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On a Hardy space approach to the analysis of spectral factors*

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The paper presents a study of several problems related to spectral factorizations. We assume only a very weak form of coercivity for the $p \times p$ spectral function Φ and look at the set $\mathcal{W}^m$ of all rectangular, $p \times m$ spectral factors. The main object is the arithmetization of the geometry of the set of minimal, stable spectral factors by employing Hardy space techniques and the arithmetic of inner functions. Particular attention is paid to the study of various partial orders associated with the set of spectral factors.

1. Introduction

The object of this paper is to present a study of the spectral factorization problem in the multivariable case. We will consider rational spectral function Φ of dimension $p \times p$, of rank $m_0 \leq p$ and degree $2n$ and we will characterize all the minimal, in the sense of the McMillan degree, spectral factors of a given size $p \times m$ in the spectral domain, removing all constraints on rank and zero location. In particular only very weak coercivity assumptions will be made.

The topic of this paper has a long history, with roots in stochastic theory. We mention Anderson (1973), Faurre *et al.* (1979), Ruckebusch (1980), and in particular the series of seminal articles of Lindquist and Picci (1979 a–c, 1985 a, 1991) and Lindquist *et al.* (1995) as well as some papers by Pavon which have been the main inspiration of our work. This problem has been widely studied in the existing literature, although a particular aspect of the problem seems to have escaped all these investigations and this was the motivation for starting this work. For example, it is well known how to characterize all the spectral factors in terms of a Riccati inequality [see Anderson (1973) and Faurre (1976)]; nevertheless, a state space approach does not allow a precise characterization of the spectral factors of *fixed* dimension m where the usual constraints ($m_0 \leq m \leq p + n$) are satisfied (these constraints come in a very natural manner from the Positive Real Lemma: we refer the reader to Anderson (1973) for details). In Lindquist and Picci (1979 c) an alternative spectral domain characterization is given for non-full-rank Φ, but only for the internal case. In Lindquist *et al.* (1995) the external, but full-rank problem is treated. In conclusion, the sets of external factors of given size m of a non-full-rank density Φ (which we will call $\mathcal{W}^m$) are not very well characterized even today. Some properties of these sets are known [for example it is known it is a manifold for each m, see Batatchart and Gombani (1994)], but the underlying structure of these sets was quite unclear. The reasons for understanding such a problem are quite practical and come from time series analysis and multichannel signal transmission. The first problem occurs in

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econometrics, when a time series with a lot of variables should be explained in terms of the dynamics of few variables (factors) and some (white) disturbance error. The second occurs in decoding the signals which come to a cellular phone from all the antennas which are operating on that cell and are of course broadcasting also to other phones. It can be shown that both situations can be brought to a non-linear minimization problem on $\mathcal{W}^m$ (and the non-full-rank assumption on Φ is crucial), and this explains why it is interesting to know that it is manifold and why, more generally, it is important to understand its structure. We do so in continuous time (even if the problems we just mentioned have a discrete time formulation), because of some simplified features of this setting. The discrete time version can be obtained by a suitable version of the Cayley transform.

With this problem in mind, we were led in a natural way to an approach which differs from that of Lindquist and Picci. Since we have removed the rank constraints we obtain some technical results which are more general than those previously obtained. The main difference however is one of emphasis. In the Lindquist–Picci approach to stochastic realization theory, geometry is reigning supreme. The approach is abstract with all the corresponding advantages and disadvantages. The first big advantage is that working with a space directly constructed from the stochastic process, stochastic objects like the past and future spaces are conceptually clearly defined and are independent of any representation. The disadvantages arise from the difficulties in the computational process: since every subspace has its own functional representation, the geometry of the stochastic setting is lost in the frequency domain. For better or worse, we do away with all that. The basic idea is to map the stochastic domain with a single isomorphism to the frequency domain. With each spectral factor we get corresponding representation spaces which are subspaces of a fixed vectorial L^2 space of the imaginary axis. For normalization purposes we single out the minimal, stable, maximum phase spectral factor W_+. We study all other spectral factors in terms of their relation to W_+. The big advantage from our point of view is that the functional representations of these spaces in terms of inner functions, obtained using Beurling's theorem, are all geometrically correlated (the Hilbert space structure is preserved). This allows us to replace the abstract geometrical constructs by the arithmetic properites of inner functions, making computations much easier. A second important feature of our approach that it solves the problem we considered in the beginning, namely the characterization of $\mathcal{W}^m$.

The paper is structured as follows. We introduce some notation in §2. In §3, given a spectral function, we study the four minimal, extremal spectral factors. These factors are completely characterized by the requirement that all their zeros and poles lie in either the open left or right half-planes. Section 4 is devoted to a study of Toeplitz operators whose symbol is a quotient of inner function. This class of Toeplitz operators is important because of the many connections to geometry. For some different applications to system theoretic problems, see Fuhrmann and Helmke (1997).

In §5 we extend the scene by looking at the set of all minimal, stable rectangular spectral factors of a given size. Here the ground is laid for the use of factorization theory in the analysis of spectral factors. This is done via the use of the algebra of inner functions. The fine structure of the various factorizations associated with a given, stable spectral factor is analysed in Theorem 4. Since, via Beurling's theorem, inner functions are closely related to the geometry of invariant subspaces in Hardy spaces, this leads to many geometric relations. These connections are studied in depth, as well as various associated projection operators. Some of this is summarized in Proposition 15. Special attention is given to two extreme cases, those of internal and external factors and characterizations of these factors are given. This study is closely related to geometric control theory, however the results obtained are beyond the scope of this paper. These results are relegated to future publications, see Gombani and Fuhrmann (1998) and Fuhrmann (1998).

We pass on, in §6, to the study of a functional representation of the set of all minimal Markovian splitting subspaces, a key element in stochastic realization theory. With each minimal, stable spectral factor we associate a canonical state space, which serves also as the state space of a shift based realization. This allows us to link the geometry of splitting subspaces with the arithmetic properties of inner functions. One of the principal themes is to study a partial order relation in the set of all minimal, stable spectral factors. We begin, in Theorem 8, by studying the standard case of square, non-singular spectral factors. While it does not generalize trivially to the singular, rectangular case, it indicates the direction. In keeping with the spirit of this paper, which is functional oriented, we avoid the use of the Riccati equation and inequality. This theme will be picked up in a different publication. Once the various partial orders, related to spectral factors, are introduced, we proceed to show the equivalence of these orders. This is the heart of this section and one of the principal results of the whole paper. This is summarized in Theorem 9. We proceed, in Theorem 10, to characterize, functionally, the various spaces that appear in the scattering approach to stochastic realization theory. We conclude by studying some lattice properties of the set of all $p \times m$ stable, spectral factors. We note however that this set is not a complete lattice. The set of all minimal Markovian splitting subspaces spans a canonical subspace of H^2_+. We give a characterization of this space in §7.

The reader who studies this paper will notice immediately that there are many important topics that have been omitted. In particular, little emphasis has been placed on state space techniques. Also, topics relating to geometric control theory, in the context of Hardy spaces, had to be avoided. The reason for this is simple. Inclusion of these topics, important as they are, would have been beyond the scope of a single paper. As indicated above, these topics will be treated in subsequent papers.

In a sense, the content of this paper is an attempt to understand stochastic realization theory in the style of Lindquist and Picci from a different point of view. Our debt to their work is evident throughout.

2. Preliminaries and notation

We work in the Hilbert space setting of the plane; we define, see Hoffman (1962), $L^2(\mathbb{I})$ to be the set of the vector or matrix valued (the proper dimension will be clear from the context) square integrable functions on the imaginary axis, and H^2_+ to be the subspace of L^2 of functions that are the boundary values of analytic functions in the right half-plane and such that

$$\sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr}\left[F^*(x+iy)F(x+iy)\right] dy < \infty$$

where * denotes transposed conjugate. If F and G are column vectors, the inner product in H^2_+ is

$$\langle F, G\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(i\omega)F(i\omega)\,\mathrm{d}\omega$$

Analogously, H_+^∞ is the subspace of L^2 of functions analytic in the right half-plane and such that

$$\sup_{\mathrm{Re}\, s>0} \|F(s)\| < \infty$$

where $\|F(x+iy)\|$ denotes the usual matrix norm. (H_-^2 and H_-^∞ are defined similarly on the left half-plane).

Let F be function of L^2: we denote by P_+F (P_-F) the orthogonal projection of L^2 onto $H_+^2\,(H_-^2)$.

A $p\times m$ matrix valued function $\hat{R}\in H_+^2$ is said to be *row rigid* if $\hat{R}\hat{R}^* = I_p$ (clearly this entails $p\leq m$). It is *column rigid* if $p\geq m$ and $\hat{R}^*\hat{R} = I_m$. A function $Q\in H_+^\infty$ is inner if it is square and $Q^*Q = QQ^* = I$. It is well known that a column vector space M in H_+^2 is invariant under multipication by $\mathrm{E}^{-i\omega t}$ for $t\geq 0$ if and only if it is of the form $M = QH_+^2$ (Beurling's Theorem). Similarly, a row space N is invariant if and only if it is of the form $N = H_+^2 Q$. We set $H_c(Q) := \{QH_+^2\}^\perp$ and $H_r(Q) := \{H_+^2 Q\}^\perp$. Analogously, we define $\bar{H}_c(Q^*) := \{Q^* H_-^2\}^\perp$ and $\bar{H}_r(Q^*) := \{H_-^2 Q^*\}^\perp$, the orthogonal complement in these cases taken in H_-^2. Clearly, we have $P_+ \mathrm{e}^{i\omega t} H_c(Q) \subset H_c(Q)$ and similarly for $H_r(Q)$. A full column-rank $p\times m$ rational matrix function G in H_+^∞ is said to be *minimum-phase* or *outer* (on the right) if rank $G(s) = m$ for $\mathrm{Re}\, s > 0$. It is well known that a rational function F in H_+^∞ admits an outer inner factorization

$$F = F_0 Q$$

This factorization is unique up to a unitary constant matrix. For any W in L^2 or in L^∞ we define $W^*(s) := \overline{W(\bar{s})}^{\mathrm{T}}$, where $^-$ denotes conjugation and $^{\mathrm{T}}$ denotes transposition. It should be noted that for an inner function $Q^* = Q^{-1}$.

We say that an inner function Q_1 divides Q_2 on the right (left) if $Q_1^* Q_2 \in H_+^\infty$ $(Q_2 Q_1^* \in H_+^\infty)$. Given two inner functions Q_1 and Q_2 we denote the greatest common right (left) divisor by $Q_1 \wedge_R Q_2$ $(Q_1 \wedge_L Q_2)$ and the least common right (left) multiple by $Q_1 \vee_R Q_2$ $(Q_1 \vee_L Q_2)$. Two inner matrices are right (left) coprime if their greatest common right (left) divisor is the identity.

Two $m\times m$ inner matrix X and Y are *equivalent* if there exist inner matrices U, V such that $X \wedge_L V = I$ and $Y \wedge_R U = I$ and $XU = VY$. Given an inner matrix X, it can be shown that there exists an (essentially) unique diagonal inner matrix $D = \mathrm{diag}\{d_1, \ldots, d_m\}$ such that $d_i | d_{i+i}$ for $i = 1, \ldots, m-1$ which is similar to X. The inner factors d_i which are not identically 1 are called invariant factors of X.

Remark: To distinguish between operators on row and column spaces, we will denote the operations on columns in the usual manner, whereas operations on row spaces are denoted by writing the argument before the operator. For example, given an operator T acting in a row space X with a subspace V, we would write $V|T$ for the restriction of T to V. Ker $V|T$ would denote the kernel of this map. This notation is immediately applied in the following definition.

Definition 1: Let G be a $p\times m$ matrix function in L^∞. We define the Hankel operator H_G^c acting on the column Hardy space as

$$\left.\begin{aligned} H_G^c : H_+^2 &\to H_-^2 \\ f &\mapsto P_+ Gf \end{aligned}\right\} \tag{1}$$

Similarly, we define the Hankel operator H_G^r acting on row Hardy space by

$$\left.\begin{aligned} H_G^r : H_+^2 &\to H_-^2 \\ f &\mapsto fGP_+ \end{aligned}\right\} \tag{2}$$

The function G is called the *symbol* of the Hankel operator. Similarly, *conjugate Hankel operators* are defined, with the role of H_+^2 and H_-^2 interchanged. We will use the notation $\hat{H}_G^r$ and $\hat{H}_G^c$ for these operators.

From now on we assume all the functions to be rational. We denote a *realization* of a rational function W as

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

With $A^{\#}$ we denote the Moore–Penrose inverse of a matrix A.

3. Extremal spectral factors

Our object is to study the set of all minimal, not necessarily square, spectral factors of a given spectral function.

We assume we are given a rational spectral function Φ, that is a $p \times p$ proper rational matrix function which is non-negative on the imaginary axis. We do not assume, as is usual, that Φ is regular on the imaginary axis. We do assume however that it is *weakly coercive*, namely that Φ has constant rank, $m_0 \leq p$, on the extended imaginary axis, i.e. including at the point of infinity. Furthermore, we assume that Φ, which clearly satisfies $\Phi(s) = \Phi(-\bar{s})^*$, has McMillan degree $2n$.

A $p \times m$ proper rational matrix function W is called a *spectral factor* if $\Phi = WW^*$. Here, as elsewhere $W^*(s) = W(-\bar{s})^*$. A spectral factor W is called stable (antistable) if $W \in H_+^\infty$ ($W \in H_-^\infty$).

It turns out that the study of the set of minimal spectral factors is facilitated if we study it in relation to four extremal spectral factors. These four spectral factors are determined by the requirement that all their poles are located in either the left or right half-planes, and the same for the zeros.

It is well known that spectral factors exist. Moreover, using the Beurling–Lax–Halmos theorem, there exists a stable, minimum phase, or outer spectral factor, which we denote by W_-. Our weak coercivity assumption implies actually that W_- is left invertible over H_+^∞. In a completely analogous way, there exists an essentially unique antistable and maximum phase spectral factor W_+, which has also dimension $p \times m_0$. By the same argument as before $\bar{W}_+$ has an antistable left inverse.

Let K_+ and K_- be the minimal essentially unique, that is unique up to a right constant unitary factor, $m_0 \times m_0$ inner functions for which $W_+ = \bar{W}_+ K_+$ is stable and $\bar{W}_- = W_- K_-^*$ is antistable. $\bar{W}_-$ is the minimal, antistable minimum phase spectral factor and W_+ is the minimal, stable maximum phase spectral factor, and the above factorizations are just the Douglas, Shapiro, Shields (DSS) factorizations of W_+ and $\bar{W}_-$ over H_-^∞ and H_+^∞ respectively. For more on this see Fuhrmann (1981).

We will assume in the following that the multiplicity of K_+ equals m_0. This basically means that if

$$W_- = \left(\begin{array}{c|c} A & B_- \\ \hline C & D_- \end{array}\right)$$

then rank $B_- = m_0$. In other words, if we seek decompositions of the form $\Phi = \hat{\Phi} + \Delta$ with Δ constant and Φ of rank lower than m_0, we rule out that the problem has trivial solutions, namely those given by internal spectral factors. For more on this, see Baratchart and Gombani (1994).

Proposition 1: *Let W be a minimal, stable $p \times m$ spectral factor. Then there exists an essentially unique $m_0 \times m$ row rigid function $\hat{Q}$ for which*

$$W = W_- \hat{Q} \tag{3}$$

Without loss of generality, we will assume $\hat{Q}(\infty) = (I \;\; 0)$.

Proof: We consider $H_+^2 W$ which is an invariant subspace. Since rank $W = m_0$ then, by Beurling's theorem, there exists a rigid function $\hat{Q}$ for which

$$H_+^2 W = H_+^2 \hat{Q}$$

Since W_- is outer, it follows that

$$H_+^2 W = H_+^2 \hat{Q} = H_+^2 W_- \hat{Q}$$

This implies $W = W_- \hat{Q}$. □

The following proposition gives a characterization of the image of all Hankel operators induced by minimal, stable, not necessarily square, spectral factors.

Proposition 2: *Let W_- be the minimal, stable, minimum phase, spectral factor of a rational spectral function. Let W be any minimal, stable, not necessarily square, spectral factor. Let $\hat{H}_W^c : H_-^2 \to H_+^2$ be the associated conjugate Hankel operator. Then*

(1)

$$\operatorname{Im} \hat{H}_W^c = \operatorname{Im} \hat{H}_{W_-^c} \tag{4}$$

In particular we have

$$\operatorname{Im} \hat{H}_{W_+}^c = \operatorname{Im} \hat{H}_{W_-}^c \tag{5}$$

(2) *Let*

$$W_- = K H_-^* = (K^*)^{-1} H_-^* \tag{6}$$

be a right coprime DSS factorization of W_- over H_-^∞. Let W be any stable, minimal spectral factor. Then it has a right coprime DSS factorization of the form

$$W = K H^* \tag{7}$$

Proof:

(1) Let W be any minimal, stable, not necessarily square, spectral factor. We know, by Proposition 1, that there exists a row-rigid function $\hat{Q}$ such that $W = W_- \hat{Q}$ or, equivalently, that $W_- = W\hat{Q}^*$. By Lemma 3.5.7 in Fuhrmann (1981), we conclude that

$$\operatorname{Im} \hat{H}^c_{W_-} \subset \operatorname{Im} \hat{H}^c_W. \tag{8}$$

Now $\operatorname{Im} \hat{H}^c_W$ can be taken, using the shift realization, as the state space of a minimal realization. Since W is, by assumption, a minimal spectral factor we have $\dim \operatorname{Im} \hat{H}^c_{W_-} = \dim \operatorname{Im} \hat{H}^c_W$. Thus (4) follows. Equality (5) is a special case.

(2) We have $\operatorname{Im} \hat{H}^c_W = \{KH^2_+\}^\perp$. Since $\operatorname{Im} \hat{H}^c_W = \hat{H}^c_{W_-}$, this implies a factorization, necessarily left coprime, of the form (7). □

Equality (4) has practical implications for the realization of minimal spectral factors.

Corollary 1: *Let*

$$W_- = \left(\begin{array}{c|c} A & B_- \\ \hline C & D_- \end{array}\right) \tag{9}$$

the realization being minimal and let W be any, not necessarily square, minimal stable spectral factor. Then W has a minimal realization of the form

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{10}$$

Proof: We consider the shift realization, see Fuhrmann (1981, 1995), based on the left coprime factorizations (6) and (7). Since the (C, A) operators in such a realization depend only on the inner function Q, the result follows. That (10) is a minimal realization follows from the minimality of W. □

It might be tempting to use the previous result to try and study the set of all minimal, stable, spectral factors of given size by using the realizations with the same state space. This is not necessarily a good idea. In fact the main contribution of this paper comes from following very different route, as will become apparent in §6.

Proposition 3: *Let W_+ be the minimal, stable, maximum phase, spectral factor. If S is any non-trivial row rigid function, i.e. $SS^* = I$, then*

$$\delta(W_+S) > \delta(W_+) \tag{11}$$

Proof: Since W_+S is a stable spectral factor, and $W_+ = (W_+S)S^*$, we have

$$\operatorname{Im} \hat{H}^c_{W_+S} \supset \operatorname{Im} \hat{H}^c_{W_+} \tag{12}$$

If $\delta(W_+S) = \delta(W_+)$ then necessarily $\operatorname{Im} \hat{H}^c_{W_+S} = \operatorname{Im} \hat{H}^c_+$. By Proposition 1, there exist H^∞ functions R, T such that

$$W_+S = W_+R^* + T^*$$

Now W_+ has an antistable left inverse W^{-L}, so $S = R^* + W^{-L}T^*$. Thus S is both in H_+^∞ and in H_-^∞ hence necessarily a constant coisometry, contrary to our assumption that S is non-trivial. □

We proceed to derive a characterization of W_+ which is analogous to Propositon 1. For this we will need the following proposition.

Proposition 4: *Let*

$$K_1^{-1}H_1 = HK^{-1} \tag{13}$$

be coprime DSS factorizations over H_+^∞. Then if H_1 is outer, so is H.

Proof: We prove it by contradiction. Assume $H = E\bar{H}$ with E non-trivial. Let

$$\begin{pmatrix} Y_1 & -X_1 \\ -H_1 & K_1 \end{pmatrix}\begin{pmatrix} K & X \\ H & Y \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

be a doubly coprime factorization. This implies also

$$\begin{pmatrix} K & X \\ H & Y \end{pmatrix}\begin{pmatrix} Y_1 & -X_1 \\ -H_1 & K_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

In particular we get $HY_1 = YH_1$ and hence $E\bar{H}Y_1 = YH_1$, which can be rewritten as $\bar{H}Y_1 = E^{-1}YH_1$. Now E, Y are left coprime as H, Y are. Let $\hat{Y}\hat{E}^{-1}$ be a right coprime factorization of $E^{-1}Y$. So

$$\bar{H}Y_1 = E^{-1}YH_1 = \hat{Y}\hat{E}^{-1}H_1$$

Now the right coprimeness of $\hat{Y}$, $\hat{E}$ implies the existence of H^∞ functions P, Q which solve the following Bezout identity

$$P\hat{Y} + Q\hat{E} = I \tag{14}$$

This implies $P\hat{Y}\hat{E}^{-1}H_1 + QH_1 = \hat{E}^{-1}H_1$. Now $L = P\hat{Y}\hat{E}^{-1}H_1 + QH_1 = P\bar{H}Y_1 + QH_1$ is clearly in H^∞, so $H_1 = \hat{E}L$ contradicting the outerness of H_1. □

Proposition 5: *Let W_+ be the minimal, stable, maximum phase, spectral factor. Let $W_+ = KH_+^*$ be a left coprime DSS factorization over H_-^∞. Then H_+ is outer.*

Proof: Let $\bar{W}_+$ be the minimal, maximum phase antistable spectral factor. Thus, in analogy to W_-, $\bar{W}_+$ is conjugate outer, i.e. left invertible in H_-^∞. Let W_+ be the minimal, maximum phase stable spectral factor. We have $W_+ = \bar{W}_+K_+ = \bar{W}_+K_+^{-*}$, which is a right coprime DSS factorization over H_-^∞. Let $W_+ = K^{-*}H_+^*$ be a left coprime factorization of W_+ over H_-^∞.

