beta \oplus \alpha$. Then (by Equation (1))

$$\gamma = \mu \oplus \alpha_0 \text{ where } \mu \leq \alpha \odot \beta \text{ and } \alpha_0 \leq \alpha .$$

If $\alpha_0 = \alpha$ then necessarily $\mu < \alpha \odot \beta$ and the inductive hypothesis applied to μ gives $\mu = \alpha_1 \oplus \cdots \oplus \alpha_k \oplus \alpha \odot \beta_0$ with $\beta_0 + k \leq \beta$. Then

$$\gamma = \alpha_1 \oplus \cdots \oplus \alpha_k \oplus \alpha \odot \beta_0 \oplus \alpha = \alpha_1 \oplus \cdots \oplus \alpha_k \oplus \alpha \odot (\beta_0 + 1) ,$$

and $\beta_0 + 1 + k \leq \beta + 1$ as required.

If $\alpha_0 < \alpha$, then there are two subcases: $\mu = \alpha \odot \beta$ and $\mu < \alpha \odot \beta$. The first subcase is obvious (giving $\gamma = \alpha_0 \oplus \alpha \odot \beta$), and the inductive hypothesis is applied in the second subcase.

The "Moreover" part of the lemma is a conclusion of its first part. If

$$\gamma' < [\alpha_1 \oplus \cdots \oplus \alpha_k] \oplus [\alpha \odot \beta_0]$$

Then apply (1) to the Hessenberg sum of the two ordinals enclosed in the square brackets, and write $\gamma' = \gamma_0 \oplus \gamma_1$ with $\gamma_0 \leq \alpha_1 \oplus \cdots \oplus \alpha_k$, and $\gamma_1 \leq \alpha \odot \beta_0$. In case $\gamma_1 < \alpha \odot \beta_0$, use the first part of the lemma. $\square$

Ranks of posets

Any well-founded poset A has an associated rank function $\mathrm{rk} : A \to Ord$ satisfying

$$\mathrm{rk}(a) = \mathrm{Sup}\{rk(x) \mid x <^A a\}.$$

The range of rk, namely $\mathrm{Sup}_{a \in A}\mathrm{rk}(a)$ is called the *length* of A and is denoted $\mathrm{rk}(A)$ for simplicity. If $\rho : A \to Ord$ is any order preserving function, then $\mathrm{rk}(a) \le \rho(a)$ for every $a \in A$. Thus $\mathrm{rk}(A)$ is the least ordinal into which there is an order preserving function from A. The rank function "leaves no gaps": if $\gamma < \mathrm{rk}(a)$ then for some $a' <^A a$ $\gamma = \mathrm{rk}(a')$.

The basic reason that the natural sum is so useful for us is the following simple lemma.

Lemma 1.9 *If $r_X : X \to \alpha$ and $r_Y : Y \to \beta$ are the rank functions of the well-founded posets X and Y, then the rank function rk on $X \times Y$ satisfies:*

$$rk(\langle x, y \rangle) = r_X(x) \oplus r_Y(y).$$

Proof. Since $\oplus$ is order preserving in both coordinates, the expression

$$r_X(x) \oplus r_Y(y)$$

on the right-side is an order preserving function from $X \times Y$ into $\alpha \oplus \beta$. From this follows the $\le$ part of the equality. Equality follows from Equation (1), namely that if $\gamma < r_X(x) \oplus r_Y(y)$ then $\gamma = \alpha_0 \oplus \beta_0$ for some $\alpha_0 \le r_X(x)$, $\beta_0 \le r_Y(y)$. So if 7 is violated, then it is violated by some minimal pair $\langle x, y \rangle$ is $X \times Y$. But then $\mathrm{rk}(\langle x, y \rangle) = \gamma < r_X(x) \oplus r_Y(y)$ implies that $\gamma = \alpha_0 \oplus \beta_0$ for some $\alpha_0 \le r_X(x)$, $\beta_0 \le r_Y(y)$ and strict inequality must hold at least once. Say $\alpha_0 < r_X(x)$ and then there is $x_0 < x$ with $\alpha_0 = \mathrm{rk}_X(x_0)$ and $y_0 \le y$ with $\mathrm{rk}_Y(y_0) = \beta_0$. Now, as $\langle x_0, y_0 \rangle < \langle x, y \rangle$, $\mathrm{rk}(\langle x_0, y_0 \rangle) = \mathrm{rk}_X(x_0) \oplus \mathrm{rk}_Y(y_0) = \gamma$ which is a contradiction! (compare with Fraïssé [4], and see Abraham [1] for related results.) $\square$