Now, by construction, $\bar{W}_+$ is outer in H_-^∞, i.e. it is left invertible over H_-^∞. Applying Proposition 4 in its H_-^∞ version, we conclude that H_+^* is left invertible in H_-^∞. Equivalently, H_+ is right invertible in H_+^∞, or it is row outer. □

We proceed with the characterization of W_+.

Proposition 6: *Let W be any minimal, stable not necessarily square spectral factor. Then there exists an $m \times m_0$ column rigid function $\hat{Q}''$ for which $W\hat{Q}'' = W_+$.*

Proof: Let $W_+ = KH_+^*$ and $W = KH^*$ be right coprime DSS factorizations. Since both W and W_+ are spectral factors, we obtain

$$H^* H = H_+^* H_+$$

Applying Proposition 1 in its H_-^∞ form, we infer the existence of a row rigid function $(\hat{Q}'')^* \in H_-^\infty$, that is satisfying $(\hat{Q}''{}^* \hat{Q}'' = I$, for which $H^* = H_+^* (\hat{Q}'')^*$. This implies

$$W = KH^* = KH_+^* (\hat{Q}'')^* = W_+ (\hat{Q}'')^*$$

and hence $W_+ = W\hat{Q}''$. □

Proposition 7: *There exist unique, $m_0 \times m_0$ square inner functions Q_- and Q_+ for which*

$$\left.\begin{aligned} W_+ &= W_- Q_+ \\ \bar{W}_+ &= \bar{W}_- Q_- \end{aligned}\right\} \tag{15}$$

Moreover, we have

$$K_- Q_+ = Q_- K_+ \tag{16}$$

Q_+, K_+ are right coprime and Q_-, K_- are left coprime.

Proof: W_-, $\bar{W}_+$ are both of dimension $p \times m_0$. Since $\bar{W}_- = W_- K_-^*$ and $W_+ = \bar{W}_+ K_+$ with K_-, K_+ inner of dimension $m_0 \times m_0$, it follows that W_+, $\bar{W}_-$ are also of dimension $p \times m_0$. By Proposition 1, there exists an $m_0 \times m_0$ rigid, hence inner, function Q_+ such that $W_+ = W_- Q_+$. We easily compute that

$$W_+ = \bar{W}_- K_- Q_+ = \bar{W}_- Q_- K_+$$

The left invertibility of $\bar{W}_-$ implies (16).

That Q_+ and K_+ are right coprime follows from the fact that K_+ is the minimal inner function for which $W_+ K_+$ is stable. The second assertion is proved similarly.

A non-trivial right common factor for Q_+, K_+ would contradict the fact that $\bar{W}_+ = W_+ K_+^*$ is a right coprime DSS factorization over H_+^∞. Similarly, for the left coprimeness of Q_- and K_-. □

It is convenient to arrange the four spectral factors as in figure 1.

To get some intuition into the role of the inner functions $Q_\pm$, $K_\pm$, we recall that one convenient way to describe zeros and poles of rational functions is in terms of polynomial coprime factorizations.

Consider the minimal stable, mininum phase $p \times m_0$ spectral factor W_-. If we consider a polynomial left coprime factorization, then $W_- = D_-^{-1} E_-$. Here D_- is a $p \times p$ stable non-singular polynomial matrix, whereas E_- is a $p \times m_0$ polynomial matrix all whose invariant factors are stable polynomials. The $p \times m_0$ numerator

$$\begin{tikzcd} W_- \arrow[r, "Q_+"] & W_+ \\ \bar{W}_- \arrow[u, "K_-"] \arrow[r, "Q_-"] & \bar{W}_+ \arrow[u, "K_+"'] \end{tikzcd}$$

Figure 1.

polynomial matrix admits a factorization EE_- with E a $p \times m_0$ polynomially left invertible matrix and E_- stable. Solving the polynomial spectral factorization problem

$$E_- E_-^* = F_+ F_+^* \tag{17}$$

with F_+ antistable and $E_-^{-1}F_+$ biproper, we conclude that $Q_+ = E_-^{-1}F_+$ is an $m_0 \times m_0$ inner function and

$$W_- Q_+ = \bar{D}_-^{-1}\bar{E}_-(E_-^{-1}F_+) = \bar{D}_-^{-1}EE_-(E_-^{-1}F_+) = \bar{D}_-^{-1}(EF_+)$$

Obviously all the invariant factors of EF_+ are antistable. So $W_- Q_+ = W_+$, where W_+ is the minimal stable maximum phase spectral factor.

Thus, the inner function Q_+ describes the flipping of zeros from the open left half plane to the open right half plane. Given a factorization $Q_+ = Q_1 Q_2$ into square inner factors, the left factor Q_1 describes the set of all zeros of W_- flipped to the right half plane by Q_1, i.e. the set of antistable zeros of the spectral factor $W_- Q_1$. Similarly, the right factor Q_2 of Q_+ describes the set of stable zeros of $W = W_- Q_1 = W_+ Q_2^*$ that remain to be flipped. The sets of zeros of this type are called *internal zeros* and the corresponding spectral factors are called *internal spectral factors*. We will discuss their role in more detail in §5.

As usual, we define the *phase function* T_0 corresponding to the spectral functon Φ by

$$T_0 = \bar{W}_+^{-L} W_- \tag{18}$$

Here, since $\bar{W}_+$ is a $p \times m_0$ function which has full column rank on the extended imaginary axis, $\bar{W}_+^{-L}$ denotes any left inverse. The phase function is an $m_0 \times m_0$ all-pass function. It plays an important role in understanding the set of all minimal spectral factors.

The next lemma relates the inner functions $K_\pm$, $Q_\pm$ to the phase function.

Lemma 1: *Let W_-, W_+, $\bar{W}_-$, $\bar{W}_+$ be the extremal spectral factors corresponding to the spectral function Φ and let T_0 be the associated phase function. Then we have*

$$\bar{W}_+ \bar{W}_+^{-L} W_+ = W_+ \tag{19}$$

$$\bar{W}_+ \bar{W}_+^{-L} W_- = W_- \tag{20}$$

$$W_+ W_+^{-L} W_- = W_- \tag{21}$$

$$T_0 = \bar{W}_+^{-L} W_- = K_+ Q_+^{-1} = Q_-^{-1} K_- \tag{22}$$

Proof: Since $\bar{W}_+^{-L}\bar{W}_+ = I$, we have $\bar{W}_+ \bar{W}_+^{-L} \bar{W}_+ = \bar{W}_+$. Now

$$\bar{W}_+ \bar{W}_+^{-L} W_+ = \bar{W}_+ \bar{W}_+^{-L} \bar{W}_+ K_+ \bar{W}_+ K_+ = W_+ \tag{23}$$

This proves (19). (20) follows, using the fact that $W_- Q_+ = W_+$. Substituting $W_+ = W_- Q_+$ in the equality $W_+ W_+^{-L} W_+ = W_+$ and eliminating Q_+ leads to (21). Finally, using (21), we compute

$$T_0 = \bar{W}_+^{-L} W_- = \bar{W}_+^{-L}(W_+ W_+^{-L} W_-) = (\bar{W}_+^{-L} W_+)(W_+^{-L} W_-) = K_+ Q_+^{-1}$$

That $K_+ Q_+^{-1} = Q_-^{-1} K_-$ was proved in (16). □

As a result of this lemma, it is clear that the phase function is inner if and only if $Q_\pm = I$, i.e. if and only if there are no internal zeros. The principal difference from

$$\begin{tikzcd}
W^e_- \arrow[r, "Q^e_+"] & W^e_+ \\
\overline{W}^e_- \arrow[u, "K^e_-"] \arrow[r, "Q^e_-"] & \overline{W}^e_+ \arrow[u, "K^e_+"']
\end{tikzcd}$$

Figure 2.

the regular case is that the degrees of the determinants of Q_+ and K_+ are no longer necessarily equal. This difference manifests itself geometrically, as well as in the invertibility properties of Toeplitz operators. This will be studied in §4. The phase function is a powerful tool for the analysis of spectral factors. It turns out that for a finer analysis we need localized versions of the phase function corresponding to ordered pairs of internal spectral factors. We postpone the introduction of these functions until, in §5, we have constructed the right notation for it.

We will find it convenient to deal with square inner functions. To this end we define the extended, extremal spectral factors, W^e_-, W^e_+, $\overline{W}^e_-$, $\overline{W}^e_+$, as the $p \times p$ matrix functions which are obtained by augmenting with $p - m_0$ zero columns. For example $W^e_- = (W_- \;\; 0)$. Similarly, we extend the inner functions $Q_\pm$, $K_\pm$ in an obvious way to $m \times m$ inner functions by letting

$$Q^e_\pm = \begin{pmatrix} Q_\pm & 0 \\ 0 & I \end{pmatrix}, \qquad K^e_\pm = \begin{pmatrix} K_\pm & 0 \\ 0 & I \end{pmatrix} \tag{24}$$

Clearly, we obtain the commutativity of figure 2.

The previous coprimeness conditions extend also in a natural way to the extended inner functions. Thus Q^e_+, K^e_+ are right coprime and Q^e_-, K^e_- are left coprime.

4. On Toeplitz operators

In this section we will turn our attention to a study of the phase function and some Toeplitz operators related to it. The invertibility of Toeplitz operators has both goemetric and arithmetic characterizations. By arithmetic characterizations we refer to Wiener–Hopf factorizations and the corresponding factorization indices, whereas the geometry refers to the invariant and coinvariant subspaces of Hardy spaces.

We recall the definition of Toeplitz operators. As we are working with both row and column spaces we must distinguish between two types of Toeplitz operators.

Definition 2: Let G be an $m \times m$ matrix function in L^∞. We define the Toeplitz operator $\mathcal{T}^c_G$ acting on column Hardy space by

$$\left.\begin{aligned} \mathcal{T}^c_G : H^2_+ &\to H^2_+ \\ f &\mapsto p_+ G f \end{aligned}\right\} \tag{25}$$

Similarly, we define the Toeplitz operator $\mathcal{T}^r_G$ acting on row Hardy space by

$$\left.\begin{aligned} \mathcal{T}^c_G : H^2_+ &\to H^2_+ \\ f &\mapsto fGP_+ \end{aligned}\right\} \tag{26}$$

In both cases G is called the *symbol* of the corresponding Toeplitz operator.

The adjoints of Toeplitz operators are also Toeplitz operators. In fact we have

$$(\mathcal{T}^r_G)^* = \mathcal{T}^r_{G^*}$$

$$(\mathcal{T}^c_G)^* = \mathcal{T}^c_{G^*}$$

We say that

$$G = G_- \Delta_r G_+ \tag{27}$$

is a *right Wiener–Hopf factorization* if $G_-^{\pm 1} \in H^\infty_-$, $G_+^{\pm 1} \in H^\infty_+$ and

$$\Delta_r(s) = \mathrm{diag}\left[\left(\frac{s-1}{s+1}\right)^{\kappa_1}, \ldots, \left(\frac{s-1}{s+1}\right)^{\kappa_m}\right] \tag{28}$$

with $\kappa_1 \geq \cdots \geq \kappa_m$. The indices $\kappa_1, \ldots, \kappa_m$ are called the *right Wiener–Hopf factorization indices*. Left factorizations and left factorization indices are similarly defined.

It is well known that if $G \in L^\infty$ is continuous, and in particular if it is rational, then Wiener–Hopf factorization exist and the factorization indices, though not necessarily the factorizations, are uniquely defined.

Proposition 8: *Let $G \in L^\infty$ and $G = G_- \Delta_r G_+$ its right Wiener–Hopf factorization. Then*

(1) *The following statements are equivalent.*
  (a) *The Toeplitz operator $\mathcal{T}^c_G$ is injective.*
  (b) *The Toeplitz operator $\mathcal{T}^c_{\Delta_r}$ is injective.*
  (c) *Δ_r is full column rank and all the right Wiener–Hopf factorization indices are non-negative.*

(2) *The following statements are equivalent.*
  (a) *The Toeplitz operator $\mathcal{T}^c_G$ is surjective.*
  (b) *The Toeplitz operators $\mathcal{T}^c_{\Delta_r}$ is surjective.*
  (c) *Δ_r is full row rank and all the right Wiener–Hopf factorization indices are non-positive.*

Proof: The maps $\mathcal{T}^c_{G_-}$, $\mathcal{T}^c_{G_-^{-1}}$, $\mathcal{T}^c_{G_+}$, $\mathcal{T}^c_{G_+^{-1}}$ are all invertible. In fact $(\mathcal{T}^c_{G_-} = \mathcal{T}^c_{G_-^{-1}}$ and $(\mathcal{T}^c_{G_+})^{-1} = \mathcal{T}^c_{G_+^{-1}}$. Clearly we have the commutative diagram (figure 3) as

$$\begin{tikzcd} H^2_+ \arrow[r, "\mathcal{T}^c_G"] \arrow[d, "\mathcal{T}^c_{G_+}"'] & H^2_+ \\ H^2_+ \arrow[r, "\mathcal{T}^c_{\Delta_r}"] & H^2_+ \arrow[u, "\mathcal{T}^c_{G_-}"'] \end{tikzcd}$$

Figure 3.

$$\mathcal{T}^c_{G_-}\mathcal{T}^c_{\Delta_r}\mathcal{T}^c_{G_+}f = P_+G_-P_+\Delta_r G_+ f = P_+G_-\Delta_r G_+ f = \mathcal{T}^c_G f$$

In particular this shows that $\mathcal{T}^c_G$ is injective if and only if $\mathcal{T}^c_{\Delta_r}$ is injective. Now, if Δ_r is given by (28), then clearly

$$\mathcal{T}^c_{\Delta_r} = \mathcal{T}_{\left(\frac{s-1}{s+1}\right)^{\kappa_1}} \oplus \cdots \oplus \mathcal{T}_{\left(\frac{s-1}{s+1}\right)^{\kappa_m}}$$

So it suffices to analyse a Toeplitz operator of the form $\mathcal{T}_{\left(\frac{s-1}{s+1}\right)^{\kappa}}$ operating on the scalar H^2_+ space. Since $s-1/s+1$ is inner, we have for $\kappa \geq 0$ that $\mathcal{T}_{\left(\frac{s-1}{s+1}\right)^{\kappa}}$ is isometric and hence injective. The codimension of the image is equal to κ. For $\kappa \leq 0$ the operator $\mathcal{T}_{\left(\frac{s-1}{s+1}\right)^{\kappa}}$ is surjective and the dimension of the kernel is κ. In fact we have

$$\operatorname{Ker}\mathcal{T}_{\left(\frac{s-1}{s+1}\right)^{\kappa}} = \left\{\left(\frac{s-1}{s+1}\right)^{-\kappa} H^2_+\right\}^{\perp}$$

This implies

$$\dim\operatorname{Ker}\mathcal{T}_{\left(\frac{s-1}{s+1}\right)^{\kappa}} = -\kappa$$

Going back to the operator $\mathcal{T}^c_{\Delta_r}$ we obtain

$$\dim\operatorname{Ker}\mathcal{T}^c_{\Delta_r} = -\sum_{\kappa_i<0}\kappa_i$$

$$\operatorname{codim}\operatorname{Im}\mathcal{T}^c_{\Delta_r} = -\sum_{\kappa_i>0}\kappa_i$$

Thus injectivity is equivalent to all the right Wiener–Hopf indices being non-negative and surjectivity to all the right Wiener–Hopf indices being non-positive.

In view of the preceeding proposition, the following corollary is obvious.

Corollary 2:

(1) *The Toeplitz operator $\mathcal{T}^c_G$ is invertible if and only if G is invertible and all the right Wiener–Hopf factorization indices are trivial.*

(2) *The Toeplitz operator $\mathcal{T}^r_G$ is invertible if and only if G is invertible and all the left Wiener–Hopf factorization indices are trivial.*

The following proposition connects invertibility properties of Toeplitz operators acting in row and column Hardy spaces.

Proposition 9: *Let $G \in L^\infty$ be rational. Then*

(1) *The Toeplitz operator $\mathcal{T}^c_G$ is injective if and only if $\mathcal{T}^r_{G^*}$ is surjective.*

(2) *The Toeplitz operator $\mathcal{T}^c_G$ is surjective if and only if $\mathcal{T}^r_{G^*}$ is injective.*

(3) *The Toeplitz operator $\mathcal{T}^c_G$ is injective if and only if $\mathcal{T}^r_G$ is injective.*

Proof:

(1) Let $G = G_-\Delta_r G_+$ be a right Wiener–Hopf factorization. Then $G^* = G_+^*\Delta_r^*G_-^*$ is a left Wiener–Hopf factorization of G^*. The factorization indices of G are all non-negative if and only if the factorization indices of G^* are all non-positive.

(2) The proof is similar.
(3) Since $(\mathcal{T}^c_G)^* = \mathcal{T}^c_{G^*}$, $\mathcal{T}^c_G$ is injective if and only if $\mathcal{T}^r_{G^*}$ is surjective. By part (2) this is the case if and only if $\mathcal{T}^r_G$ is injective. □

Note that, if $G = G_- \Delta_r G_+$ is a right Wiener–Hopf factorization with non-negative factorization indices, then we have an explicit formula for the left inverse of $\mathcal{T}^c_G$, namely

$$(\mathcal{T}^c_G)^{-L} g = G_+^{-1} P_+ \Delta^{-1} G_-^{-1} g$$

We do not have such a formula for the right inverse of $\mathcal{T}^r_G$ based on the same right Wiener–Hopf factorization.

The principal tool in our analysis are Theorem 1 and Proposition 11. Most of the content can be found in Nikolskii (1985). The connection with Wiener–Hopf factorization indices is a direction application of well known results in the study of Toeplitz operators, see for example Gohberg and Feldman (1971). In the proofs we need the following simple lemma.

Lemma 2:

(1) *Let V, W be two subspaces of a Hilbert space H, with* $\dim W < \infty$. *Let P_V be the orthogonal projection of H on V. Then*

$$P_V W = V$$

if and only if

$$H = V^{\perp} + W$$

(2) *Let U, V, W be subspaces of a Hilbert spce H. If W is orthogonal to both U and V, then*

$$U \cap V = U \cap (V + W) \tag{29}$$

Proof:

(1) If $H = V^{\perp} + W$, then clearly $V = P_V H = P_V(V^{\perp} + W) = P_V W$. Conversely, assume $P_V W = V$. Let $x \in (V^{\perp} + W)^{\perp} = V \cap W^{\perp}$. Let $w \in W$ be such that $P_V w = x$, then necessarily $(x, w) = 0$. Therefore

$$0 = (x, w) = (P_V w, w) = \|P_V w\|^2 = \|x\|^2$$

So $x = 0$, i.e. $\overline{V^{\perp} + W} = H$. Since W is finite dimensional, we have also $V^{\perp} + W = H$.

(2) Clearly $U \cap V \subset U \cap (V + W)$. Assume conversely that $f \in U \cap (V + W)$, i.e. $f = u = v + w$. Since $w = u - v$, we have $w \in (U + V) \cap (U^{\perp} \cap V^{\perp}) = (U + V) \cap (U + V)^{\perp}$. So necessarily $w = 0$ and $u = v$, i.e. $f \in U \cap V$. □

Proposition 10: *Let Q, $\bar{Q}$, K_1, K_2 be rational inner functions satisfying*

$$K_1 Q = \bar{Q} K_2 \tag{30}$$

with K_1, $\bar{Q}$ left coprime and Q, K_2 right coprime. Then

$$\dim \operatorname{Ker} \mathcal{T}^r_{QK_2^*} = \dim H_r(K_1) Q \cap H_r(K_2) = \dim K_+^2 Q \cap H_r(K_2) \tag{31}$$

Specifically, the map $g \mapsto gQ$ *is a bijective linear map from* $\operatorname{Ker} \mathcal{T}^r_{QK_2^*}$ *to* $H_r(K_1)Q \cap H_r(K_2)$.

We have

$$\dim \operatorname{Ker} \mathcal{T}^r_{QK_2^*} = \dim \left\{ f \mid \| H^r_{QK_2^*} f \| = \| f \| \right\}.$$

Proof: Assume $f \in H_r(K_1)Q \cap \underline{H}_r(K_2)$, i.e. $f = g_1 Q = g_2 = h_2 K_2$ with $g_1 \in H_r(K_1)$, $g_2 \in H_r(K_2)$ and $h_2 \in \bar{H}_r(K_2^*)$. Thus $g_1 Q K_2^* = h_2$. Applying the orthogonal projection P_+, we obtain $g_1 Q K_2^* P_+ = h_2 P_+ = 0$, i.e. $g_1 \in \operatorname{Ker} \mathcal{T}^r_{QK_2^*}$.

Conversely, assume $g_1 \in \operatorname{Ker} \mathcal{T}^r_{QK_2^*}$, i.e. $g_1 Q K_2^* P_+ = 0$. This means that $g_1 Q K_2^* = h_2$ with $h_2 \in H^2_-$. From the equality $f = g_1 Q = h_2 K_2$ we conclude that $f \in H^2_+ Q \cap H_r(K_2)$. Now, using the equality $K_1 Q = \bar{Q} K_2$, we can write $H^2_+ = H^2_+ K_1 \oplus H_r(K_1)$ and hence

$$H^2_+ Q = H^2_+ K_1 Q \oplus H_r(K_1) Q = H^2_+ \bar{Q} K_2 \oplus H_r(K_1) Q$$

Obviously $H^2_+ \bar{Q} K_2 \perp H_r(K_2)$, and hence, applying Lemma 2 (ii), we get $H_r(K_1) Q \cap H_r(K_2) = H^2_+ Q \cap H_r(K_2)$. □

For the notation used in the following theorem, as well as in the rest of the paper, the reader is advised to consult the remark preceeding Definition 1.

Theorem 1: *Let* Q_-, Q_+, K_-, K_+ *be rational inner functions satisfying*

$$Q_- K_+ = K_- Q_+ \tag{32}$$

and the coprimeness conditions

$$\left.\begin{aligned} K_- \wedge_L Q_- &= I \\ K_+ \wedge_R Q_+ &= I \end{aligned}\right\} \tag{33}$$

Then

(1) *The following statements are equivalent*

   (a) *The Toeplitz operator* $\mathcal{T}^r_{K_+ Q_+^*} + \mathcal{T}^r_{Q_-^* K_-}$ *is injective.*

   (b) *The Toeplitz operator* $\mathcal{T}^r_{Q_+ K_+^*} = \mathcal{T}^r_{K_-^* Q_-}$ *is surjective.*

   (c) *All left Wiener–Hopf factorization indices of* $K_+ Q_+^* = Q_-^* K_-$ *are nonnegative.*

   (d) *We have*

$$\operatorname{Ker} H_r(Q_+) \big| P_{H_r(K_+)} = H_r(Q_+) \cap H^2_+ K_+ = \{0\} \tag{34}$$

   (e) *We have*

$$H^2_+ Q_+ + H^2_- K_+ = L^2 \tag{35}$$

   (f) *We have*

$$H^2_+ K_+ \cap H^2_- Q_+ = \{0\} \tag{36}$$

   (g) *We have*

$$H^2_+ Q_+ + H_r(K_+) = H^2_+ \tag{37}$$

   (h) *We have*

$$H_r(K_+)P_{H_r(Q_+)} = H_r(Q_+) \tag{38}$$

(i) *All the singular values of the Hankel operator* $H^r_{K_+Q_+^*} = H^r_{K_-^*Q_-}$ *are* < 1.

(2) *The following statements are equivalent*

(a) *The Toeplitz operator* $\mathcal{T}^c_{K_+Q_+^*} = \mathcal{T}^c_{Q_-^*K_-}$ *is injective.*

(b) *the Toeplitz operator* $\mathcal{T}^c_{Q_+K_+^*} = \mathcal{T}^c_{K_-^*Q_-}$ *is surjective.*

(c) *All right Wiener–Hopf factorization indices of* $Q_+K_+^* = K_-^*Q_-$ *are non-negative.*

(d) *We have*

$$\operatorname{Ker} P_{H_c(K_-)}|H_c(Q_-) = H_c(Q_-) \cap K_- H_+^2 = \{0\} \tag{39}$$

(e) *We have*

$$H_c(K_-) + Q_- H_+^2 = H_+^2 \tag{40}$$

(f) *We have*

$$K_- H_-^2 + Q_- H_+^2 = L^2 \tag{41}$$

(g) *We have*

$$P_{H_c(Q_-)}H_c(K_-) = H_c(Q_-) \tag{42}$$

(h) *All the singular values of the Hankel operator* $H^c_{K_+Q_+^*} = H^c_{K_-^*Q_-}$ *are* < 1.

(3) *The following statements are equivalent*

(a) *The Toeplitz operator* $\mathcal{T}^c_{K_+Q_+^*} = \mathcal{T}^c_{K_-^*Q_-}$ *is surjective.*

(b) *The Toeplitz operator* $\mathcal{T}^c_{K_+Q_+^*} = \mathcal{T}^c_{K_-^*Q_-}$ *is injective.*

(c) *All right Wiener–Hopf factorization indices of* $Q_+K_+^* = K_-^*Q_-$ *are non-negative.*

(d) *We have*

$$\operatorname{Ker} P_{H_c(Q_-)}|H_c(K_-) = H_c(K_-) \cap Q_- H_+^2 = \{0\} \tag{43}$$

(e) *We have*

$$K_- H_+^2 + Q_- H_-^2 = L^2 \tag{44}$$

(f) *We have*

$$K_- H_+^2 H_c(Q_-) = H_+^2 \tag{45}$$

(g) *We have*

$$P_{H_c(K_-)}H_c(Q_-) = H_c(K_-) \tag{46}$$

(h) *All the singular values of the Hankel operator* $H^c_{K_+Q_+^*} = H^c_{K_-^*Q_-}$ *are* < 1.

(4) *The following statements are equivalent*

(a) *The Toeplitz operators* $\mathcal{T}^c_{Q_+K_+^*} + \mathcal{T}^c_{K_-^*Q_-}$ *and* $\mathcal{T}^r_{Q_+K_+^*} = \mathcal{T}^r_{K_-^*Q_-}$ *are both invertible.*

(b) *All, left and right, Wiener–Hopf factorization indices of* $Q_+K_+^* = K_-^*Q_-$ *are trivial.*

(c) *We have*

$$\left.\begin{aligned} H_c(K_-) \cap Q_- H_+^2 &= \{0\} \\ H_c(K_-) + Q_- H_+^2 &= H_+^2 \end{aligned}\right\} \tag{47}$$

(d) *We have*

$$\left.\begin{aligned} P_{H_r(K_-)} H_r(Q_-) &= H_r(K_-) \\ P_{H_c(Q_-)} H_c(K_-) &= H_c(Q_-) \end{aligned}\right\} \tag{48}$$

(e) *We have*

$$\left.\begin{aligned} K_- H_-^2 \cap Q_- H_+^2 &= \{0\} \\ K_- H_-^2 + Q_- H_+^2 &= L^2 \end{aligned}\right\} \tag{49}$$

(f) *All singular values of the Hankel operators* $H^c_{q_+K_+^*}$ *are less than* 1.

$$\dim \operatorname{Ker} \mathcal{T}^c_{Q_+K_+^*} = \dim \left\{ f \,\middle|\, \left\| H^c_{Q_+K_+^*} f \right\| = \|f\| \right\} \tag{50}$$

Proof:

(1) The equivalence of (a)–(c) follows from Proposition 8.

(a) $\Leftrightarrow$(d)

We have $f \in \operatorname{Ker} \mathcal{T}^r_{K_+Q_+^*}$ if and only if $0 = fK_+Q_+^*P_+$, i.e. if and only if $fK_+ \in H_-^2 Q_+$, that is if and only if $g = fK_+ \in H_-^2 Q_+ \cap H_+^2 K_+$. Since K_+ is inner, the equivalence follows.

(e) $\Rightarrow$(g)

Applying P_+ to equality (35) yields (37).

(g) $\Rightarrow$(h)

We apply the projection $P_{H_r(Q_+)}$ to equality (37) to get (38).

(h) $\Rightarrow$(a)

Assume $H_r(K_+)\big|P_{H_r(Q_+)}$ is surjective, i.e. $H_r(Q_+)\big|P_{H_r(K_+)}$ is injective. This implies (34), and hence also (a).

(a) $\Leftrightarrow$(i)

Since for any L^∞ matrix A and $f \in H_+^2$ we have

$$\|fA\|^2 = \|fAP_-\|^2 + \|fAP_+\|^2 = \|fH^r_A\|^2 + \|f\mathcal{T}^r_A\|^2$$

it is clear that if A is an all-pass function, i.e. isometric valued, then we have

$$\|f\|^2 = \|fH^r_A\|^2 + \|f\mathcal{T}^r_A\|^2$$

In particular we have $f \in \operatorname{Ker} \mathcal{T}^r_A$ if and only if $\|f\| = \|fH^c_A\|$. Specializing to the all-pass function $K_+Q_+^* = Q_-^*K_-$, we get

$$\operatorname{Ker} \mathcal{T}^r_{K_+Q_+^*} = \left\{ f \mid \|fH^r_{K_+Q_+^*}\| = \|f\| \right\}$$

In particular $\mathcal{T}^r_{K_+Q_+^*}$ is injective if and only if all the singular values of the Hankel operator $H^r_{K_+Q_+^*} = H^r_{Q_-^*K_-}$ are < 1.

All other statements are proved analogously. □

We would like to point out that, in view of Proposition 9, the statements in parts (1) and (3) of the previous theorem are equivalent. The equivalence of some of the relations can be proved directly. As an example we prove the equivalence of (41) and (35), using the equality (32) and the fact that multiplication by all pass functions is a unitary map in L^2.

$$L^2 = K_- H_-^2 + Q_- H_+^2 = H_-^2 + K_-^* Q_- H_+^2$$

$$= H_-^2 + Q_+ K_+^* H_+^2 = Q_+^* H_-^2 + K_+^* H_+^2$$

The equality $L^2 = Q_+^* H_-^2 + K_+^* H_+^2$ for column spaces is clearly equivalent to the equality $L^2 = H_+^2 Q_+ + H_-^2 K_+$ for row spaces.

One other thing worth noting is the close connection between Toeplitz operators and projection operators. In the context of polynomial models it was put to effective use in Fuhrmann (1981).

The previous theorem was essentially about a special class of Toeplitz operators, but did not relate to a spectral function or a spectral factorization problem. In that case, more can be said. Heutistically, based on our assumptions, the number of zeros of a spectral factor, as measured by $Q_\pm$ cannot exceed the number of poles as measured by $K_\pm$. We state the result as a one sided invertibility of the Toeplitz operators with the symbol equal to the phase function.

Proposition 11: *Let Φ be a $p \times p$, rank m_0, weakly coercive, spectral function, and let $W_\pm$, $\bar{W}_\pm$ be the $p \times m_0$ extremal spectral factors. Then both Toeplitz operators $\mathcal{T}^c_{\bar{W}_+^{-L} W_-} = \mathcal{T}^c_{K_+ Q_+^*} = \mathcal{T}^c_{Q_-^* K_-}$ and $\mathcal{T}^r_{\bar{W}_+^{-L} W_-}$ are injective.*

Proof: For the injectivity of $\mathcal{T}^c_{\bar{W}_+^{-L} W_-}$ we show an explicit left inverse, namely the map $f \mapsto W_-^{-L} P_+ \bar{W}_+ f$. Indeed, using (20), we compute

$$W_-^{-L} P_+ \bar{W}_+ O_+ \bar{W}_+^{-L} W_- f = W^{-L} P_+ \bar{W}_+ \bar{W}_+^{-L} W_- f$$

$$= W_-^{-L} P_+ W_- f = W_-^{-L} W_- f = f$$

The injectivity of $\mathcal{T}^r_{\bar{W}_+^{-L} W_-}$ follows from Proposition 9. □

Corollary 3: *With the notation of Proposition 11, we have the following.*

(1) *The maps $H_r(Q_+) | P_{H_r(K_+)}$ and $P_{H_c(K_-)} | H_c(Q_-)$ are injective.*
(2) *The maps $H_r(K_+) | P_{H_r(Q_+)}$ and $P_{H_c(Q_-)} | H_c(K_-)$ are surjective.*

Proof: Follows from Proposition 11 and Theorem 1. The second statement follows by duality. □

5. Rectangular spectral factors

In this section we study the main object of the paper, namely the set $\mathcal{W}^m$ of all minimal stable, rectangular spectral factor of size $p \times m$ for $m_0 \leq m \leq p + n$. In particular, we are interested in the parametrization of this set for a given m. We will study first two of these sets of minimal, stable rectangular spectral factors: those that are internal (i.e. such that $m = m_0$) and those that are external (in a sense that will be made precise later). The understanding of these, in a sense opposite, special cases enables us to fully understand the general rectangular case. Our approach is relating the set of spectral factors to the arithmetic of inner functions and their factorizations. This will also be interpreted later on geometrically.

As in the case of regular spectral factors, the four extremal spectral factors studied in §3 provide the framework for our analysis of all rectangular spectral factors.

The next proposition studies the embedding of rigid functions in inner ones. This is a special case of Darlington synthesis [see Dewilde (1976)]

Proposition 12:

(1) *Let $\hat{R}$ be a $p \times m_0$ column rigid function, that is $\hat{R}^* \hat{R} = I$. Then there exists a $p \times (p - m_0)$ column rigid function $\tilde{R}$ such that*

(a) $R = (\hat{R} \;\; \tilde{R})$ *is inner.*

(b) *We have the equality of McMillan degrees*

$$\delta(R) = \delta(\hat{R}) \tag{51}$$

$\tilde{R}$ *is uniquely determined up to a right constant unitary factor.*

(2) *Let $\tilde{R}$ be a $m_0 \times p$ row rigid function, that is $\hat{R}\hat{R}^* = I$. Then there exists a $(p - m_0) \times p$ row rigid function $\tilde{R}$ such that*

(a) $R = \begin{pmatrix} \hat{R} \\ \tilde{R} \end{pmatrix}$ *is inner.*

(b) *We have the equality of McMillan degrees*

$$\delta(R) = \delta(\hat{R}) \tag{52}$$

$\tilde{R}$ *is uniquely determined up to a left constant unitary factor.*

Proof:

(1) Assume $\hat{R}^* \hat{R} = I$, then $\hat{R}\hat{R}^* \leq I$. Assume an inner embedding of $\hat{R}$ exists, i.e. there exists an $\tilde{R}$ such that $R = (\hat{R} \;\; \tilde{R})$ is inner. Since $R^* R = RR^* = I$, we have $\tilde{R}\tilde{R}^* = I - \hat{E}\hat{R}^* \geq 0$. Thus $\tilde{R}$ is a spectral factor of $I - \hat{R}\hat{R}^*$. To minimize the McMillan degree of R, which is clearly bounded below by $\delta(\hat{R})$, we take $\tilde{R}$ to be the outer spectral factor of $I - \hat{R}\hat{R}^*$.

We show now that in this case

$$\operatorname{Im} H_{\tilde{R}} \subset \operatorname{Im} H_{\hat{R}} \tag{53}$$

This inclusion is equivalent to

$$\ker H^*_{\hat{R}} \subset \ker H^*_{\tilde{R}} \tag{54}$$

Note that $f \in \ker H^*_{\hat{R}}$ if and only if $\hat{R}^* f \in H^2_+$ and similarly $f \in \ker H^*_{\tilde{R}}$ if and only if $\tilde{R}^* f \in H^2_+$. If the inclusion (54) does not hold, there exists an $f \in H^2_+$ such that $\hat{R}^* f \in H^2_+$ and $\tilde{R}^* f = g + h$, with $g \in H^2_+$ and $0 \neq h \in H^2_-$. Now

$$f = (\hat{R} \quad \tilde{R}) \begin{pmatrix} \hat{R}^* \\ \tilde{R}^* \end{pmatrix} f = \hat{R}(\hat{R}^* f) + \tilde{R}(g + h)$$

which shows that there exists a non-zero $h \in H^2_-$ for which $\tilde{R}h \in H^2_+$ or, equivalently, $\tilde{R}h \perp H^2_-$. However this implies $h \perp \tilde{R}^* H^2_-$. Since $\tilde{R}$ is row outer, we have $H^2_+ \tilde{R} = H^2_+$ and hence $\tilde{R}^* H^2_- = H^2_-$. This shows that $h = 0$, contradicting our assumption. Thus the inclusion (53) holds. This inclusion immediately implies

$$\operatorname{Im} H_{(\hat{R}\ \tilde{R})} = \operatorname{Im} H_{\hat{R}}$$

and hence, by realization theory, that $\delta((\hat{R}\ \tilde{R})) = \delta(\hat{R})$.

(2) The proof is similar or can be obtained from the first part by duality considerations. □

We will refer to the extensions obtained in the previous proposition as *minimal inner embeddings*.

The next proposition relates the extremal factors to other minimal, stable and antistable, spectral factors via the use of inner functions. Since inner functions are, by Beurling's theorem, intimately related to invariant subspaces, this result opens up the possibility of a geometric approach to the study of spectral factors.

Proposition 13:

(1) *Let W_-^e and W_+^e be the extended, stable, minimum and maximum phase respectively, spectral factors. Given any minimal stable spectral factor W, there exist, essentially unique, inner functions Q', Q'', of minimal McMillan degree, for which*

$$\left.\begin{aligned} W &= W_-^e Q' \\ W_+^e &= W Q'' \end{aligned}\right\} \tag{55}$$

The inner functions Q', Q'' are uniquely determined by the normalization $Q'(\infty) = Q''(\infty)I$. We shall refer to the factorization $W = W_-^e Q'$ as an outer-inner factorization.

(2) *Let $\bar{W}_-^e$ and $\bar{W}_+^e$ be the extended, antistable, minimum and maximum phase respectively, spectral factors. Given any minimal antistable spectral factor $\bar{W}$, there exist essentially unique inner functions $\bar{Q}'$, $\bar{Q}''$ for which*

$$\left.\begin{aligned} \bar{W} &= \bar{W}_-^e \bar{Q}' \\ \bar{W}_+^e &= \overline{WQ''} \end{aligned}\right\} \tag{56}$$

The inner functions $\bar{Q}'$, $\bar{Q}''$ are uniquely determined by the normalization $Q'(\infty) = Q''(\infty)I$.

Proof:

(1) Let W be a minimal $p \times m$ stable spectral factor. Then, by Proposition 1, there exists an essentially unique row rigid $m_0 \times m$ function $\hat{Q}'$ for which $W = W_- \hat{Q}'$. Let

$$Q' = \begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix}$$

be the minimal inner extension of $\hat{Q}'$. This extension exists by Proposition 12 and is unique up to a constant left unitary factor of the form

$$\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}$$

We clearly have

$$W = W_- \hat{Q}' = (W_- \quad 0)\begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix} = W_-^e Q'$$

By Proposition 12, we have $\delta(Q') = \delta(\hat{Q}')$.

Conversely, let $W = W_-^e Q_1$ with Q_1 inner of minimal McMillan degree. Set

$$Q_1 = \begin{pmatrix} \hat{Q}_1 \\ \tilde{Q}_1 \end{pmatrix}$$

Then $W = W_- \hat{Q}_1 = W_- \hat{Q}'$, which by the left invertibility of W_- implies $\hat{Q}_1(\hat{Q}_1)^* = \hat{Q}'(\hat{Q}')^* = I$, it follows by minimality that $\hat{Q}_1 = V\hat{Q}'$ for some inner function V. If $\delta(\hat{Q}_1) = \delta(\hat{Q}')$ then this implies that V is necessarily constant.

(2) The proof is similar and we omit it. □

As a consequence of the previous proposition, we have the following characterization of minimal, stable, spectral factors.

Theorem 2: *Given a rational spectral function Φ, let W_+^e be its extended maximum phase, stable spectral factor. Then $W \in H_+^\infty$ is a minimal, stable, spectral factor if and only if there exists an inner function Q'' such that $W = W_+^e (Q'')^*$.*

Proof: By Proposition 13, if W is a minimal, stable, spectral factor, we have $W = W_+^e (Q'')^*$ for some inner function Q''.

Conversely, assume $W = W_+^e (Q'')^* \in H_+^\infty$ for an inner function Q''. Clearly, W is a stable spectral factor and it remains to show that it is a minimal factor. Applying Lemma 3.5.7 in Fuhrmann (1981), we have $\operatorname{Im} \hat{H}_W^c \subset \operatorname{Im} \hat{H}_{W_+^e}^c$ and hence

$$\delta(W) = \dim \operatorname{Im} \hat{H}_W^c \leq \dim \operatorname{Im} \hat{H}_{W_+^e}^c = \delta(W_+^e)$$

In turn, this implies

$$2\delta(W_+^e) = \delta(\Phi) \leq 2\delta(W) \leq 2\delta(W_+^e)$$

Thus we must have equality throughout, which proves the minimality of W. □

Proposition 13 dealt with the zeros of a stable or antistable spectral factor. The next theorem utilizes the DSS factorization to bring in the pole structure of spectral factors and its relation to zeros.

Theorem 3: *Let W be a minimal, $p \times m$, stable, spectral factor W, and let*

$$W = \bar{W}K = \bar{W}\bar{K}^* \tag{57}$$

be its right DSS factorization over H_-^∞. Then

(1) *$\bar{W}$ is a minimal, antistable spectral factor. Moreover*

$$
\begin{tikzcd}
& W \arrow[dr, "Q''"] & \\
W_-^e \arrow[ur, "Q'"] \arrow[rr, "Q_+^e"'] & & W_+^e \\
\overline{W}_-^e \arrow[u, "K_-^e"] \arrow[rr, "Q_-^e"] \arrow[dr, "\overline{Q}'"'] & & \overline{W}_+^e \arrow[u, "K_+^e"'] \\
& \overline{W} \arrow[uuu, "K"] \arrow[ur, "\overline{Q}''"'] &
\end{tikzcd}
$$

Figure 4.

$$\overline{W} = WK^{-1} = WK^* \tag{58}$$

is a right DSS factorization of $\overline{W}$ over H_+^∞.

(2) *Let Q', Q'', $\overline{Q}'$, $\overline{Q}''$ be the inner functions whose existence is guaranteed by Proposition* 13. *Then figure 4 is commutative.*

(3) *The following coprimeness conditions hold. $K_-^e \wedge_L \overline{Q}' = I$, $K \wedge_L \overline{Q}'' = I$, $K \wedge_R Q' = I$, $K_+^e \wedge_R Q'' = I$.*

Proof:

(1) Since K is inner, $\overline{W}$ is clearly an antistable, spectral factor. From (57) we get (58). Since $\overline{W}$ is a spectral factor, we have

$$\delta(W) \leq \delta(\overline{W}) = \delta(WK^{-1}) \leq \delta(K^*) = \delta(K) = \delta(W)$$

Thus we have equality throughout and $\overline{W}$ is a minimal spectral factor. This also shows the right coprimeness of W, K, i.e. (58) is a right coprime DSS factorization.

(2) The upper triangle of figure 4 follows from (15) and (55). Similarly, the lower triangle follows from (15) and (56). The square is the same as figure 2. Finally, (55), (56) and (57) yield $W_+^e = WQ'' = \overline{W}KQ'' = \overline{W}_-^e Q'KQ''$ which is the middle part of the diagram.

(3) The first relation follows from the fact that from (56) and (58), we can write $\overline{W}_-^e \overline{W}(\overline{Q}')^* = W_-^e (K_-^e)^*$. If $\overline{Q}'$ and K_-^e were not left coprime, then we could multiply on the right by the inverse of the common factor, obtaining another factorization of W_-^e; but then (58) would not be a DSS factorization. A similar argument applies to the relation $\overline{W} = \overline{W}_+^e (\overline{Q}'')^* = WK^*$ [derived also from (56) and (58)], to $W = W_-^e Q' = \overline{W}K$ and to $W_+^e = WQ'' = \overline{W}_+^e K_+^e$ [both from (55) and (57)]. □

We will find the following lemma useful in the sequel.