On the antichain rank

Observe that P satisfies the FAC iff the set of antichains of P under inverse inclusion forms a well-founded poset. That is, there is no infinite strictly $\subset$-increasing sequence of antichains in P. When P satisfies the FAC, we denote by $\langle \mathcal{A}(P), \supset \rangle$ the poset of all non-empty antichains of P under inverse inclusion. The rank function on $\mathcal{A}(P)$ is denoted $\mathrm{rk}_{\mathcal{A}}$, and its image (which is an ordinal) is called the length (or the rank) of $\mathcal{A}(P)$. For simplicity of notation we denote by $\mathrm{rk}_{\mathcal{A}}(P)$ this rank. (This may be confusing if P itself is a finite antichain.) Denote by $\mathcal{A}^{+}(P) = \mathcal{A}(P) \cup \{\emptyset\}$ the collection of all antichains of P, including the empty antichain which is not in $\mathcal{A}(P)$, and extend $\mathrm{rk}_{\mathcal{A}}$ on the empty antichain by defining $\mathrm{rk}_{\mathcal{A}}(\emptyset) = \mathrm{rk}_{\mathcal{A}}(P)$.

We have the following results:

Lemma 1.10 *If P and Q are FAC posets with antichain rank functions r_P and r_Q, then the perpendicular sum $R = P \perp Q$ is also a FAC poset and its rank function $rk_{\mathcal{A}}$ satisfies:*

$$rk_{\mathcal{A}}(X) = r_P(X \cap P) \oplus r_Q(X \cap Q)$$

for every antichain X in R (including the empty antichain). In particular $rk_{\mathcal{A}}(R) = r_P(P) \oplus r_Q(Q)$.

Proof. The proof is a is a direct consequence of Lemma 1.9 and the fact that $\mathcal{A}^{+}(P \perp Q) = \mathcal{A}^{+}(P) \times \mathcal{A}^{+}(Q)$. (Recall that $\mathcal{A}^{+}(X) = \mathcal{A}(X) \cup \{\emptyset\}$ is the set of all antichains of X, including the empty one.) So $\mathrm{rk}_{\mathcal{A}}(R) = \mathrm{rk}_{\mathcal{A}}(\emptyset) = r_P(\emptyset) \oplus r_Q(\emptyset) = r_P(P) \oplus r_Q(Q)$. □

Lemma 1.11 *If P and Q are FAC posets, and $R = P \cdot Q$ is the lexicographical product (that is the lexicographic sum of copies of P along Q), if r_P and r_Q are the antichain rank functions on $\mathcal{A}(P)$ and $\mathcal{A}(Q)$, then R is a FAC poset too and the length of $\mathcal{A}(R)$ is $r_P(P) \odot r_Q(Q)$.*

Proof. Lemma 1.8 is used in the proof. Any antichain X in R can be represented as a union of P-antichains along some Q-antichain. That is $X = \bigcup\{A_q \mid q \in B\}$ where B is an antichain in Q and each A_q is an antichain in P. If $B = \{q(1), \ldots, q(n)\}$, and $\alpha = r_P(P)$, define

$$f(X) = r_P(A_{q(1)}) \oplus r_P(A_{q(2)}) \oplus \cdots \oplus r_P(A_{q(n)}) \oplus \alpha \odot r_Q(B).$$

We shall prove that f is the rank function for R. Having done so, we can put $X = \emptyset$ to obtain the lemma. First we prove the following:

> *Claim 1. The function f is order reversing: If $X_1 \subset X_2$ are two antichains in R, then $f(X_1) > f(X_2)$.*

This claim implies that $r_R(X) \leq f(X)$ for any antichain X in R (r_R is the antichain rank for R).

Proof of Claim 1. Suppose that antichain X_1 is represented with P-antichains A^1_q for $q \in B_1$, and antichain X_2 is represented with A^2_q for $q \in B_2$. Then $X_1 \subset X_2$ implies that $B_1 \subseteq B_2$ and for $q \in B_1$ $A^1_q \subseteq A^2_q$. There are two cases to consider: When $B_1 = B_2$ and when $B_1 \subset B_2$.

In the first case it is easy to see that $f(X_1) > f(X_2)$, because the Hessenberg sum is (strictly) order preserving in all coordinates.

In the second case, let $\ell \geq 1$ be the number of points in B_2 not in B_1. Then $r_Q(B_2) + \ell \leq r_Q(B_1)$. This is so, since if $B_2 = B_1 \cup \{b_1, b_2, \ldots, b_\ell\}$, then

$$\mathrm{rk}_Q(B_1) > \mathrm{rk}_Q(B_1 \cup \{b_1\}) > \mathrm{rk}_Q(B_1 \cup \{b_1, b_2\}) > \cdots > \mathrm{rk}_Q(B_2).$$

Now

$$\begin{aligned} f(X_2) = r_P(A^2_{q(1)}) \oplus \cdots \oplus r_P(A^2_{q(n)}) \oplus r_P(A^2_{q(n+1)}) \oplus \cdots \\ \oplus r_P(A^2_{q(n+\ell)}) \oplus \alpha \odot r_Q(B_2) \end{aligned}$$

(where $B_1 = \{q(1), \ldots, q(n)\}$, $B_2 = \{q(1), \ldots, q(n+\ell)\}$). Hence

$$\begin{aligned} f(X_2) &< r_P(A^1_{q(1)}) \oplus \cdots \oplus r_P(A^1_{q(n)}) \oplus \overbrace{\alpha \oplus \cdots \oplus \alpha}^{\ell \ times} \oplus \alpha \odot r_Q(B_2) \\ &= r_P(A^1_{q(1)}) \oplus \cdots \oplus r_P(A^1_{q(n)}) \oplus \alpha \odot (r_Q(B_2) + \ell) \\ &\leq f(X_1). \end{aligned}$$

This ends the proof of Claim 1.

Applying the claim to $X_1 = \emptyset$ we get that $f(X) < \alpha \odot \beta$ for any nonempty antichain X in R (where $\alpha = r_P(P)$ and $\beta = r_Q(Q)$).

To prove equality of f and r_R, we assume (towards a contradiction) that for some antichain X, with a minimal f-value,

$$r_R(X) = \gamma < f(X).$$

We intend to extend the antichain X to some antichain X' that satisfies a corresponding formula, in contradiction to the minimality of $f(X)$.

We present $X = \bigcup \{A_q \mid q \in B\}$, where $B = \{q(i) \mid i = 1, \ldots, n\}$, and write

$$f(X) = r_P(A_{q(1)}) \oplus \cdots \oplus r_P(A_{q(n)}) \oplus \alpha \odot r_Q(B).$$

Then $\gamma < f(X)$ implies, by (1), that

$$\gamma = \alpha_1 \oplus \cdots \alpha_n \oplus \cdots \oplus \alpha_{n+\ell} \oplus \alpha \odot \beta_0$$

where $\ell \in \omega$, $\alpha_i \leq r_P(A_{q(i)})$ and $\beta_0 + \ell \leq r_Q(B)$ by Lemma 1.8.

We shall define $X' \supset X$ with $f(X') \geq \gamma$. Thus $r_R(X') < \gamma \leq f(X')$ contradicts the minimality of $f(X)$. First a simple observation.

> *Claim 2. If B is an antichain of Q such that $rk_{\mathcal{A}}(B) = \beta$ and $\beta_0 < \beta$ is an ordinal such that $\beta_0 + \ell \leq \beta$ (where $\ell \in \omega$), then there is an antichain $B' \supset B$ with $rk_{\mathcal{A}}(B') \geq \beta_0$ and such that $B' \setminus B$ contains ℓ points.*

Proof. Define inductively $B \subset B_1 \subset \cdots \subset B_\ell$ with $\mathrm{rk}_{\mathcal{A}}(B_1) = \beta_0 + (\ell - 1), \ldots, \mathrm{rk}_{\mathcal{A}}(B_\ell) = \beta_0$. Then $B_\ell \setminus B$ contains at least ℓ points, and any $B \subset B' \subseteq B_\ell$ containing exactly ℓ additional points concludes the claim.

The construction of X' us done in two steps. Since $\beta_0 + \ell \leq r_Q(B)$, by Claim 2, we first add ℓ new points to B and obtain an antichain $B' \supset B$ in Q with $r_Q(B') \geq \beta_0$. Having defined B' pick for each $q \in B'$ an antichain A'_q in P that extend A_q (if $q \in B$) and with rank as given by the corresponding ordinal α_i. Then $f(X') \geq \gamma$ follows. (To argue that $X' \neq X$ consider separately the case $l = 0$, in which $\alpha_i < r_P(A_{q(i)})$ for some i, and the case $l > 0$.) □

2 The generalized Hausdorff hierarchy

Fix an ordinal $\rho \geq 1$ which will denote antichain ranks. We shall define and study in this section the class $\mathcal{H}^\rho$ which is the least class of posets that contain all the well-founded posets with antichain rank $\leq \rho$ and is closed under inversion and lexicographical sums. We shall stratify $\mathcal{H}^\rho$ in the form $\mathcal{H}^\rho = \bigcup_{\alpha \in Ord} \mathcal{H}^\rho_\alpha$ and investigate the properties of $\mathcal{H}^\rho_\alpha$. In particular we prove here that for any $P \in \mathcal{H}^\rho_\alpha$ $\mathrm{rk}_{\mathcal{A}}(P) \leq \rho^\alpha$ (Hessenberg based exponentiation).