Lemma 3: *Let W_1, W_2 be two minimal, stable spectral factors having the right coprime DSS factorizations, over H^∞_-, given by $W_i = \bar{W}_i K_i$. Assume $W_2 = W_1 T$ for some inner function T, and assume T is the inner function with smallest McMillan degree such that the relation holds (in the sense of Proposition 12). Then there exists a unique inner function $\bar{T}$ for which $K_1 T = \bar{T} K_2$ and $\bar{W}_2 = \bar{W}_1 \bar{T}$.*

Proof: By our assumption we have $\bar{W}_2 K_2 = W_2 = W_1 T = \bar{W}_1 K_1 T$. Thus we have two right DSS factorizations of W_2 of which $\bar{W}_2 K_2$ is right coprime. There exists therefore a unique inner function $\bar{T}$ for which $\bar{W}_1 = \bar{W}_2 \bar{T}^*$ and $K_1 T = \bar{T} K_2$. This also implies the equality $\bar{W}_1 \bar{T} = \bar{W}_2$. □

The rigid functions, characterized by Proposition 13, can be further factored and this leads to a finer analysis of the set of spectral factors.

Thus we consider next the subsidiary factorizations

$$\left.\begin{aligned} \hat{Q}' &= \hat{Q}'_i \hat{Q}'_e \\ \hat{Q}'' &= \hat{Q}''_e \hat{Q}''_i \end{aligned}\right\} \tag{59}$$

Here $\hat{Q}'_i$, $\hat{Q}''_i$ are $m_0 \times m_0$ inner whereas $\hat{Q}'_e$, $\hat{Q}''_e$ are right and left outer, i.e. are right and left invertible respectively over H^∞_+. That such factorizations are possible is a direct result of Beurling's theorem. We will say that $\hat{Q}'_i$ and $\hat{Q}''_i$ describe the internal antistable and stable zeros respectively of W, whereas $\hat{Q}'_e$ and $\hat{Q}''_e$ describe the external antistable and stable zeros respectively of W. We shall refer to the factorizations (59) as the *internal–external* and *external–internal factorizations* of $\hat{Q}'$ and $\hat{Q}''$ respectively. The terminology is fitting as the factorizations in (59) are inner–outer and outer–inner factorizations.

We recall that we need not deal with rigid functions if we utilize the results of Proposition 13. Let W be a $p \times m$ minimal stable spectral factor. Let W^e_-, W^e_+ be the appropriately extended, i.e. of dimension $p \times m$, extremal spectral factors. The extension is obtained by adding $m - m_0$ zero columns to W_- and W_+ respectively. Let Q', Q'' be the inner functions whose existence is guaranteed by Proposition 13. Thus

$$Q' = \begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix}, \qquad Q'' = \begin{pmatrix} \hat{Q}'' & \tilde{Q}'' \end{pmatrix}$$

with $\tilde{Q}'$, $\tilde{Q}''$ given by the embedding procedure described in Proposition 12.

The factorizations (59) induce the following factorizations

$$Q' = \begin{pmatrix} \hat{Q}'_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{Q}'_e \\ \tilde{Q}' \end{pmatrix} = Q'_i Q'_e \tag{60}$$

and

$$Q'' = \begin{pmatrix} \hat{Q}''_e & \tilde{Q}'' \end{pmatrix} \begin{pmatrix} \hat{Q}''_i & 0 \\ 0 & I \end{pmatrix} = Q''_e Q''_i \tag{61}$$

We shall refer to these factorizations as *internal–external* and *external–internal factorizations* of Q' and Q'' respectively.

The interesting phenomena emerging in the study of rectangular spectral factors is the fact that the internal zeros, namely the zeros parametrized by the inner functions Q'_i and Q''_i no longer account for all the (stable) zeros of W_- or the (antistable) zeros of W_+. These are measured by the inner function Q_+. So some zeros have been externalized. This is a central theme in the parametrization of the set of stable spectral factors. The extreme situation is encountered in the set of external factors, a class that will be introduced in Definition 4.

The following theorem analyses the situation in the general case, and brings in a new inner function. It will be seen later that this inner function is a measure of the number of external zeros of the spectral factor. A geometric interpretation of this inner function in terms of reachability subspaces will be given in a subsequent paper.

We turn our attention now to the study of arbitrary, minimal $p \times m$ spectral factors. Essentially, the next theorem provides a parametrization of the set of all $p \times m$, minimal, stable spectral factors. This will be made explicit in Theorem 7.

Theorem 4: *Let Φ be a $p \times p$, rational spectral function of rank m_0. Let W be a minimal, $p \times m$, stable spectral factor. Let $W = \bar{W}K$ be the DSS factorization over H^∞_-. Let Q_+, Q_- be the $m_0 \times m_0$ inner functions for which*

$$\left.\begin{aligned} W_- Q_+ &= W_+ \\ \bar{W}_- Q_- &= \bar{W}_+ \end{aligned}\right\} \tag{62}$$

Let Q', Q'', $\bar{Q}'$, $\bar{Q}''$ be determined by Proposition 13 have the internal–external factorizations

$$Q' = Q'_1 Q'_2, \qquad \bar{Q}' = \bar{Q}'_1 \bar{Q}'_2 \tag{63}$$

and the external–internal factorizations

$$Q'' = Q''_2 Q''_1, \qquad \bar{Q}'' = \bar{Q}''_2 \bar{Q}''_1 \tag{64}$$

Then

(1) (a) *We have Q'_1 is the greatest common left inner factor of Q' and Q^e_+ and Q''_1 is the greatest common right inner factor of Q'' and Q^e_+. We have $\bar{Q}'_1$ is the greatest common left inner factor of $\bar{Q}'$ and $\bar{Q}^e_+$ and $\bar{Q}''_1$ is the greatest common right inner factor of $\bar{Q}''$ and $\bar{Q}^e_+$.*

(b) *$\bar{Q}''_1$ is the greatest common right inner factor of $\bar{Q}''$ and $\bar{Q}^e_-$ and Q''_1 is the greatest common right inner factor of Q'' and Q^e_+.*

(2) *We have*

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} \tag{65}$$

with R inner. We have

$$\bar{Q}'\bar{Q}'' = \begin{pmatrix} Q_- & 0 \\ 0 & \bar{R} \end{pmatrix} \tag{66}$$

with $\bar{R}$ inner.

(3) *There exists a unique inner function Q such that*

$$Q^e_+ = Q'_1 Q Q''_1 \tag{67}$$

A similar factorization holds for $\bar{Q}^e_+$, namely

$$\bar{Q}^e_+ = \bar{Q}'_1 \overline{Q Q''_1} \tag{68}$$

(4) *We have the following equalities*

$$\left.\begin{aligned} K^e_- Q' &= \bar{Q}' K \\ \bar{Q}'' K^e_+ &= K Q'' \end{aligned}\right\} \tag{69}$$

Here K^e_-, $\bar{Q}'$ *are left coprime and* Q', K *right coprime. Similarly,* K, $\bar{Q}''$ *are left coprime and* K^e_+, Q'' *right coprime.*

(5) *Define, after Lindquist and Picci* (1991),

$$\left.\begin{aligned} W_{0-} &= W^e_- Q'_1 \\ W_{0+} &= W^e_+ (Q''_1)^* \end{aligned}\right\} \tag{70}$$

Then we have

$$\left.\begin{aligned} W &= W_{0-} Q'_2 \\ W Q''_2 &= W_{0+} \end{aligned}\right\} \tag{71}$$

(6) *Let*

$$\left.\begin{aligned} W_{0-} &= \bar{W}_{0-} K_{0-} \\ W_{0+} &= \bar{W}_{0+} K_{0+} \end{aligned}\right\} \tag{72}$$

be right coprime DSS factorizations over H^∞_-. *Then*

$$\left.\begin{aligned} K^e_- Q'_1 &= \bar{Q}'_1 K_{0-} \\ \bar{Q}''_1 K^e_+ &= K_{0+} Q''_1 \end{aligned}\right\} \tag{73}$$

and

$$K_{0-} Q = \bar{Q} K_{0+} \tag{74}$$

(7) *We have*

$$\bar{Q}' K Q'' = \begin{pmatrix} Q_- K_+ & 0 \\ 0 & R_W \end{pmatrix} \tag{75}$$

with R_W *inner, and moreover*

$$R = \bar{R} = R_W \tag{76}$$

(8) (a) *The inner functions* Q *and* Q'_2 *are left coprime and* Q *and* Q''_2 *are right coprime.*
 (b) *The inner functions* $\bar{Q}$ *and* $\bar{Q}'_2$ *are left coprime and* $\bar{Q}$ *and* $\bar{Q}''_2$ *are right coprime.*
 (c) *The inner functions* Q'_2, R^e *are left coprime and* Q''_2, R^e *are right coprime.*
 (d) *The inner functions* $\bar{Q}'_2$, R^e *are left coprime and* $\bar{Q}''_2$, R^e *are right coprime.*

(9) *The inner functions* Q, $\bar{Q}$, Q'_2, Q''_2, $\bar{Q}'_2$, $\bar{Q}''_2$, R *are all equivalent.*

$$\begin{tikzcd}
 & & W \arrow[dr, "Q_2''"] \arrow[drr, "Q''"] & & \\
W_-^e \arrow[r, "Q_1'"'] \arrow[urr, "Q'"] & W_{0-} \arrow[rr, "Q"'] \arrow[ur, "Q_2'"] & & W_{0+} \arrow[r, "Q_1''"'] & W_+^e \\
\overline{W}_-^e \arrow[u, "K_-^e"] \arrow[r, "\overline{Q}_1'"] \arrow[drr, "\overline{Q}'"'] & \overline{W}_{0-} \arrow[u, "K_{0-}"] \arrow[rr, "\overline{Q}"] \arrow[dr, "\overline{Q}_2'"'] & & \overline{W}_{0+} \arrow[u, "K_{0+}"] \arrow[r, "\overline{Q}_1''"] & \overline{W}_+^e \arrow[u, "K_+^e"] \\
 & & \overline{W} \arrow[uuu, "K"] \arrow[ur, "\overline{Q}_2''"'] \arrow[urr, "\overline{Q}''"'] & &
\end{tikzcd}$$

Figure 5.

(10) *The inner functions K_-^e, K_{0-}, K, K_{0+}, K_+^e are all equivalent.*

(11) *Figure 5 is commutative.*

Proof:

(1) (a) Recall that in the factorization $\hat{Q}' = \hat{Q}_1'\hat{Q}_2'$ the factor $\hat{Q}_2'$ is outer. Since $\hat{Q}'\hat{Q}'' = Q_+$, $\hat{Q}_1'$ is a left inner factor of Q_+. This implies that

$$Q' = \begin{pmatrix} \hat{Q}_1' & 0 \\ 0 & I \end{pmatrix}$$

is a common left inner factor of Q', Q_+^e. Any left inner factor of Q_+^e is, up to a constant right unitary factor, of the form

$$\begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix}$$

Since $\hat{Q}_2'$ is outer, it is clear that Q_1' is the greatest common left inner factor of Q' and Q_+^e. The second assertion is proved analogously.

(b) The second part follows by duality considerations, working over H_-^∞ and starting the analysis from W_+.

(2) Note that from

$$Q' = \begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix}, \qquad Q''(\hat{Q}'' \quad \tilde{Q}'')$$

we obtain

$$Q'Q'' = \begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix} (\hat{Q}'' \quad \tilde{Q}'') = \begin{pmatrix} \hat{Q}'\hat{Q}'' & \hat{Q}'\tilde{Q}'' \\ \tilde{Q}'\hat{Q}'' & \tilde{Q}'\tilde{Q}'' \end{pmatrix}$$
$$= \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

Here we used the fact that $W_- \hat{Q}'\hat{Q}'' = W_+$ and hence $\hat{Q}'\hat{Q}'' = Q_+$. Since $Q'Q''$ is inner, it follows, necessarily, that $\hat{Q}'\tilde{Q}'' = 0$ and $\tilde{Q}'\hat{Q}'' = 0$. Finally, this also implies that $R = \tilde{Q}'\tilde{Q}''$ is inner.

(3) Note that $\hat{Q}'\tilde{Q}'' = 0$ implies $\hat{Q}'_2\tilde{Q}'' = 0$ and $\tilde{Q}'\hat{Q}'' = 0$ implies $\tilde{Q}'\hat{Q}''_2 = 0$. Using this, we compute

$$W_+^e = W_-^e Q'Q'' = W_-^e \begin{pmatrix} \hat{Q}'_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{Q}'_2 \\ \tilde{Q}' \end{pmatrix} \begin{pmatrix} \hat{Q}''_2 & \tilde{Q}'' \end{pmatrix} \begin{pmatrix} \hat{Q}''_1 & 0 \\ 0 & I \end{pmatrix}$$

$$= W_-^e \begin{pmatrix} \hat{Q}'_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{Q}'_2\hat{Q}''_2 & \hat{Q}'_2\tilde{Q}'' \\ \tilde{Q}'\hat{Q}''_2 & \tilde{Q}'\tilde{Q}'' \end{pmatrix} \begin{pmatrix} \hat{Q}''_1 & 0 \\ 0 & I \end{pmatrix}$$

$$= W_-^e \begin{pmatrix} \hat{Q}'_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{Q} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \hat{Q}''_1 & 0 \\ 0 & I \end{pmatrix}$$

But we also have

$$W_+^e = W_-^e \begin{pmatrix} Q_+ & 0 \\ 0 & I \end{pmatrix}$$

This implies the equality $Q_+ = \hat{Q}'_1\hat{Q}\hat{Q}''_1$.

(4) Since W is a minimal, stable spectral factor, we have $W = W_- \hat{Q}' = \bar{W}_- K_- \hat{Q}'$. But we also have $W = \bar{W}K = \bar{W}_- \overset{\triangle}{Q}{}' K$. Using the left invertibility of $\bar{W}_-$ and comparing the two expressions, we have

$$K_- \hat{Q}' = \overset{\triangle}{Q}{}' K \tag{77}$$

Since K is the minimal inner function that stabilizes $\bar{W}$, then necessarily $\hat{Q}'$ and K are right coprime. Clearly, this implies the right coprimeness of Q' and K. Considering the respective Bezout equations, this shows also the right coprimeness of Q' and K. Similarly, K_-^* is the minimal conjugate inner function that destabilizes W_-, which implies the left coprimeness of K_- and $\overset{\triangle}{Q}{}'$. Equation (77), together with the coprimeness conditions, shows that K_- and K are equivalent inner functions. In particular, they have the same McMillan degree. Also it follows that the McMillan degrees of $\hat{Q}'$, $\overset{\triangle}{Q}{}'$ are equal. Now we look at

$$Q' = \begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overset{\triangle}{Q}{}' \\ \tilde{Q}' \end{pmatrix}$$

which are both minimal McMillan degree inner completions. So $\delta(Q') = \delta(\hat{Q}') = \delta(\overset{\triangle}{Q}{}') = \delta(\bar{Q}')$. This shows that

$$\bar{Q}'K = \begin{pmatrix} \overset{\triangle}{Q}{}'K \\ \tilde{Q}'K \end{pmatrix} = \begin{pmatrix} K_- \hat{Q}' \\ \tilde{Q}'K \end{pmatrix}$$

is a minimal inner extension of $\hat{Q}'K$. Since a minimal inner extension is unique up to a left constant inner factor which we absorb into $\hat{Q}'$, we conclude that $\tilde{Q}' = \tilde{Q}'K$. Thus the first equality in (69) follows.

The second equality follows by duality considerations, working over H^∞_- and starting the analysis for $\bar{W}_+$.

(5) We compute $W = W^e_- Q' = W^e_- Q'_1 Q'_2 = W_{0-} Q'_2$. Similarly, $W^e_+ = WQ'' = WQ''_2 Q''_1$, and so $W_{0+} = W^e_+ (Q''_1)^* = WQ''_2$.

(6) The equalities in (73) are a special case of those in (69), replacing W by W_{0-} for the first one and by W_{0+} for the second. The existence of Q follows also from that of Q by an application of Lemma 3.

(7) Using the equalities (69), we have

$$K^e_- Q' Q'' = \bar{Q}' K Q'' = \bar{Q}' \bar{Q}'' K^e_+ .$$

Substituting equalities (65) and (66), we get

$$\bar{Q}' K Q'' = K^e_- Q' Q'' = K^e_- \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} K_- Q_+ & 0 \\ 0 & R \end{pmatrix}$$

$$= \bar{Q}' \bar{Q}'' K^e_+ = \begin{pmatrix} Q_- & 0 \\ 0 & \bar{R} \end{pmatrix} K^e_+ = \begin{pmatrix} Q_- K_+ & 0 \\ 0 & \bar{R} \end{pmatrix}$$

Since we have the equality $K_- Q_+ = Q_- K_+$, it follows that $R = \bar{R}$ and that

$$\bar{Q}' K Q'' = \begin{pmatrix} Q_- K_+ & 0 \\ 0 & \bar{R} \end{pmatrix}$$

(8) (a) Since Q'_1 is the greatest common left inner factor of Q' and Q^e_+, it follows that Q'_2, QQ''_1 are left coprime and so are Q'_2, Q. In a similar way, the right coprimeness of the pair Q''_2, Q is proved.

(b) In much the same way, the left coprimeness of $\bar{Q}'_2$, $\bar{Q}$ and the right coprimeness of $\bar{Q}''_2$, $\bar{Q}$ is also proved.

(c) A common left inner factor of

$$Q'_2 = \begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}$$

is, up to a right constant unitary factor, of the form

$$\begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix}$$

If P were non-trivial, it would contradict the assumption that Q' is the minimal McMillan degree extension of $\hat{Q}'$. Hence the right coprimeness.

(d) The proof is similar.

(9) We use the equalities

$$
\left.\begin{aligned}
Q'_2 Q''_2 &= \begin{pmatrix} \hat{Q} & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix} \\
\bar{Q}'_2 \bar{Q}''_2 &= \begin{pmatrix} \hat{Q} & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix}
\end{aligned}\right\} \tag{78}
$$

These equalities, together with the proven coprimeness conditions, imply now the equivalences

$$
Q'_2 \simeq \begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix} \simeq \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \simeq Q \simeq R
$$

$$
\bar{Q}'_2 \simeq \begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix} \simeq \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \simeq \bar{Q} \simeq R
$$

By transitivity of the equivalence of inner functions, the statement is proved.

(10) The equality $KQ'' = \bar{Q}''K^e_+$, the right coprimeness of Q'', K^e_+ and the left coprimeness of K, $\bar{Q}''$ imply the equivalence of K and K^e_+. The rest follows by the transitivity of equivalence.

(11) Equalities (63), (64), (69), (73), (70), (71), (74) taken together, yield the commutativity of the diagram. □

Definition 3: We denote by π_1 the projection of $\mathbf{C}^m$ on the subspace of all vectors whose last $m - m_0$ coordinates vanish, and by π_2 its complement, i.e. $\pi_2 := I - \pi_1$.

Corollary 4: *Given the factorizations* (65) *and* (66) *with* $Q_\pm$ *both* $m_0 \times m_0$ *inner functions, then we have*

$$
\left.\begin{aligned}
H_r(Q'')\pi_1 &\subset H_r(Q_+) \\
H_r(\bar{Q}'')\pi_1 &\subset H_r(Q_-)
\end{aligned}\right\} \tag{79}
$$

Proof: The factorization (65) can be written as $Q'Q'' = Q^e_+ R^e = R^e Q^e_+$ and therefore we have

$$
H_r(Q'Q'') = H_r(Q')Q'' \oplus H_r(Q'') = H_r(R^e) \oplus H_r(Q^e_+)
$$

and hence the inclusion $H_r(Q'') \subset H_r(R^e) \oplus H_r(Q^e_+)$. Applying the projection π_1 to this inclusion, we get $H_r(Q'')\pi_1 \subset H_r(Q_+)$. The other inclusion is proved similarly. □

Theorem 4 is general but two special cases need to be pointed out. This leads us to the following definition.

Definition 4: Let W be a $p \times m$ minimal, stable spectral factor and let Q', Q'' be the minimal inner functions characterized in Proposition 13.

(1) We say W is an *internal spectral factor* if we have $Q'Q'' = Q^e_+$.

(2) We say W is an *external spectral factor* if we have

(a) Q', Q^e_+ are left coprime, and

(b) Q'', Q_+^e are right coprime.

We point out that our definition of external spectral factors differs from the way Lindquist and Picci use this term. In their work they equate external with non-internal.

A description of factorizations of inner functions is clearly needed. A complete characterization of all factorizations of inner functions is available in terms of non-negative definite solutions of a homogeneous Riccati equation or in terms of invariant subspaces of a linear transformation. For more on this, see Willems (1971), Finesso and Picci (1982), Picci and Pinzoni (1994) and Fuhrmann (1995).

External and internal spectral factors are obviously defined in an extremely opposite way. The analysis of the general spectral factors will depend to a large extent on a full understanding of the classes of both external and internal spectral factors.

The next proposition is a characterization of minimal, stable, internal spectral factors.

Proposition 14: *Let W be a $p \times m$ minimal, stable spectral factor. Let Q', Q'' be the inner functions determined via Proposition 13. Then W is an internal spectral factor if and only if we have*

$$Q'Q'' = Q_+^e = \begin{pmatrix} Q_+ & 0 \\ 0 & I \end{pmatrix} \tag{80}$$

and, up to a right unitary factor for Q' and the inverse left factor for Q'', the factorizations (80) are in a bijective correspondence with normalized inner factorizations of Q_+, that is factorizations $Q_+ = Q_1 Q_2$ with Q_i normalized inner functions.

Proof: From $W = W_-^e Q'$ and the fact that the last $m - m_0$ columns of both W and W_-^e are zero, it follows that

$$Q' = \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix}$$

for some $m_0 \times m_0$ inner function Q_1. A similar argument holds for Q''. Clearly, we have $Q_1 Q_2 = Q_+$. □

Corollary 5: *Let W be a $p \times m_0$ minimal, stable spectral factor. Then W is necessarily internal.*

Proof: As in the proof of Proposition 14, we have $W = W_-^e Q'$ with

$$Q' = \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix}$$

for some $m_0 \times m_0$ inner function Q_1. Necessarily Q_1 is a left factor of Q_+, i.e. W is internal. □

From the proof of Theorem 4 it becomes clear that minimal, stable, external spectral factors of size $p \times m$ of a given spectral density Φ are related to factorizations of an inner function of the form

$$\begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$$

having special coprimeness properties. We proceed to formalize these properties in the following definition.