Definition 2.1 *Let $\rho \geq 1$ be our fixed ordinal. By induction on $\alpha \in Ord$ define the class $\mathcal{H}^\rho_\alpha$ as follows.*

1. *$\mathcal{H}^\rho_0 = \{1\}$ consists of a single poset 1 containing just one element.*

2. *$\mathcal{H}_1^\rho$ is the class of all posets P with $rk_{\mathcal{A}}(P) \leq \rho$ and such that P or P^* (the inverse of P) is well-founded. So $\mathcal{H}_1^1$ is the class of all well-ordered sets and their inverses. (namely, order types of ordinals and inverses of ordinals).*

3. *For any limit ordinal δ, $\mathcal{H}_\delta^\rho = \bigcup_{\alpha<\delta} \mathcal{H}_\alpha^\rho$.*

4. *If $\beta = \alpha + 1$ is a successor ordinal, then $\mathcal{H}_\beta^\rho$ is the class of all posets P that are a lexicographical sum of the form*
$$P = \sum_{i \in I} P_i$$
where $P_i \in \mathcal{H}_\alpha^\rho$ and I is a poset with $rk_{\mathcal{A}}(I) \leq \rho$ and such that I or I^* is well founded, that is I is in $\mathcal{H}_1^\rho$.

Finally, set $\mathcal{H}^\rho = \bigcup_{\alpha \in Ord} \mathcal{H}_\alpha^\rho$, and $\mathcal{H} = \bigcup_{1 \leq \rho \in Ord} \mathcal{H}^\rho$.

We first state some simple properties concerning this hierarchy of posets $\mathcal{H}^\rho$.

Lemma 2.2 1. *$\mathcal{H}^\rho$ is the least class that contains the well-founded posets of antichain ranks $\leq \rho$ and is closed under lexicographical sums and inverses. In particular, $\mathcal{H}^1$ is the class originally defined by Hausdorff as the least class containing the ordinals that is closed under inverses and sums.*

2. *Each $\mathcal{H}_\alpha^\rho$ and $\mathcal{H}^\rho$ is closed under restrictions.*

3. *If $P \in \mathcal{H}^\rho$ then P is scattered and satisfies the FAC.*

4. *$aug(\mathcal{H}^\rho)$ (the collection of all augmentations of posets in $\mathcal{H}^\rho$) is closed under lexicographical sums, inverses, restrictions, and (naturally) augmentations. Every poset in $aug(\mathcal{H}^\rho)$ is scattered.*

Proof. For item 1, we shall only indicate the proof that $\mathcal{H}^\rho$ is closed under lexicographic sums. Prove by induction on α that if $I \in \mathcal{H}_\alpha^\rho$ and $P_i \in \mathcal{H}^\rho$ for each $i \in I$, then $\sum_{i \in I} P_i \in \mathcal{H}^\rho$. Case $\alpha = 1$ is obvious by the definition of the hierarchy, and case α a limit ordinal is trivial. Let $I \in \mathcal{H}_{\alpha+1}^\rho$ and for each $i \in I$, $P_i \in \mathcal{H}^\rho$ be given. We have $I = \sum_{j \in J} L_j$ with $L_j \in \mathcal{H}_\alpha^\rho$, and $J \in \mathcal{H}_1^\rho$. Then $P = \sum_{i \in I} P_i = \sum_{j \in J}(\sum_{i \in L_j} P_i)$. Now because $J \in \mathcal{H}_1^\rho$, $P \in \mathcal{H}^\rho$. Items 2, 3 and 4 are obvious (note that the last statement of item 4 follows from Lemma 1.2). $\square$

Theorem 2.3 *If $R \in \mathcal{H}^\rho_\alpha$ then $rk_{\mathcal{A}}(R) \leq \rho^{H\alpha}$ (the superscripted H refers to the Hessenberg based exponentiation, and we omit it below).*

Proof. The proof is by induction on α. If $\alpha = 0$ and $R \in \mathcal{H}^\rho_0$, then R consists of a single point and so $\mathrm{rk}_{\mathcal{A}}(R) = 1$. The case $\alpha = 1$ follows from the definition: That is, if $R \in \mathcal{H}^\rho_1$ then $\mathrm{rk}_{\mathcal{A}}(R) \leq \rho = \rho^1$. In case α is a limit ordinal the inductive claim follows trivially, and so we assume that $R \in \mathcal{H}^\rho_{\alpha+1}$ and the lemma holds for $\mathcal{H}^\rho_\alpha$.

So $R = \sum_{i\in I} P_i$ where $P_i \in \mathcal{H}^\rho_\alpha$ and $I \in \mathcal{H}^\rho_1$. It follows from the inductive assumption that $\mathrm{rk}_{\mathcal{A}}(P_i) \leq \rho^\alpha$, and it is not difficult to find a single poset P with $\mathrm{rk}_{\mathcal{A}}(P) \leq \rho^\alpha$ and such that $P_i \subseteq P$ for each $i \in I$ (just align the P_i's along some ordinal). Then $R \subseteq P \cdot I$ and hence $\mathrm{rk}_{\mathcal{A}}(R) \leq \mathrm{rk}_{\mathcal{A}}(P \cdot I) = \mathrm{rk}_{\mathcal{A}}(P) \odot \mathrm{rk}_{\mathcal{A}}(I) \leq \rho^\alpha \odot \rho = \rho^{\alpha+1}$. (Use Lemma 1.11.)

3 A generalization of Hausdorff's theorem

This is the main section of our paper.

It is a quite useful, well-known observation that every poset P possesses a cofinal well-founded set. (Recall that $H \subseteq P$ is *cofinal* iff every $p \in P$ has an extension in H.) To prove this, define h_ξ by induction until $\{h_\xi \mid \xi < \alpha\} = H_\alpha$ is cofinal. The inductive requirement is that for every $\xi_1 < \xi_2$ we have $h_{\xi_2} \not<^P h_{\xi_1}$; this ensures that H_α is well-founded.

Given a cofinal well-founded sequence $H = \{h_\xi \mid \xi < \alpha\} \subseteq P$ as above, we define the *stratification* of P (induced by H) as the sequence of disjoint subposets

(7) $\quad A(h_\xi) = \{p \in P \mid p \leq^P h_\xi \text{ but } p \not\leq^P h_{\xi'} \text{ for } \xi' < \xi\}, \quad \text{for} \quad h_\xi \in H.$

We consider $A(h_\xi)$ as a restriction of P.

Suppose that P satisfies FAC and H' is a well–founded subset of P. We define

$$(\leq H') = \bigcup_{h \in H'} (\leq h),$$

and

$$F(H') = P \setminus (\leq H').$$

We say that $p \in F(H')$ is *minimal over* H' if for every $q \in F(H') \cap (\leq p)$,

$$h \leq q \text{ if and only if } h \leq p \text{ for every } h \in H.$$

If P satisfies the FAC property, then the cofinal well-founded sequence H can be defined in such a way that P is an augmentation of the lexicographical sum $\sum_{h \in H} A(h)$ obtained from the stratification of P induced by H (using the existence of minimal point). Towards this aim we bring the following lemma.

Lemma 3.1 *If P is a FAC poset, then there exists a cofinal well-founded sequence $H = \{h_\xi \mid \xi < \alpha\}$ such that, for every $\xi < \alpha$, h_ξ is minimal over $H_\xi = \{h_\mu \mid \mu < \xi\}$.*

Proof. The construction of H is done inductively as before, but with the following change. Suppose that $H_\xi = \{h_\mu \mid \mu < \xi\}$ has been defined with each h_μ minimal over its set of predecessors. If H_ξ is cofinal in P, then we are done. Otherwise $F_\xi \stackrel{def}{=} F(H_\xi)$ is non–empty, and we must find some minimal element $h = h_\xi \in F_\xi$ over H_ξ. Suppose that no such h can be found, and construct an infinite descending sequence $p(0) >^P \cdots >^P p(i) > \cdots$ in F_ξ and corresponding members $h_{\mu(i)}$ of H_ξ such that for each $i < \omega$

$$h_{\mu(i)} <^P p(i) \text{ but } h_{\mu(i)} \not<^P p(i+1)). \tag{8}$$

Indeed, $p(0) \in F_\xi$ is arbitrarily chosen, and having defined $p(i)$ use the supposition that $p(i)$ is not minimal over H_ξ to find $p(i+1) <^P p(i)$ in F_ξ with a different type: for some $h_{\mu(i)}$ in H_ξ (8) holds. Since P satisfies the FAC condition and H_ξ is well–founded, H_ξ is a well-founded FAC poset, and hence there must be $i < j$ with $h_{\mu(i)} <^P h_{\mu(j)}$ (apply Lemma 1.1). But then

$$h_{\mu(i)} <^P h_{\mu(j)} <^P p(j) \leq^P p(i+1)$$

contradicts (8). □

This proves Lemma 3.1, and now we look at the stratification $\langle A(h) \mid h \in H \rangle$ obtained by a well-founded cofinal sequence H with the "minimality condition" as in the lemma. Form the lexicographical sum $S = \sum_{h \in H} A(h)$ where $A(h)$, as defined in (7), is the collection of points $\leq^P h$ that are not in any previous $A(h')$. Then the universe of S is the universe of P, since H is cofinal. We claim that P augments S. Indeed, assume that $p_1 <^S p_2$. There are two possibilities.