Definition 5: Let Q be an $m_0 \times m_0$ inner function and let R be an $(m - m_0) \times (m - m_0)$ inner function equivalent to Q. We say that a factorization

$$Q'Q'' = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \tag{81}$$

into the product of two $m \times m$ inner functions is a *balanced factorization* if Q' is left coprime and Q'' right coprime with both

$$Q^e = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}$$

Note that for the factorization (81) to be balanced, we need four coprimeness conditions. However, two of them are trivially satisfied. In fact, by the construction in Theorem 3, we have the left coprimeness of Q', R^e as well as the right coprimeness of Q'', R^e.

In the following we will need the next lemma.

Lemma 4: *Let*

$$Q'Q'' = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \tag{82}$$

Let, given an inner function Q, the unitary map $\tau_{\tilde{Q}} : H_r(Q) \to H_r(\tilde{Q})$ be defined by

$$(f\tau_{\tilde{Q}})(s) = f(-s)\tilde{Q}(s) \tag{83}$$

Here $\tilde{Q}(s) = Q(\bar{s})^*$. Then

(1) *Figures 6 and 7 are commutative*

$$\begin{tikzcd} H_r(Q'Q'') \arrow[r, "\tau_{\tilde{Q}''\tilde{Q}'}"] \arrow[d, "\pi_1"'] & H_r(\tilde{Q}''\tilde{Q}') \arrow[d, "\pi_1"] \\ H_r(Q_+) \arrow[r, "\tau_{\tilde{Q}^e_+}"] & H_r(\tilde{Q}_+) \end{tikzcd}$$

Figure 6.

$$\begin{tikzcd} H_r(Q'Q'') \arrow[r, "\tau_{\tilde{Q}''\tilde{Q}'}"] \arrow[d, "\pi_2"'] & H_r(\tilde{Q}''\tilde{Q}') \arrow[d, "\pi_2"] \\ H_r(R) \arrow[r, "\tau_{\tilde{R}}"] & H_r(\tilde{R}) \end{tikzcd}$$

Figure 7.

(2) *We have*

$$\left.\begin{aligned} H_r(Q'')\tau_{\tilde{Q}''\tilde{Q}'} &= H_r(\tilde{Q}'')\tilde{Q}' \\ H_r(Q')Q''\tau_{\tilde{Q}''\tilde{Q}'} &= H_r(\tilde{Q}') \end{aligned}\right\} \tag{84}$$

Proof:

(1) From the factorization (82) it follows that

$$\tau_{\tilde{Q}''\tilde{Q}'} = \tau_{\begin{pmatrix} \tilde{Q}_+ & 0 \\ 0 & \tilde{R} \end{pmatrix}} = \tau_{\tilde{Q}^e_+} \oplus \tau_{\tilde{R}^e}$$

Moreover, since $H_r(Q'Q'') = H_r(Q^e_+) \oplus H_r(R^e)$, any $f \in H_r(Q'Q'')$ can be written as $f = (f_1 \; f_2)$ with $f_1 \in H_r(Q^e_+)$ and $f_2 \in H_r(R^e)$. With this we compute

$$\begin{aligned} F\tau_{\tilde{Q}''\tilde{Q}'}\pi_1 &= (f_1 \quad f_2)(\tau_{\tilde{Q}^e_+} \oplus \tau_{\tilde{R}^e})\pi_1 \\ &= (f_1\tau_{\tilde{Q}^e_+} \quad f_2\tau_{\tilde{R}^e})\pi_1 \\ &= f_1\tau_{\tilde{Q}^e_+} = f\pi_1\tau_{\tilde{Q}^e_+} \end{aligned}$$

The commutativity of the second diagram is proved similarly.

(2) Let $f \in H_r(Q'')$. Then $f\tau_{\tilde{Q}''\tilde{Q}'} = (f\tau_{\tilde{Q}''})\tilde{Q}'$. Since $H_r(Q'')\tau_{\tilde{Q}''} = H_r(\tilde{Q}'')$, the first equality is proved. The proof of the second equality is similar. □

The following proposition is a geometric characterization of balanced factorizations. It is advisable, in following the arguments, to refer to figure 5.

Proposition 15: *Let W be a minimal, stable spectral factor. Then, with the notation of Theorem 4, we have*

(1) *The factorization*

$$Q_2'Q_2'' = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \tag{85}$$

is a balanced factorization.

(2) *We have*

$$\left.\begin{aligned} H_r(Q_2'')\pi_1 &= H_r(Q) \\ H_r(Q_2'')\pi_2 &= H_r(R) \end{aligned}\right\} \tag{86}$$

(3) *We have*

$$\left.\begin{aligned} H_r(\bar{Q}_2'')\pi_1 &= H_r(\bar{Q}) \\ H_r(\bar{Q}_2'')\pi_2 &= H_r(R) \end{aligned}\right\} \tag{87}$$

(4) *We have*

$$\left.\begin{aligned} \pi_1 H_c(Q_2') &= H_c(Q^e) \\ \pi_2 H_c(Q_2') &= H_c(R^e) \end{aligned}\right\} \tag{88}$$

(5) *We have*

$$\left.\begin{aligned} H_r(Q'')\pi_1 &= H_r(QQ_1'') \\ H_r(Q')Q''\pi_1 &= H_r(Q_1')Q_1''Q \end{aligned}\right\} \tag{89}$$

(6) *We have*

$$\left.\begin{aligned} H_r(Q'')\pi_2 &= H_r(R) \\ H_r(Q')Q''\pi_2 &= H_r(R) \end{aligned}\right\} \tag{90}$$

(7) *We have*

$$\left.\begin{aligned} H_r(\bar{Q}'')\pi_2 &= H_r(R) \\ H_r(\bar{Q}')\bar{Q}''\pi_2 &= H_r(R) \end{aligned}\right\} \tag{91}$$

(8) *We have*

$$H_r(QQ_1'') \cap H_r(Q_1'Q)Q_1'' = H_r(Q)Q_1'' \tag{92}$$

or equivalently

$$H_r(Q'')\pi_1 \cap H_r(Q')Q''\pi_1 = H_r(Q)Q_1''$$

(9) *The reduced maps* $H_r(Q'')|\pi_1$ *and* $H_r(Q')Q''|\pi_1$ *are injective.*

(10) *The reduced maps* $H_r(Q'')|P_{H_r(Q_+^e)}$ *and* $H_r(Q')Q''|P_{H_r(Q_+^e)}$ *are injective, and moreover, we have*

$$\left.\begin{aligned} H_r(Q'')P_{H_r(Q_+^e)} &= H_r(QQ_1'') \\ H_r(Q')Q''P_{H_r(Q_+^e)} &= H_r(Q_1'Q)Q_1'' \end{aligned}\right\} \tag{93}$$

(11) *We have*

$$H_r(Q_+^e)P_{H_r(Q'')} = H_r(Q'') \tag{94}$$

i.e. $H_r(Q_+^e)|P_{H_r(Q'')}$ *is surjective.*

Proof:

(1) That the factorization (85) is balanced was proved in Theorem 4.

(2) From the factorization (85) it follows that

$$H_r(Q_2'') \subset H_r(Q_2'Q_2'') = H_r(Q^e) \oplus H_r(R^e) \tag{95}$$

and hence

$$H_r(Q_2'')\pi_1 \subset H_r(Q_2'Q_2'') = H_r(Q^e)$$

From the fact that the factorization is balanced, we conclude the equivalence of the inner functions Q_2'' and Q, and hence it suffices to show that $\operatorname{Ker} H_r(Q_2'')|\pi_1 = \{0\}$. Indeed, assume $f = (f_1\ f_2) \in H_r(Q_2'')$ and $f\pi_1 = 0$, i.e. $f_1 = 0$. Thus, from (95), necessarily $(0\ f_2) \in H_r(R^e)$. So $f \in H_r(Q''_2) \cap H_r(R^e)$. But this intersection is zero by the right coprimeness of Q''_2 and R^e.

(3) This is proved in a completely analogous way to the proof of the previous assertion.

(4) This is proved in a completely analogous way to the proof of the previous assertion.

(5) From $Q'' = Q''_2 Q''_1$ it follows that $H_r(Q'') = H_r(Q''_2)Q''_1 \oplus H_r(Q''_1)$. We apply now π_1 to this equality, noting that π_1 commutes with multiplication by Q''_1 and that $H_r(Q''_1)\pi_1 = H_r(Q''_1)$. Hence

$$\begin{aligned} H_r(Q'')\pi_1 &= H_r(Q''_2)Q''_1\pi_1 + H_r(Q''_1) \\ &= H_r(Q''_2)\pi_1 Q''_1 \oplus H_r(Q''_1) \\ &= H_r(Q)Q''_1 \oplus H_r(Q''_1) = H_r(QQ''_1) \end{aligned}$$

Here we used the previously established equality $H_r(Q''_2)\pi_1 = H_r(Q)$.

To prove the second equality in (89), we apply the unitary map $\tau_{\tilde{Q}+}$ to the first, using the commutativity of figure 6. We compute

$$\begin{aligned} H_r(\tilde{Q}''_1\tilde{Q})\tilde{Q}_1 &= H_r(QQ''_1)\tau_{\tilde{Q}_+} = H_r(Q'')\pi_1\tau_{\tilde{Q}_+} \\ &= H_r(Q'')\tau_{\tilde{Q}''\tilde{Q}'}\pi_1 = H_r(\tilde{Q}'')\tilde{Q}'\pi_1 \end{aligned}$$

This is equivalent to the statement, using duality and starting from the factorization

$$\tilde{Q}''\tilde{Q}' = \begin{pmatrix} \tilde{Q}_+ & 0 \\ 0 & \tilde{R} \end{pmatrix}$$

(6) As before, from $Q'' = Q''_2 Q''_1$ it follows that $H_r(Q'') = H_r(Q''_2)Q''_1 \oplus H_r(Q''_1)$. To this equality we apply now the projection π_2 and use the second equality in (86) to get

$$H_r(Q'')\pi_2 = H_r(Q''_2)(Q''_1\pi_2 = H_r(Q''_2)\pi_2 = H_r(R)$$

Using Lemma 4, we apply to the previous equality the unitary map $\tau_{\tilde{R}}$ to get

$$\begin{aligned} H_r(\tilde{R}) &= H_r(R)\tau_{\tilde{R}} = H_r(Q'')\pi_2\tau_{\tilde{R}} \\ &= H_r(Q'')\tau_{\tilde{Q}''\tilde{Q}'}\pi_2 = H_r(\tilde{Q}'')\tilde{Q}'\pi_2 \end{aligned}$$

This is equivalent to the second equality.

(7) This is proved analogously.

(8) Follows directly from the equalities in (89). In fact

$$\begin{aligned} H_r(QQ''_1) \cap H_r(Q_1 Q)Q''_1 &= \left[H_r(Q)Q''_2 \oplus H_r(Q''_2)\right] \cap \left[H_r(Q_1)QQ''_2 \oplus H_r(Q)Q''_2\right] \\ &= H_r(Q)Q''_2 \end{aligned}$$

as clearly $H_r(Q''_2) \perp H_r(Q_1)QQ''_2$.

(9) Using (89), it suffices to show that $\dim H_r(Q'') = \dim H_r(QQ''_1)$. Now $Q'' = Q''_2 Q''_1$ and so $H_r(Q'') = H_r(Q''_2)Q''_1 \oplus H_r(Q''_1)$, which implies the equality $\dim H_r(Q'') = \deg\det Q''_2 + \deg\det Q''_1$. Now, the fact that the factorization (85) is balanced implies the equality $\deg\det Q''_2 = \deg\det Q$. Since $\dim H_r(QQ''_1) = \deg\det Q + \deg\det Q''_1$ the result follows.

(10) Clearly, $H_r(Q_+) \subset H^2_+\pi_1$ and therefore $P_{H_r(Q_+)} = \pi_1 P_{H_r(Q_+)}$. Applying this to the equality $H_r(Q'')\pi_1 = H_r(QQ''_1)$, and noting that $H_r(QQ''_1) \subset H_r(Q_+)$, we get

$$H_r(Q'')P_{H_r(Q_+)} = H_r(Q'')\pi_1 P_{H_r(Q_+)}$$

$$= H_r(QQ''_1)P_{H_r(Q_+)} = H_r(QQ''_1)$$

As we saw that $\dim H_r(Q'') = \dim H_r(QQ''_1)$, the statement is proved.

(11) Given a Hilbert space H with subspaces U, V, let P_U, P_V be the respective orthogonal projections. Then a simple computation yields

$$(P_V \mid U)^* = P_U \mid V \tag{96}$$

In particular, if $P_V \mid U$ is injective, $P_U \mid V$ is surjective. Thus the statement follows from the preceding one. □

We wish to emphasize one point which may be confusing exactly to those readers that have some intuition in the functional approach to geometric control theory. Given two arbitrary inner functions Q'' and Q^e_+ of the same size, then, in general, $H_r(Q'')P_{H_r(Q^e_+)}$ is not a coinvariant subspace of $H_r(Q^e_+)$, but, as will be shown in a forthcoming paper, a controlled invariant, inner stabilizable subspace with respect to a natural controllable pair associated with Q^e_+. Equation (93) shows that under our assumptions and the construction of Q', Q'' in Proposition 13, this subspace is actually coinvariant. This observation is crucial for the later analysis of the partial ordering of the set of minimal, stable spectral factors. Also, we point out that the question of existence of balanced factorizations will be addressed in §7.

Corollary 6: *With the notation of Proposition* 15, *we have*

(1)

$$\left.\begin{aligned} H_r(Q'')\pi_1 &= H_r(Q'')P_{H_r(Q^e_+)} = H_r(QQ''_1) \\ H_r(Q')Q''\pi_1 &= H_r(Q')Q''P_{H_r(Q^e_+)} = H_r(Q'_1Q)Q''_1 \end{aligned}\right\} \tag{97}$$

and

$$\left.\begin{aligned} H_r(Q'')\pi_2 &= H_r(Q'')P_{H_r(R)} = H_r(R) \\ H_r(Q')Q''\pi_2 &= H_r(Q')Q''P_{H_r(R)} = H_r(R) \end{aligned}\right\} \tag{98}$$

(2)

$$\left.\begin{aligned} H_r(\bar{Q}'')\pi_1 &= H_r(\bar{Q}'')P_{H_r(Q^e_-)} = H_r(\bar{Q}\bar{Q}''_1) \\ H_r(\bar{Q}')\bar{Q}''\pi_1 &= H_r(\bar{Q}')\bar{Q}''P_{H_r(Q^e_-)} = H_r(\bar{Q}'_1\bar{Q})\bar{Q}''_1 \end{aligned}\right\} \tag{99}$$

and

$$\left.\begin{aligned} H_r(\bar{Q}'')\pi_2 &= H_r(\bar{Q}'')P_{H_r(R)} = H_r(R) \\ H_r(\bar{Q}_2')\bar{Q}''\pi_2 &= H_r(\bar{Q}')\bar{Q}''P_{H_r(R)} = H_r(R) \end{aligned}\right\} \tag{100}$$

Proof: (97) follows from equations (89) and (93).

To prove (98) we use the equalities (90). Note that the inclusion $H_r(R) \subset H_+^2\pi_2$ implies $f\pi_2 P_{H_r(R)} = fP_{H_r(R)}$ for all $f \in H_+^2$. Therefore

$$H_r(Q')Q''P_{H_r(R)} = H_r(Q')Q''\pi_2 P_{H_r(R)} = H_r(R)P_{H_r(R)} = H_r(R)$$

and similarly

$$H_r(Q'')P_{H_r(R)} = H_r(Q'')\pi_2 P_{H_r(R)} = H_r(R)P_{H_r(R)} = H_r(R) \qquad \square$$

Corollary 7: *With the notation of Proposition* 15, *we have*

(1) $H_r(Q'')\pi_1 = H_r(Q_+^e)$ *if and only if* Q', Q_+^e *are left coprime.*
(2) $H_r(Q')Q''\pi_1 = H_r(Q_+^e)$ *if and only if* Q'', Q_+^e *are right coprime.*
(3) *Both equalities hold if and only if* W *is external.*

We saw that, given a minimal, stable spectral factor W, the inner functions Q_1' and Q_1'' parametrize the number of internal antistable and stable zeros of W respectively. Thus the previous corollary can be restated in these terms.

Corollary 8: *Let* W *be as in Proposition* 15. *Then*

(1) $H_r(Q'')\pi_1 = H_r(Q_+^e)$ *if and only if* W *has no internal antistable zeros.*
(2) $H_r(Q'')\pi_2 = H_r(R^e)$ *if and only if* W *has no internal stable zeros.*
(3) W *is an external spectral factor if and only if the factorization* (65) *is balanced which in turn is equivlent to the conditions*

$$H_r(Q'')\pi_1 = H_r(Q_+^e)$$

$$H_r(Q'')\pi_2 = H_r(R^e)$$

The previous results dealt with the geometry of coinvariant and semiinvariant subspaces related to the zeros of a spectral factor W, i.e. to the inner functions appearing in Theorem 4. The situation gets more intricate as soon as we add to this the pole structure. This means that we also consider the DSS factorizations of the spectral factor W, that is we study also spaces associated with the inner function K. Of particular interest is, using the notation of Theorem 4, the space $H_r(K)Q''$ which we associate with the spectral factor W. The full study of these spaces will be given in §6. The following results, which reflect some of the results of Corollary 6 will be used in later sections.

Lemma 5: *With the notation of Theorem* 4, *and assuming the factorizations*

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} \quad \text{and} \quad \bar{Q}'\bar{Q}'' = \begin{pmatrix} Q_- & 0 \\ 0 & R \end{pmatrix}$$

are balanced, we have

(1) *The restricted projection map* $H_r(Q')Q''|P_{H_r(K)Q''}$ *is injective. Equivalently*

$$H_r(K)Q''P_{H_r(Q')Q''} = H_r(Q')Q'' \tag{101}$$

(2) *The restricted projected map* $H_r(R^e)|P_{H_r(K)Q''}$ *is injective. Equivalently*

$$H_r(K)Q''P_{H_r(R^e)} = H_r(R^e) \tag{102}$$

(3) *We have*

$$H_r(K)Q''\pi_2 = H_r(R^e) \tag{103}$$

(4) *The restricted projection map* $H_r(\bar{Q}'')K_+^e|P_{H_r(K)Q''}$ *is injective. Equivalently*

$$H_r(K)Q''P_{H_r(\bar{Q}'')K_+^e} = H_r(\bar{Q}'')K_+^e \tag{104}$$

Proof:

(1) We show (101). By Corollary 3, we know that

$$H_r(K_+^e)P_{H_r(Q_+^e)} = H_r(Q_+^e) \tag{105}$$

Now $H_r(K)P_{H_r(Q')} = H_r(Q')$ if and only if $H_r(K)Q''P_{H_r(Q')Q''} = H_r(Q')Q''$. Since $H_r(Q'')$ is orthogonal to both $H_r(K)Q''$ and $H_r(Q')Q''$, the last equality is equivalent to $H_r(KQ'')P_{H_r(Q'Q'')} = H_r(Q'Q'')$ or to

$$H_r(KQ'')P_{H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}} = H_r(\bar{Q}''K_+^e)P_{H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}} = H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

Thus, it suffices to show that

$$H_r(\bar{Q}''K_+^e)P_{H_r(Q_+^e)} = H_r(Q_+^e) \tag{106}$$

and

$$H_r(\bar{Q}''K_+^e)P_{H_r(R^e)} = H_r(R^e) \tag{107}$$

Using the orthogonality of $H_r(K_+^e)$ and $H_r(R^e)$, we compute

$$H_r(\bar{Q}''K_+^e)P_{H_r(R^e)} = \left[H_r(\bar{Q}'')K_+^e \oplus H_r(K_+^e)\right]P_{H_r(R^e)} = H_r(\bar{Q}'')K_+^e P_{H_r(R^e)}$$
$$= H_r(\bar{Q}'')P_{H_r(R^e)} = H_r(R^e)$$

Here we used (100).

Similarly, using (105),

$$H_r(\bar{Q}''K_+^e)P_{H_r(Q^e)} = \left[H_v(\bar{Q}'')K_+^e \oplus H_r(K_+^e)\right]P_{H_r(Q^e)} \supset H_r(K_+^e)P_{H_r(Q^e)} = H_r(Q^e)$$

As the opposite inclusion is trivially satisfied, we have the equality.

(2) Since the subspaces $H_r(K)Q''$ and $H_r(Q'')$ are orthogonal, we have $H_r(K)Q''|P_{H_r(Q'')} = 0$. Using this, as well as equality (101), we compute

$$\begin{aligned} H_r(K)Q''P_{H_r(Q'Q'')} &= H_r(K)Q''P_{H_r(Q')Q''\oplus H_r(Q'')} \\ &= H_r(K)Q''\left[P_{H_r(Q')Q''} + P_{H_r(Q'')}\right] \\ &= H_r(K)Q''P_{H_r(Q')Q''} = H_r(Q')Q'' \end{aligned}$$

This can be rewritten as

$$H_r(K)Q''P_{H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}} = H_r(Q')Q''$$

To the last equality we apply the projection $P_{H_r(R)}$, noting that, for all $f \in H_+^2$, we have

$$fP_{H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}} P_{H_r(R)} = fP_{H_r(R)}$$

to obtain

$$\begin{aligned} H_r(K)Q''P_{H_r(R)} &= H_r(K)Q''P_{H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}} P_{H_r(R)} \\ &= H_r(Q')Q''P_{H_r(R)} = H_r(R) \end{aligned}$$

In the last line we used (98).

(3) The inclusion

$$H_r(K)Q'' \subset H_r\begin{pmatrix} K_-Q_+ & 0 \\ 0 & R \end{pmatrix}$$

implies the equality

$$H_r(K)Q'' = H_r(K)Q''P_{H_r\begin{pmatrix} K_-Q_+ & 0 \\ 0 & R \end{pmatrix}}$$

So, using (102), we compute

$$\begin{aligned} H_r(K)Q''\pi_2 &= H_r(K)Q''P_{H_r\begin{pmatrix} K_-Q_+ & 0 \\ 0 & R \end{pmatrix}} \pi_2 \\ &= H_r(K)Q''\left[P_{H_r(K_-^eQ_+^e)} + P_{H_r(R^e)}\right]\pi_2 \\ &= H_r(K)Q''P_{H_r(R^e)}\pi_2 = H_r(R^e)\pi_2 = H_r(R^e) \end{aligned}$$

(4) The proof is analogous to the proof of the first statement and we omit it. □

The previous result can have many equivalent restatements. Intuitively, they all state that there is an excess of poles over zeros.