1. For some $h \in H$, $p_1, p_2 \in A(h)$. Then $p_1 <^S p_2$ iff $p_1 <^P p_2$ by definition.

2. $p_1 \in A(h^1)$ and $p_2 \in A(h^2)$, and $p_1 <^S p_2$ because $h^1 <^P h^2$. Then by the "special minimality property" any point in $A(h^2)$ is above h^1 and thus any point in $A(h^1)$ is $<^P$-below any point in $A(h^2)$.

Corollary 3.2 *Suppose P is a FAC poset with $rk_{\mathcal{A}}(P) \leq \rho$. If for every $x \in P$ $(\leq x) \in aug(\mathcal{H}^\rho)$, then $P \in aug(\mathcal{H}^\rho)$.*

Proof. Find first an ordinal α such that $(\leq x) \in \text{aug}(H^\rho_\alpha)$ for every $x \in P$, and we will prove that $P \in \text{aug}(H^\rho_{\alpha+1})$. Construct a cofinal well-founded sequence $H \subseteq P$ as in Lemma 3.1, and form the lexicographical sum

$$S = \sum_{h \in H} A(h)$$

of the resulting stratification. We consider the following facts:

1. P augments S.

2. Each $A(h)$ is a restriction of $(\leq h)$ to some subset, and hence $A(h) \in \text{aug}(\mathcal{H}^\rho_\alpha)$ (as $\mathcal{H}^\rho_\alpha$ and hence $\text{aug}(\mathcal{H}^\rho_\alpha)$ are closed under restrictions). Say $A(h)$ augments $A^0(h) \in \mathcal{H}^\rho_\alpha$.

3. H is well-founded and $\text{rk}_{\mathcal{A}}(H) \leq \rho$ (as $H \subseteq P$). Thus $H \in \mathcal{H}^\rho_1$.

4. So $S^0 = \sum_{h \in H} A^0(h) \in \mathcal{H}^\rho_{\alpha+1}$. S augments S^0 since each $A(h)$ augments $A^0(h)$.

5. But S augments S^0 and P augments S imply that P augments S^0. Thus $P \in \text{aug}(\mathcal{H}^\rho_{\alpha+1})$.

This proves Corollary 3.2. □

Since $\mathcal{H}^\rho_1$ also contains all inverses of well-founded posets of antichain rank $\leq \rho$, the same result holds for every poset P with $\text{rk}_{\mathcal{A}}(P) \leq \rho$ and such that each $(\geq x)$ is in $\text{aug}(\mathcal{H}^\rho)$.

Corollary 3.3 *1. If P is a FAC poset with $rk_{\mathcal{A}}(P) \leq \rho$, but $P \notin aug(\mathcal{H}^\rho)$, then for some $p \in P$ $(\leq p) \notin aug(\mathcal{H}^\rho)$ and $(\geq p) \notin aug(\mathcal{H}^\rho)$.*

2. If P is a FAC poset with $rk_{\mathcal{A}}(P) \leq \rho$ but $P \notin aug(\mathcal{H}^\rho)$, then $\mathbb{Q}$ is embeddable in P, that is P is not scattered.

Proof. We first prove *1.* Suppose that P is a FAC poset with $\mathrm{rk}_{\mathcal{A}}(P) \leq \rho$, and let $P^0 = \{p \in P \mid (\leq p) \in \mathrm{aug}(\mathcal{H}^\rho)\}$ and $P^1 = \{p \in P \mid (\geq p) \in \mathrm{aug}(\mathcal{H}^\rho)\}$. Assume that $P = P^0 \cup P^1$. We must prove that $P \in \mathrm{aug}(\mathcal{H}^\rho)$. P^0 is an initial segment of P, and P^1 a final segment (that is $(\geq q) \subseteq P^1$ for every $q \in P^1$). By Corollary 3.2, both P^0 and P^1 are in $\mathrm{aug}(\mathcal{H}^\rho)$: Say $Q^0,\ Q^1 \in \mathcal{H}^\rho$ are such that $P^i \in \mathrm{aug}(Q^i)$ for $i = 0, 1$. There are two cases: either for every $x \in P^0$ and $y \in P^1$ $(x \leq^P y)$, or else some $x \in P^0$ and $y \in P^1$ are incomparable.

In the first case P is the sum $P^0 + (P^1 \setminus P^0)$ of two posets that are in $\mathrm{aug}(\mathcal{H}^\rho)$, and hence P is in $\mathrm{aug}(\mathcal{H}^\rho)$ as well (because $\mathcal{H}^\rho$ is closed under sums).

In the second case there are two incomparable points in P and thus $\rho > 1$. Hence $\mathcal{H}^\rho_1$ contains the poset consisting of two incomparable members, and so $\mathcal{H}^\rho$ is closed under perpendicular sums. As P is an augmentation of $Q^0 \perp (Q^1 \setminus Q^0)$ it follows that $P \in \mathrm{aug}(\mathcal{H}^\rho)$.

The second part of our lemma requires that we find an embedding f of the rational chain $\mathbb{Q}$ into P, in case P is a FAC poset with antichain rank $\leq \rho$ and such that $P \notin \mathrm{aug}(\mathcal{H}^\rho)$.

Fix an enumeration q_n of the rational numbers, and decide in the n–th step of the construction the value of $f(q_n) \in P$. The inductive assumption is that: for distinct $i, j < n$, if $q_i <^{\mathbb{Q}} q_j$ then $f(q_i) <^P f(q_j)$ and the P–interval

$$[f(q_i), f(q_j)] = \{p \in P \mid f(q_i) \leq p \leq q_j\}$$

is not in $\mathrm{aug}(\mathcal{H}^\rho)$, and similarly that $(\leq f(q_i))$ and $(\geq f(q_i))$ are not in $\mathrm{aug}(\mathcal{H}^\rho)$. The first part of the lemma is used at each step. Thus we construct f which is an embedding of $\mathbb{Q}$ into P. □

Since the second part of the corollary is our main result we restate it for conclusion:

Theorem 3.4 *aug($\mathcal{H}^\rho$), which is the closure of the class of all well-founded posets with antichain rank $\leq \rho$ under inversion, lexicographical sums, and augmentation, contains the class of all scattered FAC posets with ranks $\leq \rho$. So aug($\mathcal{H}$), which is the closure of the well-founded posets with the FAC property under inversion, lexicographical sums, and augmentation, is the class of all scattered FAC posets.*

Hausdorff's theorem is obtained for $\rho = 1$.

4 On the need for augmentations

Is it possible to prove a stronger statement in which there is no need for augmentations? Namely, is it true that $\mathcal{H}$ already contains all scattered FAC posets? We shall show in this section that this is not the case. We make first a definition.

Let P be a poset. A subset S of P is called a *strong interval of* P iff

> *for every $x, y \in S$ and $p \in P \setminus S$: $p \leq x$ iff $p \leq y$, and $p \geq x$ iff $p \geq y$.*

In other words, S is a strong interval if any $p \in P \setminus S$ is either greater than all members of S, or smaller, or incomparable with all of them. For instance, $\emptyset$, P, and every singleton are strong intervals of any poset P. Also if P is a lexicographical sum $\sum_{q \in Q} P_q$, then each P_q is a strong interval of P.

The following trivial observations are useful: (1) Any strong interval is an interval (a convex set). (2) If S is a strong interval of P, if $s, s' \in S$ and $p \in P$ is such that p is comparable with s but not with s', then $p \in S$.

We describe a FAC poset P that is not in $\mathcal{H}$. Let Let $A = \{a_n \mid n \in \omega\}$ and $B = \{b_n \mid n \in \omega\}$ be two disjoint countable sets, and let $Q = A \cup B$. Let $<$ be the ordering on Q defined by:

$$a_0 < a_1 < a_2 < a_3 < \cdots b_3 < b_2 < b_1 < b_0 \ .$$

Let $\{0, 1\}$ be the linear ordering of two points $0 < 1$, and then let

$$P = Q \times \{0, 1\}$$

be the Cartesian product. P has a least element $p^- = \langle a_0, 0 \rangle$ and a greatest element $p^+ = \langle b_0, 0 \rangle$.

Lemma 4.1 *The strong intervals of P are $\emptyset$, the singletons, and the sets of the form $P \setminus X$ where $X \subseteq \{p^-, p^+\}$. Hence every strong interval of P that contains at least two points contains a restriction isomorphic to P.*

Proof. Clearly each of these sets, except the singletons and the empty set, is a strong interval of P, and contains a restriction isomorphic to P. For example, $S = P \setminus \{p^+\}$ is a strong interval because p^+ is greater than all members of S. Also, $S \setminus \{p^-\}$ contains a restriction isomorphic to P.