Corollary 9: *Let W be a minimal, stable spectral factor. Then with the notation of Theorem 4,*

(1) *The Toeplitz operator $\mathcal{T}^r_{K(Q')^*} = \mathcal{T}^r_{(\bar{Q}')^*K_-^e}$ is injective.*

(2) *We have*

$$H_+^2Q' + H_-^2K = L^2 \tag{108}$$

(3) *We have*

$$H_+^2 K \cap H_-^2 Q' = \{0\} \tag{109}$$

(4) *We have*

$$H_+^2 K \cap H_r(Q') = \{0\} \tag{110}$$

Proof: By Lemma 5, we have the injectivity of the projection map $H_r(Q')Q''\big|P_{H_r(K)Q''}$ which is equivalent to the injectivity of the projection map $H_r(Q')\big|P_{H_r(K)}$. We apply now Theorem 1 to infer (108), (109) and (110). $\square$

We proceed to prove an important geometric result whose control significance will be stated at the end of the proof.

Theorem 5: *Let W be a minimal, stable spectral factor. Then, with the notation of Theorem* 5.3, *we have*

(1)

$$H_r(R)P_{H_r(K)Q''} \subset H_r(Q')Q''P_{H_r(K)Q''} \cap H_r(\bar{Q}'')K_+ P_{H_r(K)Q''} \tag{111}$$

(2)

$$H_r(R^e)P_{H_r(K)Q''} = H_r(Q_2')Q''P_{H_r(K)Q''} = H_r(\bar{Q}_2'')\bar{Q}_1''K_+^e P_{H_r(K)Q''} \tag{112}$$

Proof:

(1) We use the factorization

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

to conclude

$$\begin{aligned} H_r(Q')Q'' \oplus H_r(Q'') = H_r(Q'Q'') &= H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} \\ &= H_r(Q_+^e) \oplus H_r(R^e) \supset H_r(R^e) \end{aligned}$$

Therefore

$$\begin{aligned} H_r(Q')Q''P_{H_r(K)Q''} &= H - r(Q'Q'')P_{H_r(K)Q''} \\ &= H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} P_{H_r(K)Q''} \supset H_r(R^e)P_{H_r(K)Q''} \end{aligned}$$

Similarly, we have $KQ'' = \bar{Q}''K_+^e$ and hence

$$\begin{aligned} H_r(\bar{Q}'')K_+^e P_{H_r(K)Q''} &= \left[H_r(\bar{Q}')KQ'' \oplus H_r(\bar{Q}'')K_+^e\right]P_{H_r(K)Q''} \\ &= \left[H_r(\bar{Q}')\bar{Q}'' \oplus H_r(\bar{Q}'')\right]K_+^e P_{H_r(K)Q''} \\ &= H_r(\bar{Q}'\bar{Q}'')K_+^e P_{H_r(K)Q''} \\ &= H_r\begin{pmatrix} Q_- & 0 \\ 0 & R \end{pmatrix} K_+^e P_{H_r(K)Q''} \\ &= \left[H_r(Q_-^e)K_+^e \oplus H_r(R^e)\right]P_{H_r(K)Q''} \supset H_r(R^e)P_{H_r(K)Q''} \end{aligned}$$

From these two inclusions we obtain the inclusion

$$H_r(R)P_{H_r(K)Q''} \subset H_r(Q')Q''P_{H_r(K)Q''} \cap H_r(\bar{Q}'')K_+ P_{H_r(K)Q''} \tag{113}$$

(2) We have

$$Q_2'Q_2'' = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$$

Since Q_2', Q are left coprime and Q_2'', Q are right coprime, we obtain the equivalence of the inner functions Q, R, Q_2', Q_2''. Now we consider the direct sum decomposition $H_r(Q_2'Q'') = H_r(Q_2')Q'' \oplus H_r(Q'')$ which, by the orthogonality of $H_r(K)Q''$ and $H_r(Q'')$, leads to $H_r(Q_2')Q''P_{H_r(K)Q''} = H_r(Q_2'Q'')P_{H_r(K)Q''}$. But we also have

$$\begin{aligned} H_r(Q_2'Q'') = H_r(Q_2'Q_2''Q_1'') &= H\left[\begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} Q_1''\right] \\ &= H_r(Q^e)Q_1'' \oplus H_r(R^e) \supset H_r(R^e) \end{aligned}$$

Hence

$$H_r(Q_2')Q''P_{H_r(K)Q''} = H_r(Q_2'Q'')P_{H_r(K)Q''} \supset H_r(R^e)P_{H_r(K)Q''}$$

Similarly, note that

$$H_r(\bar{Q}_2')\bar{Q}_2''\bar{Q}_1''K_+^e = H_r(\bar{Q}_2')\bar{Q}''K_+^e = H_r(\bar{Q}_2')KQ''$$

So $H_r(\bar{Q}_2')KQ'' \subset \operatorname{Ker} P_{H_r(K)Q''}$ and hence

$$\begin{aligned} H_r(\bar{Q}_2'')\bar{Q}_1''K_+^e P_{H_r(K)Q''} &= \left[H_r(\bar{Q}_2')\bar{Q}_2''\bar{Q}_1''K_+^e \oplus H_r(\bar{Q}_2'')\bar{Q}_1''K_+^e\right]P_{H_r(K)Q''} \\ &= \left[H_r(\bar{Q}_2')\bar{Q}_2'' \oplus H_r(\bar{Q}_2'')\right]\bar{Q}_1''K_+^e P_{H_r(K)Q''} \\ &= H_r(\bar{Q}_2'\bar{Q}_2'')\bar{Q}_1''K_+^e P_{H_r(K)Q''} \\ &= \left[H_r\begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \bar{Q}_1''K_+^e\right]P_{H_r(K)Q''} \\ &= \left[H_r(Q^e)(\bar{Q}_1'')^e K_+^e \oplus H_r(R^e)\right]P_{H_r(K)Q''} \\ &\supset H_r(R^e)P_{H_r(K)Q''} \end{aligned}$$

These two inclusions prove the inclusion

$$H_r(R^e)P_{H_r(K)Q''} \subset H_r(Q_2')Q''P_{H_r(K)Q''} \cap H_r(\bar{Q}_2'')\bar{Q}_1''K_+^e P_{H_r(K)Q''} \tag{114}$$

By Lemma 5, the projection $P_{H_r(K)Q''}$ is injective on $H_r(R^e)$, which implies

$$\dim\left[H_r(R^e)P_{H_r(K)Q''}\right] = \dim\left[H_r(R^e)\right]$$

Also, by Theorem 4, the inner functions R, Q_2', $\bar{Q}_2''$ are equivalent and hence in particular

$$\dim\left[H_r(R^e)\right] = \dim H_r(Q'_2)Q'' = \dim H_r(\bar{Q}''_2)\bar{Q}''_1 K^e_+$$

Therefore the inclusion (114) forces the equality (112). □

For the case of external spectral factors, the inclusion (111) can be strengthened to give.

Corollary 10: *Let W be a minimal, stable, external spectral factor. Then*

$$H_r(R^e)P_{H_r(K)Q''} = H_r(Q')Q''P_{H_r(K)Q''} = H_r(\bar{Q}'')K^e_+ P_{H_r(K)Q''} \tag{115}$$

Proof: By Theorem 5, we have

$$H_r(R^e)P_{H_r(K)Q''} = H_r(Q')Q''P_{H_r(K)Q''} \cap H_r(\bar{Q}'')K^e_+ P_{H_r(K)Q''}$$

But using the injectivity of the projection $P_{H_r(K)Q''}$ on the three subspaces and the fact that they all have the same dimension, the equality follows. □

In a sense not made explicit in this paper, the space $H_r(K)Q''$ can be used as a natural state space for a state space realization of the spectral factor W. This is based on the realization theory developed in Fuhrmann (1981) and we refer to it as the shift realization. Thus the subspaces $H_r(Q')Q''P_{H_r(K)Q''}$, $H_r(\bar{Q}'')K_+P_{H_r(K)Q''}$, $H_r(Q'_2)Q''P_{H_r(K)Q''}$, $H_r(\bar{Q}''_2)\bar{Q}''_1 K^e_+ P_{H_r(K)Q''}$ and $H_r(R^e)P_{H_r(K)Q''}$, are all special subspaces of $H_r(K)Q''$, and it is natural to inquire as to their system theoretic significance. Without going into the details, we just state here that, with respect to the shift realization, the first two are the maximal output nulling, inner stabilizable subspace and the maximal output nulling, inner antistabilizable subspace respectively, whereas the last three are different representations of the maximal output nulling, reachability subspace. Moreover the inclusion in (111) can be shown to be actually an equality. A state space approach to these results will be given in Gombani and Fuhrmann (1998), while a functional approach to these characterizations will be presented in Fuhrmann (1998).

So far, we have bypassed the question of the existence of balanced factorizations. The following proposition shows that plenty of balanced factorizations exist.

Proposition 16: *Let Q_+ be an inner function and let R be an inner function equivalent to it. Then there exists a balanced factorization*

$$\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} = Q'Q'' \tag{116}$$

Proof: By the assumption of equivalence, there exist, necessarily inner, matrix functions T, S in H^∞_+ such that the intertwining relation

$$Q_+T = SR \tag{117}$$

holds, with Q_+, S left coprime and T, R right coprime. Equation (117), see Fuhrmann (1981) for the details, gives rise to an H^∞_+-isomorphism $Z : H_r(Q_+) \to H_r(R)$, defined by

$$fZ = fTP_{H_r(R)}$$

The isomorphism is in the sense that, for every $\phi \in H^\infty_+$ and $f \in H_r(Q_+)$, we have

$$f\phi P_+ TP_{H_r(R)} = fTP_{H_r(R)}\phi P_+$$

The adjoint map, $Z^*: H_r(R) \to H_r(Q_+)$, is given by

$$gZ^* = gT^*P_+$$

and we have, again for every $\phi \in H_+^\infty$, that

$$g\phi^*P_+T^*P_+ = gP_+T^*P_+\phi^*$$

Next, we define a subspace of

$$H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} = H_r(Q_+^e) \oplus H_r(R^e)$$

by

$$\mathcal{V} = \left\{ (gZ^* \quad g) \middle| g \in H_r(R) \right\}$$

Clearly, $\mathcal{V}$ is a coinvariant subspace. In fact, given $\phi \in H_+^\infty$, we compute

$$\begin{aligned} (gZ^*P_+ \quad g)\phi^*P_+ &= (gT^*P_+\phi^*P_+ \quad g\phi^*P_+) \\ &= (g\phi^*P_+T^*P_+ \quad g\phi^*P_+) \\ &= (g_1Z^* \quad g_1) \in \mathcal{V} \end{aligned}$$

where $g_1 = g\phi^*P_+ \in H_r(R)$. Thus, by Beurling's theorem, there exists an inner function Q'' for which $\mathcal{V} = H_r(Q'')$. This implies the factorization (116).

We show that necessarily, the factorization (116) is balanced. Clearly, by construction, we have

$$\left.\begin{aligned} \mathcal{V}\pi_1 &= \mathcal{V}P_{H_r(Q_+^e)} = H_r(Q_+^e) \\ \mathcal{V}\pi_2 &= \mathcal{V}P_{P_r(R_+^e)} = H_r(R^e) \end{aligned}\right\} \tag{118}$$

If Q'', Q_+^e are not right coprime, we set $Q'' \wedge_R Q_+^e = Q_1''$. Thus there exist a factorization $Q'' = Q_2''Q_1''$ and hence $H_r(Q'') = H_r(Q_2'')Q_1'' \oplus H_r(Q_1'')$. Clearly this implies that $H_r(Q'')\pi_2 = H_r(Q_2'')Q_1''\pi_2$ cannot be equal to $H_r(R)$, in contradiction to (118). The other coprimeness conditions are proved similarly. □

We can give now a characterization of minimal, stable, external spectral factors.

Theorem 6: *There is a bijective correspondence between*

(1) *Minimal, external $p \times m$ spectral factors.*

(2) *Balanced factorizations of inner functions of the form*

$$\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} \tag{119}$$

with $R \simeq Q_+$ being an $(m - m_0) \times (m - m_0)$ inner function.

(3) *The set of all H_+^∞-isomorphisms from $H_r(Q'')$ onto $H_r(R)$, with $R \simeq Q_+$ being an $(m - m_0) \times (m - m_0)$ inner function.*

Proof: Assume W is a minimal, stable, $p \times m$ external spectral factor. By Theorem 4, and using its notaton, we have the existence of a factorization of the form (119) which, by Definition 5, is balanced.

Conversely, let (119) be any balanced factorization. Such factorizations exist by Proposition 16. Define now $W = W_-^e Q'$. Clearly W is a stable spectral factor. Since $WQ'' = W_+^e$, it has to be minimal. Clearly this correspondence is bijective.

From Proposition 16 it follows that to each such H_+^∞-isomorphism there corresponds an essentially unique balanced factorization.

Conversely, the balanced factorization (119), can be written as $Q'Q'' = Q_+^e R$. Using the coprimeness conditions, we can embed this equality in a doubly coprime factorization. This implies the existence of rational matrices L, M satisfying $Q''L = MR^e$. Define a map $Z : H_r(Q'') \to H_r(R^e)$ by $fZ = fLP_{H_r(R^e)}$. Using the left coprimeness of Q'', M and the right coprimeness of L, R^e which follow from the construction of a doubly coprime factorization, the map Z is indeed an H_+^∞-isomorphism. Note that the doubly coprime factorization is not unique, but the intertwining map Z is. Thus there exists a corresponding unique isomorphism between the set of balanced factorizations and the set of H_+^∞-isomorphisms from $H_r(Q'')$ onto $H_r(R)$. Now, by Theorem 4 and using the fact that W is external, we have the equivalence of Q'' and Q_+. Fix any H_+^∞-isomorphism of $H_r(Q'')$ and $H_r(R^e)$ and the result follows by transitivity. □

Theorem 6 leads immediately to a full parametrization of the set $\mathcal{W}^m$.

Theorem 7: *There is a bijective correspondence between*

(1) *The set $\mathcal{W}^m$ of all $p \times m$, minimal, stable spectral factors.*

(2) *Factorizations of the form*

$$Q_1'Q_2'Q_2''Q_1'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

where $Q_+^e = Q_1'QQ_1''$ with all factors block diagonal of the form

$$\begin{pmatrix} * & 0 \\ 0 & I_{m-m_0} \end{pmatrix}$$

and

$$Q_2'Q_2'' = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$$

is a balanced factorization with R any $(m - m_0) \times (m - m_0)$ inner function satisfying $Q \simeq R$. All inner functions are assumed to be normalized at infinity.

Proof: Follows from Theorems 4 and 6. □

As a result of Theorem 6, it is clear that the existence of a $p \times m$ external spectral factor is dependent on m. Indeed we have the following.

Corollary 11: *Let Φ be a rank m_0 spectral function. Then there exists a $p \times m$ external spectral factor if and only if $m - m_0$ is greater or equal to the number of non-trivial inner invariant factors of Q_+.*

Proof: The inner function R has to be chosen to be equivalent to Q_+, thus necessarily it has the same number of non-trivial inner invariant factors as Q_+ and its dimension has to be at least equal to that number. □

We promised in §3 to give a localized version of the phase function and this we proceed to do. To this end, we need first the following.

Definition 6: Let W_1 and W_2 be two internal spectral factors. We say that the $W_1 \leq W_2$ if there exists an inner function Q such that $W_2 = W_1 Q$.

Lemma 6: *Let W_1, W_2 be two, minimal stable internal spectral factors. Let $W_i = \bar{W}_i K_i$ be the DSS factorizations over H^∞_-. Assume $W_1 \leq W_2$, i.e. $W_1 Q = W_2$, and let $\bar{Q}$ be the inner function satisfying $\bar{Q} K_2 = K_1 Q$. Defining the local phase function by*

$$T_0(W_1, W_2) = \bar{W}_2^{-L} W_1 \tag{120}$$

we have

$$\bar{W}_2 \bar{W}_2^{-L} W_2 = W_2 \tag{121}$$

$$\bar{W}_2 \bar{W}_2^{-L} W_1 = W_1 \tag{122}$$

$$W_2 W_2^{-L} W_1 = W_1 \tag{123}$$

$$T_0 = \bar{W}_2^{-L} W_1 = K_2 Q^{-1} = \bar{Q}^{-1} K_1 \tag{124}$$

Proof: By assumption, we have $W_1 Q = W_2$ and

$$\bar{W}_2 = W_2 K_2^* = W_1 Q K_2^* = W_1 K_1^* \bar{Q} = \bar{W}_1 \bar{Q}$$

Since $\bar{W}_2^{-L} \bar{W}_2 = I$, we have $\bar{W}_2 \bar{W}_2^{-L} \bar{W}_2 = \bar{W}_2$. Now

$$\bar{W}_2 \bar{W}_2^{-L} W_2 = \bar{W}_2 \bar{W}^{-L} \bar{W}_2 K_2 = \bar{W}_2 K_2 = W_2 \tag{125}$$

This proves (121). (122) follows, using the fact that $W_1 Q = W_2$. Substituting $W_2 = W_1 Q$ in the equality $W_2 W_2^{-L} W_2 = W_2$ and eliminating Q leads to (123). Finally, using (123), we compute

$$T_0(W_1, W_2) = \bar{W}_2^{-L} W_1 = \bar{W}_2^{-L}(W_2 W_2^{-L} W_1) = (\bar{W}_2^{-L} W_2)(W_2^{-L} W_1) = K_2 Q^{-1}$$

and use the equality $K_2 Q^{-1} = \bar{Q}^{-1} K_1$. □

Next we proceed to prove the following version of Proposition 11.

Proposition 17: *Let Φ be a $p \times p$, rank m_0, weakly coercive, spectral function, and let $W_1 \leq W_2$ be two, minimal, stable, internal spectral factors. Let $W_i = \bar{W}_i K_i$ be the respective DSS factorizations over H^∞_-. Let Q, $\bar{Q}$ be the minimal McMillan degree inner functions satisfying $\bar{Q} K_2 = K_1 Q$. Then both Toeplitz operators $\mathcal{T}^c_{\bar{W}_2^{-L} W_1} = \mathcal{T}^c_{K_2 Q^*} = \mathcal{T}^c_{\bar{Q}^* K_1}$ and $\mathcal{T}^r_{\bar{W}_2^{-L} W_1}$ are injective.*

Proof: The proof follows that of Proposition 11. For the injectivity of $\mathcal{T}^c_{\bar{W}_2^{-L} W_1}$ we show an explicit left inverse, namely the map $f \mapsto W_1^{-L} P_+ \bar{W}_2 f$. Indeed, using (122), we compute

$$\begin{aligned} W_1^{-L} P_+ \bar{W}_2 P_+ \bar{W}_2^{-L} W_1 f &= W_1^{-L} P_+ \bar{W}_2 \bar{W}_2^{-L} W_1 f \\ &= W_1^{-L} P_+ W_1 f = W_1^{-L} W_1 f = f \end{aligned}$$

The injectivity of $\mathcal{T}^r_{\bar{W}_2^{-L} W_1}$ follows from Proposition 9. □

The previous result indicates that the definition of the phase function can be extended to all spectral factors. In fact, given a Lindquist–Picci pair W, $\bar{W}$ of minimal spectral factors, then, with the notation of Theorem 4, we can define the associated phase function to be $K(Q')^* = (\bar{Q}')^* K_-^e$. Note that we already proved, in Corollary 9, the injectivity of the Toeplitz opeator $\mathcal{T}^r_{K(Q')^*} = \mathcal{T}^r_{(\bar{Q}')^* K_-^e}$.

6. Minimal Markovian splitting subspaces

Each minimal, stable $p \times m$ spectral factor W has two, coprime, DSS factorizations. With each one there is associated a coinvariant subspace that is a natural state space for a shift based realization of W. If we choose to work with a left coprime factorization of W over H_-^∞ then, as a consequence of Proposition 2, all minimal, stable spectral factors have the same left Douglas–Shapiro–Shields factor. This amounts to the fact that in a realization of any such spectral factor W, the matrices A, C can be chosen to be the same. This is the uniform choice of basis, see Caines and Delchamps (1980). It turns out that for our purpose we have to work with right coprime DSS factorizations; in this case, the right denominators, and hence the associated state spaces, do not need to be the same, this fact is peculiar of the multivariable case. This enables us to study the zeros in purely geometric terms. We proceed to explain this association. By Proposition 13, there exists an essentially unique inner function Q'' such that $W_+^e = \bar{W}Q''$. On the other hand, W has a right coprime DSS factorization $W = \bar{W}K = \bar{W}\bar{K}^*$, with $\bar{W}$ necessarily a minimal anti-stable spectral factor. Let $H_r(K) = \{H_+^2 K\}^\perp$ be the coinvariant subspace associated with the inner function K. We say that

$$X_W = H_r(K)Q'' \tag{126}$$

is the *minimal Markovian splitting subspace associated with the minimal spectral factor* W. Note that all the spaces $H_r(K)$ carry isomorphic H_+^∞-module structures. This is due to the fact that the pole structure of all minimal, stable spectral factors is the same, and they differ only in their zero structure. The particular definition, given in (126) is a result of a normalization, where with W_+ we associate the state space $H_r(K_+^e)$. We denote by $\mathcal{X}^m$ the family of all minimal Markovian splitting subspaces associated to minimal, stable, spectral factors of size $p \times m$. In an analogous way we can associate with the antistable factor $\bar{W}$ the state space $X_{\bar{W}} = H_r(\bar{K}^*)(\bar{Q}')^*$. We denote by $\bar{\mathcal{X}}^m$ the set of all such spaces corresponding to factors of size $p \times m$. At this point our definiton of splitting subspaces is rather formal and devoid of any stochastic interpretation. But it turns out that this is the simplest way to transpose in the Hardy space setting the geometric structure of the stochastic domain, however we omit the details.

The following proposition organizes the information concerning the geometry of the splitting subspaces associated with an arbitrary minimal, stable spectral factor.

Proposition 18: *Let W be a minimal, stable spectral factor. Let $W = \bar{W}K$ be its DSS factorization over H_-^∞. Let the inner functions Q', Q'', $\bar{Q}'$, $\bar{Q}''$ be determined by Proposition* 13. *Then we have*

(1) *The following are orthogonal direct sum decompositions of the coinvariant subspace $H_r(Q'K)$*

$$\begin{aligned}
H_r(\bar{Q}'KQ'') &= H_r(Q^e_-)K^e_+ \oplus H_r(R^e) \oplus H_r(K^e_+) \\
&= H_r(\bar{Q}')\bar{Q}''K^e_+ \oplus H_r(\bar{Q}'')K^e_+ \oplus H_r(K^e_+) \\
&= H_r(\bar{Q}')KQ'' \oplus H_r(K)Q'' \oplus H_r(Q'') \\
&= H_r(K^e_-)Q'Q'' \oplus H_r(Q')Q'' \oplus H_r(Q'') \\
&= H_r(K^e_-)Q^e_+ \oplus H_r(R^e) \oplus H_r(Q^e_+)
\end{aligned} \tag{127}$$

(2) *For L^2 we have the following direct sum decomposition.*

$$\begin{aligned}
L^2 &= H^2_+Q'' \oplus H^2_-Q'' \\
&= H^2_+KQ'' \oplus H_r(K)Q'' \oplus H^2_-Q''
\end{aligned} \tag{128}$$

(3) *Defining the L^2 subspaces by*

$$\left.\begin{aligned}
S_W &= H^2_+Q'' \\
\bar{S}_W &= H^2_-KQ''
\end{aligned}\right\} \tag{129}$$

we have

$$X_W = H_r(K)Q'' = S_W \cap \bar{S}_W \tag{130}$$

The pair $(S_W, \bar{S}_W)$ will be called an extended scattering pair *for the spectral factor W.*

(4) *We have*

$$\left.\begin{aligned}
S_W^\perp &= H^2_-Q'' \\
\bar{S}_W^\perp &= H^2_+KQ''
\end{aligned}\right\} \tag{131}$$

and

$$L^2 = \bar{S}_W^\perp \oplus X_W \oplus S_W^\perp \tag{132}$$

(5) *The projection operators*

$$\mathcal{R} = H_r(K^e_+)\big|P_{H_r(K)Q''} : H_r(K^e_+) \to H_r(K)Q'' \tag{133}$$

and

$$\mathcal{O} = H_r(K)Q''\big|P_{H_r(K^e_-)Q'Q''} : H_r(K)Q'' \to H_r(K^e_-)Q'Q'' \tag{134}$$

are bijective.

(6) *The projection operators*

$$H_r(K)Q''\big|P_{H_r(K^e_+)} : H_r(K)Q'' \to H_r(K^e_+) \tag{135}$$

and

$$H_r(K)Q''\big|P_{H_r(K^e_-)Q'Q''} : H_r(K)Q'' \to H_r(K^e_-)Q'Q'' \tag{136}$$

are bijective.

$$\begin{tikzcd}
 & H_r(K_\alpha)Q''_\alpha \arrow[dl, "\mathcal{O}_\alpha"'] & H_r(K_+^e) \arrow[l, "\mathcal{R}_\alpha"'] \arrow[dl, "\mathcal{R}_\beta"] \\
H_r(K_-^e)Q_+^e & H_r(K_\beta)Q''_\beta \arrow[u, "Z"] \arrow[l, "\mathcal{O}_\beta"] &
\end{tikzcd}$$

Figure 8.

(7) *We have the following factorization*

$$H_r(K_+^e)\big|P_{H_r(K_-^e)Q'Q''} = \left(H_r(K_+^e)\big|P_{H_r(K)Q''}\right)\cdot\left(H_r(K)Q''\big|P_{H_r(K_-^e)Q'Q''}\right) \tag{137}$$

(8) *Let W_i, $i=\alpha, \beta$ be two minimal, stable spectral factors and let $X_{W_i} = H_r(K_i)Q_i''$. Then, with $\mathcal{R}_i$, $\mathcal{O}_i$ defined as above, there exists a unique map $Z\colon H_r(K_\beta)Q''_\beta \to H_r(K_\alpha)Q''_\alpha$ which makes figure 8 commutative.*
thus

$$Z = \mathcal{R}_\beta^{-1}\mathcal{R}_\alpha = \mathcal{O}_\beta\mathcal{O}_\alpha^{-1} \tag{138}$$

(9) *We have*

$$\left\|fP_{H_r(K_\alpha)Q''_\alpha}\right\|^2 \le \left\|fP_{H_r(K_\beta)Q''_\beta}\right\|^2 \qquad \forall f \in H_r(K_+^e)$$

if and only if

$$\left\|gP_{H_r(K_\alpha)Q''_\alpha}\right\|^2 \ge \left\|gP_{H_r(K_\beta)Q''_\beta}\right\|^2 \qquad \forall g \in H_r(K_-^e)Q_+^e$$

These inequalities characterize the contractivity of the map Z.

Proof:

(1) From the commutativity of figure 4, we obtain the equality

$$\bar{Q}'KQ'' = \bar{Q}'\bar{Q}''K_+^e = K_-^e Q'Q'' = Q_-^e K_+^e R^e = K_-^e Q_+^e R^e$$

and the result follows.

(2) We use the equalities $L^2Q'' = L^2$, $H_+^2 = H_+^2K \oplus H_r(K)$ as well as the fact that multiplication by Q'' is a unitary map in L^2.

(3) We compute

$$\begin{aligned} S_W \cap \bar{S}_W &= H_+^2 Q'' \cap H_-^2 KQ'' \\ &= (H_+^2 \cap H_-^2 K)Q'' = H_r(K)Q'' = X_W \end{aligned}$$

(4) Since $S_W = H_+^2Q''$ and $L^2 = H_+^2Q'' \oplus H_-^2Q''$, we conclude that $S_W^\perp = H_-^2Q''$. Similarly, $\bar{S}_W = H_-^2KQ''$ implies $\bar{S}_W^\perp = H_+^2KQ''$. Thus (132) is equivalent to the second direct sum decomposition in (128).

(5) Since, using Theorem 4, the equivalence of K_+^e and K implies that the dimension of the state spaces $H_r(K_+^e)$ and $H_r(K)Q''$ is equal, it suffices to prove that the projection $P_{H_r(K)Q''}\big|H_r(K_+^e)$ is injective. Now

$$\begin{aligned}\operatorname{Ker} P_{H_r(K)Q''} &= H_+^2 KQ'' \oplus H_r(Q'') \\ &= H_+^2 \bar{Q}'' K_+^e \oplus H_r(Q'')\end{aligned}$$

Since $H_r(K_+^e)$ is orthogonal to $H_+^2\bar{Q}''K_+^e$, we can compute

$$\begin{aligned}\operatorname{Ker} H_r(K_+^e)\big|P_{H_r(K)Q''} &= H_r(K_+^e) \cap (H_+^2 KQ'' \oplus H_r(Q'')) \\ &= H_r(K_+^e) \cap H_r(Q'') \\ &= H_r(K_+^e \wedge_R Q'') = \{0\}\end{aligned}$$

We clearly have $K_+^e \wedge_R Q''$, the greatest common right inner divisor of K_+^e and Q'' equal to the identity because of the right coprimeness of these two inner functions. In this connection, see Theorem 3.

To prove the injectivity of (134), we show first that figure 9 commutes.

We note that, for $f \in H_+^2$, we have $fP_{H_r(K)Q''} = fQ''^* P_+ K^* P_- KQ''$. If $f \in H_r(K^e)Q'Q''$ then $f = gQ'Q''$ with $g \in H_r(K_-^e)$.

We compute

$$\begin{aligned}fP_{H_r(K)Q''}(Q'')^* K^*(\bar{Q}')^* &= gQ'Q''Q''^* P_+ K^* P_- KQ''(Q'')^* K^*(\bar{Q}')^* \\ &= gQ'K^*P_-(\bar{Q}')^*\end{aligned}$$

Going the other way, and noting that $P_{\bar{H}_r(K^*)(\bar{Q}')^*}$ restricted to H_-^2 is given by $h \mapsto h\bar{Q}'P_- KP_+ K^*(\bar{Q}')^*$, we compute

$$\begin{aligned}fQ''^* P_{\bar{H}_r(K^*)\bar{Q}'^*} &= gQ'Q''Q''^* Q'^* K_-^* \bar{Q}' P_- KP_+ K^*(\bar{Q}')^* \\ &= gK_-^* \bar{Q}' P_- KP_+ K^*(\bar{Q}')^* \\ &= gQ'K^* P_- KP_+ K^*(\bar{Q}')^* \\ &= gQ'K^* P_- (\bar{Q}')^*\end{aligned}$$

$$\begin{tikzcd}
H_r(K_-^e)Q'Q'' \arrow[r, "P_{H_r(K)Q''}"] \arrow[d, "Q''^*Q'^*K_-^*"'] & H_r(K)Q'' \arrow[d, "Q''^*K^*\overline{Q}'^*"] \\
\overline{H}_r((K_-^e)^*) \arrow[r, "P_{\overline{H}_r(K^*)\overline{Q}'^*}"] & \overline{H}_r(K^*)\overline{Q}'^*
\end{tikzcd}$$

Figure 9.

for $gQ'K^*P_- \in \bar{H}_r(K^*)$ and hence $gQ'K^*P_-KP_+K^* = gQ'K^*P_-P_{\bar{H}_r(K^*)} = gQ'K^*P_-$.

The bijectivity of $P_{H_r(K)Q''} : H_r(K_+^e) \to H_r(K)Q''$ is equivalent to the bijectivity of $P_{\bar{H}_r(K^*)(\bar{Q}')^*} : \bar{H}_r(K_-^*) \to \bar{H}_r(K^*)(\bar{Q}')^*$, and this we proceed to prove. As in the first part,

$$\begin{aligned}\operatorname{Ker} P_{\bar{H}_r(K^*)(\bar{Q}')^*}\big|\bar{H}_r(K_-^*) &= \bar{H}_r(K_-^*) \cap (\bar{H}_r((\bar{Q}')^*) \oplus H_-^2 K^*(\bar{Q}')^*)\\ &= \bar{H}_r(K_-^*) \cap \bar{H}_r((\bar{Q}')^*) = \{0\}\end{aligned}$$

The last equality follows from the left coprimeness of K_-^e and $\bar{Q}'$.

(6) Follows from the previous part by taking adjoints.

(7) This follows from the three direct sum decompositions in (127). For $F \in H_r(\bar{Q}'KQ'')$, let

$$F = f_1 + f_2 + f_3 = g_1 + g_2 + g_3 = h_1 + h_2 + h_3$$

be the representations with respect to these direct sum decompositions. Clearly, we have $f_1 = g_1$ and $g_3 = h_3$. Assume now that $F \in H_r(K_+^e)$, so we have $F = f_3 = g_2 + g_3 = g_2 + h_3$. This implies $g_2 = f_3 P_{H_r(K)Q''}$. Now $g_2 = H_1 + h_2$ and so $h_1 = g_2 P_{H_r(K_-^e)Q'Q''}$. But $f_3 = g_2 + h_3 = h_1 + h_2 + h_3$ and so we have also $h_1 = f_3 P_{H_r(K_-^e)Q'Q''}$.

(8) By part (5), the maps $\mathcal{R}_1$, $\mathcal{R}_2$ are invertible. Computing the adjoints of the maps $\mathcal{O}_1$, $\mathcal{O}_2$ we obtain $\mathcal{O}_i^* = P_{H_r(K_i)Q_i''}\big|H_r(K_-^e)Q_i'Q_i''$ and these are also invertible. Hence so are the $\mathcal{O}_i$. By part (7) we have $\mathcal{R}_1\mathcal{O}_1 = \mathcal{R}_2\mathcal{O}_2$, for both are factorizations of $P_{H_r(K_-^e)Q'Q''} : H_r(K_+^e) \to H_r(K_-^e)Q'Q''$. This implies the equality $Z = \mathcal{R}_2^{-1}\mathcal{R}_1 = \mathcal{O}_2\mathcal{O}_1^{-1}$. This part of the proof is adapted from Lindquist and Picci (1991).

(9) Assume $\|fP_{H_r(K_1)Q_1''}\|^2 \le \|fP_{H_r(K_2)Q_2''}\|^2$ for all $f \in H_r(K_+^e)$, i.e. $\|f\mathcal{R}_1\| \le \|f\mathcal{R}_2\|$. Setting $g = d\mathcal{R}_2$, we get $\|gZ\| = \|g\mathcal{R}_2^{-1}\mathcal{R}_1\| \le \|g\|$, i.e. Z is contractive. But then so is Z^*, which implies $\|fZ^*\| = \|f\mathcal{O}_1^{-*}\mathcal{O}_2^*\| \le \|f\|$, for all $f \in H_r(K_-^e)Q_+^e$. Setting $g = f\mathcal{O}_1^*$, we get $\|f\mathcal{O}_2^*\| \le \|f\mathcal{O}_1^*\|$, i.e. $\|fP_{H_r(K_2)Q_2''}\|^2 \le \|fP_{H_r(K_1)Q_1''}\|^2$ for all $f \in H_r(K_-^e)Q_+^e$. □

In (129) we have defined the extended scattering pairs; this is because in the literature, see Lindquist and Picci (1991), a different object is called *scattering pair*. We briefly introduce some basic facts about it.

First, we need a simple lemma. Let $x, y \in L_r^2$: we say that x and y are pointwise orthogonal if $x(i\omega)y^*(i\omega) = 0$ for all $\omega \in \mathbb{R}$. The pointwise orthogonal complement Y of a subspace X is the set of all vectors $y \in L_r^2$ which are pointwise orthogonal to all vectors $x \in X$. Let K be an inner function. We set $\pi_K^\perp$ to be the orthogonal projection of L^2 onto the pointwise orthogonal complement of $H_r(K)$ and we set $\pi_K := I - \pi_K^\perp$. We also set $L_K^2 := \overline{span}\{e^{i\omega s}H_r(K);\, s \in \mathbb{R}\}$.

Lemma 7: *We have the following relations:*

(1)

$$\operatorname{Im}\pi_K = L_K^2 \tag{139}$$

and

(2)

$$K\pi_K = \pi_K K \tag{140}$$

Proof:

(1) We show that

$$\operatorname{Im}\pi_K^{\perp} = \overline{span}\{e^{i\omega s}H_r(K); s \geq 0\}^{\perp}$$

If $f \in \operatorname{Im}\pi_K^{\perp}$, then f is pointwise orthogonal to any $g \in H_r(K)$; that is $g(i\omega)f(i\omega)^* \equiv 0$. But then also $e^{i\omega s}g(i\omega)f(i\omega)^* \equiv 0$ for all $s \in \mathbb{R}$, which implies that f is orthogonal to L_K^2. Conversely, if $f \in (L_K^2)^{\perp}$, it means that for each $g \in l_K^2$, it is $\int_{\mathbb{I}} g(i\omega)f(i\omega)^* d(\omega) = 0$. In particular, since L_K^2 is invariant, we also have $\int_{\mathbb{I}} e^{i\omega s}g(i\omega)f(i\omega)^* d(\omega) = 0$ which implies that $g(i\omega)f(i\omega)^* \equiv 0$.

(2) Let $V_r = \{v_1^r, \ldots, v_k^r\}$ be a row basis for $H_r(K)$. It is well known that there exist column vectors c_i^r in $\mathbb{C}^m$ such that $K = I + \sum_{i=1}^r c_i^r v_i^r$. Similarly if $V_c = \{v_1^c, \ldots, v_k^c\}$ is a basis for $H_c(K)$ then $K = I + \sum_{i=1}^r v_i^c c_i^c$ for suitable row vectors c_i^c. We now exploit the fact that for $f \in H_r(K)$, we have $f\pi_K = f$ and similarly, for $g \in H_c(K)$, we hve $\pi_K g = g$. Then

$$\begin{aligned} K(s)\pi_K &= \pi_K + \sum_{i=1}^r c_i^r v_i^r \pi_K + \sum_{i=1}^r c_i^r v_i^r \\ &= \pi_K + \sum_{i=1}^r v_i^c c_i^c = \pi_K + \pi_K \sum_{i=1}^r v_i^c c_i^c = \pi_K K(s) \end{aligned}$$

□

Proposition 19: *Let W be a minimal, stable spectral factor. Let $W = \bar{W}K$ be its DSS factorization over H_-^∞. Let the inner functions Q', Q'', $\bar{Q}'$, $\bar{Q}''$ be determined by Proposition* 13. *Then we have*

(1) *Defining the L_K^2 subspaces by*

$$\left.\begin{aligned} \hat{S}_W &= H_+^2 \pi_K Q'' \\ \overset{\triangle}{S}_W &= H_-^2 \pi_K K Q'' \end{aligned}\right\} \tag{141}$$

we have

$$X_W = H_r(K)Q'' = \hat{S}_W \cap \overset{\triangle}{S}_W \tag{142}$$

The pair $(\hat{S}_W, \overset{\triangle}{S}_W)$ *will be called a* scattering pair *for the spectral factor W.*

(2) *We denote by $\hat{S}_W^{\perp}$ the orthogonal complement of $\hat{S}_W$ in L_K^2. Similarly, we define $\overset{\triangle}{S}{}_W^{\perp}$ to be the orthogonal complement in L_K^2 of $\overset{\triangle}{S}_W$. Then we have*

$$\left.\begin{aligned} \hat{S}_W^{\perp} &= H_-^2 \pi_K Q'' \\ \overset{\triangle}{S}{}_W^{\perp} &= H_+^2 K \pi_K Q'' \end{aligned}\right\} \tag{143}$$

and

$$L^2_K = \overset{\triangle}{S}{}^{\perp}_W \oplus X_W \oplus \hat{S}^{\perp}_W \tag{144}$$

Proof:

(1) We compute, in view of Lemma 7

$$\hat{S}_W \cap \overset{\triangle}{S}_W = H^2_+ \pi_k Q'' \cap H^2_- \pi_k KQ'' = (H^2_+ \pi_k \cap H^2_- K\pi_k)Q''$$

$$= \left[(H_r(K) \oplus H^2 K\pi_k) \cap (H^2_- \pi_k \oplus H_r(K)\right] Q'' = H_r(K)Q'' = X_W$$

(2) Since $\hat{S}_W = H^2_+ \pi_k Q''$ and $L^2_K = H^2_+ \pi_k Q'' \oplus H^2_- \pi_k Q''$, again in view of Lemma 7 we conclude that $\hat{S}^{\perp}_W = H^2_- \pi_k Q''$. Similarly, $\overset{\triangle}{S}_W = H^2_- \pi_k KQ''$ implies $\overset{\triangle}{S}_W = H^2_+ \pi_k KQ''$. Thus

$$L^2_K Q'' = \hat{S}_W \oplus \hat{S}^{\perp}_W = H^2_+ \pi_k Q'' \oplus \hat{S}^{\perp}_W = H^2_+ \pi_k KQ'' \oplus H_r(K)\pi_k Q'' \oplus \hat{S}^{\perp}_W$$

$$= H^2_+ \pi_k KQ'' \oplus H_r(K)Q'' \oplus \hat{S}^{\perp}_W = \overset{\triangle}{S}{}^{\perp}_W \oplus H_r(K)Q'' \oplus \hat{S}^{\perp}_W \qquad \square$$

The above proposition illustrates quite clearly the difference of our approach with respect to the one of Lindquist and Picci. We simply take 'larger' spaces to represent our scattering pair: since the main use of scattering pairs is in the state space they generate by their intersection, the results we are obtaining are exactly the same. But, as can be seen comparing the equivalent results in Propositions 18 and 19 the proofs are simpler. So, in the following, a *scattering pair* will implicitly assume to be *extended*. For the scattering theory origins of scattering pairs, see Lax and Phillips (1967).

We proceed with a study of the zero structure of the set $\mathcal{W}^m$ of minimal, stable spectral factors. This is done mainly through a geometric study of the associated set $\mathcal{X}^m$ of minimal Markovian splitting subspaces. Since in our approach the splitting subspaces have a unified representation, given by (126), the study reduces to a great extent to the arithmetic of inner functions.

We are interested mainly in the parametrization of the set $\mathcal{W}^m$, for $m_0 \leq m \leq n + p$. Moreover, the set $\mathcal{W}^m$ or the corresponding set $\mathcal{X}^m$ carry with them a natural partial order. In the internal case, i.e. $m = m_0$, this partial order is closely related to the factorization of certain inner functions and hence also, see Fuhrmann (1995), to the set of solutions of a certain homogeneous algebraic Riccati equation.

We feel it is instructive to begin our study of the set $\mathcal{W}^m$ in the special case where the spectral function is of full rank and coercive and the factors are internal, i.e. $m = m_0 = p$. We denote this set by $\mathcal{W}^p$. This case shows the general scheme, and also clarifies the new concepts that have to be introduced for the analysis of the general case.

Without loss of generality, we can assume the normalization $\Phi(\infty) = I$. Thus all regular spectral factors can be assumed to satisfy $W(\infty) = I$ as well. This set has been fully studied, e.g. Fuhrmann (1995). The inner functions Q_-, Q_+, K_-, K_+ are defined as before. Heuristically, we shall say, given spectral factors W_1 and W_2, that $W_1 \leq W_2$ if W_2 is closer to being maximum phase. This can be expressed precisely by saying $W_2 = W_1 Q$ for some inner function Q, see Definition 6. So, if $W_1 Q''_i = W_+$, this can be expressed also by saying that Q''_2 is a right factor of Q''_1. Equivalently, Q''_1 is a left factor of Q''_2. This leads to other, geometric, characterizations that turn out to be useful in the generalization to the non-regular case. Moreover, given an inner function Q_+, the set of left inner factors of Q_+ is clearly partially ordered with $Q_1 \leq Q_2$ if Q_1 is a left factor of Q_2. In this set Q_+ is clearly the maximal element and I is the minimal. This order has been expressed in Fuhrmann (1995) via the set of non-negative definite solutions of a homogeneous Riccati equation. It is not surprising that all these orders are equivalent. We express this equivalence in the following.