Now let S be any strong interval of P containing at least two points, and we shall prove that S has the required form. Let $Q_0 = Q \times \{0\}$, $Q_1 = Q \times \{1\}$.

We claim that it is not possible for S to be contained in Q_0. Indeed, in such a case pick two points in S: they are of the form $\langle i, 0\rangle$, $\langle j, 0\rangle$ with $i <^Q j$. But then $e = \langle i, 1\rangle \notin S$ contradicts the assumption that S is a strong interval, because e is above $\langle i, 0\rangle$ but is incomparable with $\langle j, 0\rangle$. In a similar vein S cannot be contained in Q_1. Thus we conclude that S contains a point $\langle i, 0\rangle$ from Q_0 and a point $\langle j, 1\rangle$ from Q_1. The proof proceeds now in four cases.

Case 1. Both p^- and p^+ are in S: Then $S = P$ because S is convex.

Case 2. $p^+ \notin S$, $p^- \in S$: Since $p^+ \notin S$, but $\langle j, 1\rangle \in S$ by our conclusion, necessarily $j \neq b_0$ and then $\langle b_0, 0\rangle$ is incomparable with $\langle j, 1\rangle$. Yet it is above p^-, and thus $\langle b_0, 0\rangle \in S$. Since $p^- \in S$ and S is an interval $Q_0 \subset S$. A similar argument can show that $Q_1 \setminus \{p^+\} \subset S$: Because any $\langle k, 1\rangle$ is incomparable with $\langle b_0, 0\rangle \in S$ but is above p^-. Thus $S = P \setminus \{p^+\}$, and in particular S contains an isomorphic copy of P.

Case 3. $p^+ \in S$, $p^- \notin S$. This case is symmetric to Case 2, and $S = P \setminus \{p^-\}$ contains again a restriction isomorphic to P.

Case 4. $p^-, p^+ \notin S$. Then $\langle b_0, 0\rangle$ is in S as it is incomparable with $\langle j, 1\rangle$ but comparable with $\langle i, 0\rangle$. Similarly $\langle a_0, 1\rangle \in S$. Except for p^-, any pair in Q_0 is incomparable with $\langle a_0, 1\rangle$ and comparable with $\langle b_0, 0\rangle$, and thus $Q_0 \setminus \{p^-\} \subset S$. Similarly, $Q_1 \setminus \{p^+\} \subset S$. Thus $S = P \setminus \{p^-, p^+\}$, and again S contains an isomorphic copy of P. □

Corollary 4.2 *$P \notin \mathcal{H}$.*

Proof. Look at

$$\mathcal{H}' = \{Q \in \mathcal{H} \mid \text{ no restriction of } Q \text{ is isomorphic to } P \ \}.$$

Prove that $\mathcal{H}'$ contains all FAC well-founded posets and their inverses, and that it is closed under restrictions, inverses, and lexicographical sums. (Hence $\mathcal{H}' = \mathcal{H}$ and so $P \notin \mathcal{H}$.)

Indeed, the main point is the proof that if P can be represented as a lexicographical sum $R = \sum_{q \in Q} P_q$, then either a restriction of Q is isomorphic to P, or else a restriction of some P_q is isomorphic to P. To prove this, consider first the case that any P_q is a singleton or the empty poset. Then Q is isomorphic to P. Otherwise some P_q contains two points. But each P_q is a strong interval of R, and hence this P_q contains a restriction isomorphic to P.

References

[1] U. Abraham. A note on Dilworth's theorem in the infinite case. *Order* **4**, 1987.107–125.

[2] R. Bonnet et M. Pouzet. Extension et stratification d'ensembles dispersés. C.R.A.S., Paris, Série A, 168, (1969), 1512-1515.

[3] P. W. Carruth. Arithmetic of ordinals with applications to the theory of ordered abelian groups. *Bulletin Amer. Math. Soc.* Vol. 48, 262–271, 1942.

[4] R. Fraïssé. *Theory of relations.* North-Holland, 1986 (Studies in Logic, Volume 118).

[5] F. Hausdorff. Grundzüge einer Theorie der Geordnete Mengen. *Math. Ann.* **65**, 435–505, 1908.

[6] G. Hessenberg. Grundbegriffe der Mengenlehre. *Abhandlungen der Friesschen Schule N.S.* Vol. 1.4 n. 75 Göttingen, 1906.

[7] A. Levy. *Basic Set theory.* Springer 1979 (Ω Series: Perspectives in Mathematical Logic).

[8] W. Sierpiński, Cardinal and Ordinal Numbers. (Monografie Matematyczne 34) Państwowe Wydawnictwo Naukowe, Warszawa 1958.