The set $\mathcal{W}^p$ can be parametrized in seveal different ways. In fact, by the results of §3, given a normalized, minimal, stable spectral factor, there exists a normalized factorization (i.e. both factors normalized and inner) $Q_+ = Q'Q''$ such that

$$\left.\begin{aligned} W_- Q' &= W \\ WQ'' &= W_+ \end{aligned}\right\} \tag{145}$$

So normalized factorizations of Q_+ are in a bijective correspondence with the elements of the set $\mathcal{W}^p$, i.e. they parametrize $\mathcal{X}^p$. Clearly, if $Q_i'Q_i''$ are the factorizations of Q_+ associated with W_i, then $W_1 \leq W_2$ if and only if Q_1' is a left factor of Q_2'. On the other hand, see Fuhrmann (1995) for the details, if

$$Q_+ = \left(\begin{array}{c|c} A & B \\ \hline -B^*X_+ & I \end{array}\right)$$

with the realization minimal and X_+ the unique positive definite solution of the homogeneous Riccati equation

$$A^*X + XA + XBB^*X = 0 \tag{146}$$

then an arbitrary normalized right inner factor of Q_+ is given by

$$Q = \left(\begin{array}{c|c} A & B \\ \hline -B^*X & I \end{array}\right)$$

where X is a non-negative definite solution of the Riccati equation (146). We denote by $\mathcal{P}$ the set of non-negative definite solutions of this Riccati equation. $\mathcal{P}$ has a natural partial order induced by the standard order on Hermitian matrices.

We want to remark that, for a minimal spectral factor W that satisfies (145), the factorization $Q_+ = Q'Q''$ represents a parametrization of the set of stable and antistable zeros of W. In fact Q'' determines the set of stable zeros of W that can still be moved to the right half plane, via multiplication by Q'', whereas Q' determines the set of antistable zeros of W, that is zeros of W_- that have been moved already to the right half plane, via multiplication by Q'. This issue will be addressed formally in a subsequent paper. For a full parametrization of the set of all minimal spectral factors, see Fuhrmann (1955).

There is a third order that can be considered and it is a geometric one. It is a variation on an order introduced by Lindquist and Picci (1991). For each $W \in \mathcal{W}^p$ there is a unique right coprime DSS factorization $W = \bar{W}K$ with K normalized inner and $\bar{W} \in H_-^\infty$. If $WQ'' = W_+$ we associate with the spectral factor W the state space $X_W = H_r(K)Q''$. We denote by $\mathcal{X}^p$ the set of these spaces. We say $X_{W_1} \leq X_{W_2}$ if

$$\|fP_{X_{W_1}}\| \leq \|fP_{X_{W_2}}\|, \qquad \forall f \in X_{W_+} \tag{147}$$

Of course we have $X_{W_+} = H_r(K_+)$.

In the next theorem we show that all these partial orders are equivalent.

Theorem 8: *Let Φ be a full rank, normalized, coercive spectral density. Let $\mathcal{W}^p$ be the set of all minimal, stable internal spectral factors of Φ, let $\mathcal{X}^p$ be the set of all state spaces corresponding to these factors. Let the inner function Q_+ be defined as before and assume it has a minimal realization*

$$Q_+ = \left(\begin{array}{c|c} A & B \\ \hline -B^*X_+ & I \end{array}\right)$$

with X_+ the positive definite solution of the Riccati equation

$$A^*X + XA + XBB^*X = 0$$

Let $\mathcal{P}$ be the set of all non-negative definite solutions of this Riccati equation. Let W_α, $W_\beta \in \mathcal{W}^p$.

Then the following conditions are equivlent.

(1)

$$W_1(s)\,W_1(s)^* \leq W_2(s)\,W_2(s)^*, \qquad \mathrm{Re}\, s > 0 \tag{148}$$

(2) *For some inner function Q, we have $W_2 = W_1 Q$.*

(3) *For some inner function Q, we have $QQ_2 = Q_1$.*

(4)

$$Q_1(s)^* Q_1(s) \leq Q_2(s)^* Q_2(s), \qquad \mathrm{Re}\, s > 0 \tag{149}$$

(5) *With $H_r(K_1)Q_1$, $H_r(K_2)Q_2$ the state spaces associated with W_1, W_2 respectively, we have*

$$\|fP_{H_r(K_1)Q_1}\| \leq \|fP_{H_r(K_2)Q_2}\|, \qquad \forall f \in H_r(K_+) \tag{150}$$

(6) *We have*

$$\|gP_{H_r(Q_1)}\| \geq \|gP_{H_r(Q_2)}\|, \qquad \forall g \in H_r(Q_+) \tag{151}$$

(7) *Q_1, Q_2 have unique representations of the form*

$$Q_i = \left(\begin{array}{c|c} A & B \\ \hline -B^*X_i & I \end{array}\right)$$

with X_i non-negative definite solutions of the Riccati equation satisfying $X_1 \geq X_2$.

Proof: We prove the following implications:

(1) $\Leftrightarrow$(2)

Assume statement (1) holds. This implies $\|\xi W_1(s)\| \geq \|\xi W_2(s)\|$, in $\mathrm{Re}\, s > 0$, and hence $Q(s) = W_1(s)^{-1}W_2(s)$ has at most a finite number of removable singularities in the right half plane. Thus $Q(s)$ is analytic and contractive in the right half plane. Moreover, as the W_i are spectral factors, we have $W_1(s)\,W_1(s)^* = W_2(s)\,W_2(s)^*$ on the boundary, i.e. $Q(s)$ is unitary on the imaginary axis, which means it is inner. So (2) holds.

Conversely, $W_2 = W_1 Q$ implies that, for every s in the open right half plane,

$$W_2(s)W_2(s)^* = W_1(s)Q(s)Q(s)^* W_1(s)^* \le W_1(s)W_1(s)^*$$

as $Q(s)$, being inner, is contractive in the open right half plane.

(3) $\Leftrightarrow$(5)
Assume $QQ_2^r = Q_1^r$ holds. We clearly have $H_r(Q_1^r) = H_r(QQ_2^r) = H_r(Q)Q_2^r \oplus H_r(Q_2^r)$ and this implies that for every $f \in H(K_+)$

$$\|fP_{H_r(Q_1^r)}\| \ge \|fP_{H_r(Q_2^r)}\| \tag{152}$$

Note that we have $K_i Q_i^r = \bar{Q}_i^r K_+$ and hence

$$H_r(K_i)Q_i^r \subset H_r(K_i)Q_i^r \oplus H_r(Q_i^r) = H_r(K_iQ_i^r) = H_r(\bar{Q}_i^r K_+) = H_r(\bar{Q}_i^r)K_+ \oplus H_r(K_+)$$

This implies the equality

$$\|f\|^2 = \|fP_{H_r(K_i)Q_i^r}\|^2 + \|fP_{H_r(Q_i^r)}\|^2$$

for all $f \in H_r(K_+)$. As a consequence of (152), we conclude that statement (2) implies $\|fP_{H_r(K_1)Q_1^r}\| \le \|fP_{H_r(K_2)Q_2^r}\|$.

(2) $\Leftrightarrow$(3)
We have $W_1 Q_1^r = W_+ = W_2 Q_2^r$. Since $W_2 = W_1 Q_1$, it follows that $W_1 Q_1^r = W_1 QQ_2^r$, and, by the invertibility of W_1, that $Q_1^r = QQ_2^r$.

Conversely, assume $QQ_2^r = Q_1^r$. We compute

$$W_2 Q_2^r = W_+ = W_1 Q_1^r = W_1 QQ_2^r$$

which clearly implies the equality $W_2 = W_1 Q$.

(3) $\Leftrightarrow$(4)
Inequality (149) follows from $QQ_2^r = Q_1^r$ by the contractivity of Q in the open right half plane.

Conversely, inequality (149) shows that, with the exception of a finite number of points, the function $Q = Q_1^r(Q_2^r)^{-1}$ is analytic and contractive in the open right half plane. Thus the singularities are removable. As Q has unitary boundary values, it is necessarily inner.

(2) $\Leftrightarrow$(6)
The factorizations $Q_+ = Q_1^l Q_1^r = Q_2^l Q_2^r = Q_1^l QQ_2^r$ show that $Q_1^r = QQ_2^r$. We have therefore the inclusions $H_r(Q_2^r) \subset H_r(Q_1^r) \subset H_r(Q_+)$. This implies

$$\|gP_{H_r(Q_1^r)}\| \ge \|gP_{H_r(Q_2^r)}\|, \qquad \forall g \in H_r(Q_+) \tag{153}$$

To prove the converse, note that $H_r(Q_i^r) \subset \{H_+^2 Q_+\}^\perp$. Here it follows that the inequality $\|gP_{H_r(Q_1^r)}\| \ge \|gP_{H_r(Q_2^r)}\|$ holds for all $g \in H_+^2$. We conclude that $H_r(Q_2^r) \subset H_r(Q_1^r)$ and therefore $Q_1^r = QQ_2^r$ for some inner function Q.

(3) $\Leftrightarrow$(7)
This has been proved in Fuhrmann (1995).

(5) $\Leftrightarrow$(6)
Assume $\|fP_{H_r(K_1)Q_1^r}\| \le \|fP_{H_r(K_2)Q_2^r}\|$ for all $f \in H_r(K_+)$. By a previous computa-

tion, this is equivalent to $\|fP_{H_r(Q_1'')}\| \geq \|fP_{H_r(Q_2'')}\|$ for $f \in H_r(K_+)$. Now, if $M \subset N$ are subspaces of a Hilbert space and P_M, P_N the respective orthogonal projections, then we have $P_M = P_M P_N$. We apply this in the following way. We clearly have $Q_i' Q_i'' = Q_+$ and therefore $H_r(Q_i'') \subset H_r(Q_+)$. So $P_{H_r(Q_i'')} = P_{H_r(Q_i'')} P_{H_r(Q_+)}$. Now, we claim that

$$H_r(K_+)P_{H_r(Q_+)} = H_r(Q_+) \tag{154}$$

To see this we consider the inverse of the phase function $T_0^{-1} = W_-^{-1}\bar{W}_+ = Q_+K_+^*$. The first factorization is a left Wiener–Hopf factorization with trivial factorization indices. Hence the Toeplitz operator $T^r_{W_-^{-1}\bar{W}_+} = T^r_{Q_+K_+^*}$ is invertible. Applying Theorem 1, we conclude that (154) holds. Now (152) implies

$$\|fP_{H_r(Q_+)}P_{H_r(Q_1'')}\| \geq \|fP_{H_r(Q_+)}P_{H_r(Q_2'')}\|$$

and hence that

$$\|gP_{H_r(Q_1'')}\| \geq \|gP_{H_r(Q_2'')}\|$$

for all $g \in H_r(Q_+)$.

(6) $\Longleftrightarrow$ (3)
From (6), using the factorizations $Q_+ = Q_i'Q_i''$, we conclude that

$$\begin{aligned} H_r(Q_1')Q_1'' &= H_r(Q_+) \cap H_+^2 Q_1'' = \operatorname{Ker} H_r(Q_+)\big|P_{H_r(Q_1'')} \\ &\subset \operatorname{Ker} P_{H_r(Q_1'')}\big|H_r(Q_+) = H_r(Q_+) \cap H_+^2 Q_2'' \\ &= H_r(Q_2')Q_2'' \end{aligned}$$

This implies that $Q_1'' = QQ_2''$ for some inner function Q, and hence also $Q_2' = Q_1'Q$. So $W_2 = W_- Q_2' = W_- Q_1'Q = W_1Q$, i.e. $W_1 \leq W_2$. □

We pass now to the study of the set $\mathcal{W}^m$ of minimal, stable, $p \times m$ spectral factors in the general case ($m_0 \leq m \leq m_0 + p$), which is the central theme of this paper. We denote by $\mathcal{W}^{m_0}$ the set of all minimal, stable $p \times m_0$ internal spectral factors of Φ. We also denote by $\bar{\mathcal{W}}^m$ the set of all minimal, antistable $p \times m$ spectral of Φ. Again $\mathcal{W}^m$ can be partially ordered. Unhappily, defining $W_1 \leq W_2$ if $W_2 = W_1Q$ for some inner function Q is too coarse an order, and it can be refined. Thus we proceed differently, defining a partial order in the sets $\mathcal{W}^m$, $\bar{\mathcal{W}}^m$, $\mathcal{X}^m$ and $\bar{\mathcal{X}}^m$.

Definition 7: Let W_1, W_2 be minimal, stable, $p \times m$ spectral factors and let $X_i = H_r(K_i)Q_i'' \in \mathcal{X}^m$, $i = 1, 2$ be the respective minimal Markovian splitting subspaces. Let $W_i = \bar{W}_iK_i$ be the right coprime DSS factorizations over H_-^∞. Let $\bar{Q}_i$ be defined via figure 4.

(1) We say that $W_\alpha \leq W_\beta$ if

$$W_\alpha(s)\, W_\alpha(s)^* \geq W_\beta(s)\, W_\beta(s)^*, \qquad \operatorname{Re} s > 0 \tag{155}$$

(2) If $\bar{W}_\alpha$, $\bar{W}_\beta \in \bar{\mathcal{W}}^m$, we say that $\bar{W}_\alpha \leq \bar{W}_\beta$ if

$$\bar{W}_\alpha(s)\, \bar{W}_\alpha(s)^* \geq \bar{W}_\beta(s)\, \bar{W}_\beta(s)^*, \qquad \operatorname{Re} s > 0 \tag{156}$$

(3) We say that $X_1 \leq X_2$ if

$$\|fP_{X_1}\| \le \|fP_{X_2}\|, \qquad \forall f \in H_r(K_+^e) \tag{157}$$

Here P_X denotes the orthogonal projection of H_+^2 onto X.

(4) Given two minimal Markovian splitting subspaces

$$\bar{X}_i = \bar{H}_r(K_i^*)(K_i^*)(\bar{Q}_i')^* \in \bar{\mathcal{X}}^m,$$

we say that $\bar{X}_1 \le \bar{X}_2$ if

$$\|hP_{\bar{X}_1}\| \ge \|hP_{\bar{X}_2}\|, \qquad \forall h \in \bar{H}_r\left[(K_-^e)^*\right] \tag{158}$$

Here $P_{\bar{X}}$ denotes the orthogonal projection of H_-^2 onto $\bar{X}$.

The definition of the partial order in the set $\mathcal{W}$ is customary, see Anderson (1973). We point out the inversion of the direction of the inequality, compared with (157). Also, we note that, since multiplication by inner functions is a unitary map in L^2 and therefore preserves orthogonality, (158) is equivalent to

$$\|fP_{H_r(K_1)Q_1''}\| \ge \|fP_{H_r(K_2)Q_2''}\|, \qquad \forall f \in H_r(K_-^e)Q'Q'' \tag{159}$$

We recall that, assuming W_- is $p \times m_0$ and of full column rank, the inner functions Q_-, Q_+, K_-, K_+ are all $m_0 \times m_0$, whereas the spectral factors we study are all of size $p \times m$. We also recall that we have defined π_1, see Definition 3, to be the projection on the first m_0 components. For following the statement and proof of the next theorem, it is advisable to refer to figure 4.

Theorem 9: *Let Φ be a rank m_0, spectral function, having no zeros on the extended imaginary axis. Let $\mathcal{W}^m$ be the set of all minimal, stable spectral factors of Φ, let $\mathcal{X}^m$ be the set of all minimal Markovian splitting subspaces corresponding to these factors. Let W_α, $W_\beta \in \mathcal{W}^m$. Then the following statements are equivalent.*

(1)

$$W_\alpha \le W_\beta \tag{160}$$

(2)

$$\bar{W}_\alpha \le \bar{W}_\beta \tag{161}$$

(3)

$$\pi_1 Q_\alpha''(s)^{-1} Q_\alpha''(s)^{-*}\pi_1 \ge \pi_1 Q_\beta''(s)^{-1} Q_\beta''(s)^{-*}\pi_1, \qquad \operatorname{Re} s > 0 \tag{162}$$

(4)

$$\pi_1 Q_\alpha''(s)^* Q_\alpha''(s)\pi_1 \le \pi_1 Q_\beta''(s)^* Q_\beta''(s)\pi_1, \qquad \operatorname{Re} s > 0 \tag{163}$$

(5) *We have*

$$\|P_{H_c((Q_\alpha'')^*)}g\| \ge \|P_{H_c((Q_\beta'')^*)}g\|, \qquad \forall g \in \bar{H}_c\left[(Q_+^e)^*\right] \tag{164}$$

(6) *We have*

$$\|gP_{H_r(Q_\alpha'')}\| \ge \|gP_{H_r(Q_\beta'')}\|, \qquad \forall g \in H_r(Q_+^e) \tag{165}$$

(7) *With the maps T_i defined by*

$$T_i = H_r(Q_+^e)\big|P_{H_r(Q_i'')}, \qquad i = \alpha, \beta \tag{166}$$

there exists a unique, contractive map Y for which $T_\alpha Y = T_\beta$.

(8) *We have*

$$\left\| f P_{H_r(K_\alpha)Q_\alpha''} \right\| \leq \left\| f P_{H_r(K_\beta)Q_\beta''} \right\|, \qquad \forall f \in H_r(K_+^e) \tag{167}$$

(9) *Let the maps* R_i: $H_r(K_+^e) \to H_r(K_i)Q_i''$ *be defined by* (133) *for* $i = \alpha, \beta$. *The map*

$$Z: H_r(K_\beta)Q_\beta'' \to H_r(K_\alpha)Q_\alpha''$$

defined by $Z = R_\beta^{-1} R_\alpha$ *is contractive.*

Proof:

(1) $\Leftrightarrow$(2)

Since $W_i = \bar{W}_i K_i$, we have

$$W_i W_i^* = \bar{W}_i K_i K_i^* \bar{W}_i^* = \bar{W}_i \bar{W}_i^*$$

which clearly implies the equivalence.

(1) $\Leftrightarrow$(3)

We note that, on the imaginary axis, we have $W_\alpha = W_+^e (Q_\alpha'')^*$, which clearly has a meromorphic extension to the right half plane, given by $W_\alpha(s) = W_+^e(s) Q_\alpha''(s)^{-1}$. Assume $W_\alpha \leq W_\beta$, i.e. inequality (155) holds. Let $W_+^{-L}(s)$ be a left inverse of $W_+(s)$. We clearly have the following implications.

$$W_\alpha(s) W_\alpha(s)^* \geq W_\beta(s) W_\beta(s)^*$$

$$\Rightarrow$$

$$W_+^{-L}(s) W_\alpha(s) W_\alpha(s)^* W_+^{-L}(s)^* \geq W_+^{-L}(s) W_\beta(s) W_\beta(s)^* W_+^{-L}(s)^*$$

$$\Leftrightarrow$$

$$W_+^{-L}(s) \begin{pmatrix} W_+(s) & 0 \end{pmatrix} Q_\alpha''(s)^{-1} Q_\alpha''(s)^{-*} \begin{pmatrix} W_+(s)^* \\ 0 \end{pmatrix} W_+^{-L}(s)^*$$

$$\geq W_+^{-L}(s) \begin{pmatrix} W_+(s) & 0 \end{pmatrix} Q_\beta''(s)^{-1} Q_\beta''(s)^{-*} \begin{pmatrix} W_+(s)^* \\ 0 \end{pmatrix} W_+^{-L}(s)^*$$

$$\Leftrightarrow$$

$$\begin{pmatrix} I & 0 \end{pmatrix} Q_\alpha''(s)^{-1} Q_\alpha''(s)^{-*} \begin{pmatrix} I \\ 0 \end{pmatrix} \geq \begin{pmatrix} I & 0 \end{pmatrix} Q_\beta''(s)^{-1} Q_\beta''(s)^{-*} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$\Leftrightarrow$$

$$\pi_1 Q_\alpha''(s)^{-1} Q_\alpha''(s)^{-*} \pi_1 \geq \pi_1 Q_\beta''(s)^{-1} Q_\beta''(s)^{-*} \pi_1$$

Our only remaining task is to invert the first implication. To this end we show that $W_+(s) W_+(s)^{-L} W_\alpha(s) = W_\alpha(s)$. Indeed, by Lemma 6, we have $W_+(s) W_+(s)^{-L} W_-(s) = W_-(s)$. Therefore

$$
\begin{aligned}
W_+(s)W_+(s)^{-L}W_\alpha(s) &= W_+(s)W_+(s)^{-L}W_-^e(s)Q'_\alpha(s) \\
&= W_-^e(s)Q'_\alpha(s) = W_\alpha(s)
\end{aligned}
$$

Extending $(I\ 0)$ in a natural way to a projection proves the reverse implication.

(6) $\Leftrightarrow$(4)

By our assumption, we have

$$\|gP_{H_r(Q''_\alpha)}\| \geq \|gP_{H_r(Q''_\beta)}\|, \qquad \forall g \in H_r(Q_+) \tag{168}$$

We show now this inequality holds for all $g \in H_+^2\pi_1$. Clearly, for all $f \in H_+^2$, we have $f = f\pi_1 + f\pi_2$. In particular, since by equation (89) $H_r(Q'')\pi_1 \subset H_r(Q_+^e)$, we have

$$H_r(Q'') = H_r(Q'')\pi_1 + H_r(Q'')\pi_2 \subset H_r(Q_+^e) + H_+^2\pi_2$$

So $gP_{H_r(Q'')} = gP_{H_r(Q'')}(P_{H_r(Q_+^e)} + P_{H_+^2\pi_2})$. Assume $g \in H_+^2\pi_1$, i.e. $g = f\pi_1$, then it follows that, for all $f \in H_+^2$,

$$
\begin{aligned}
f\pi_1 P_{H_r(Q'')} &= f\pi_1(P_{H_r(Q_+^e)} + P_{H_+^2\pi_2})P_{H_r(Q'')} \\
&= f\pi_1 P_{H_r(Q_+^e)}P_{H_r(Q'')}
\end{aligned}
$$

This implies

$$\|f\pi_1 P_{H_r(Q''_\alpha)}\| = \|f\pi_1 P_{H_r(Q_+^e)}P_{H_r(Q''_\alpha)}\| \geq \|f\pi_1 P_{H_r(Q_+^e)}P_{H_r(Q''_\beta)}\| = \|f\pi_1 P_{H_r(Q''_\beta)}\|$$

Thus we have obtained

$$\|f\pi_1 P_{H_r(Q''_\alpha)}\| \geq \|f\pi_1 P_{H_r(Q''_\beta)}\| \qquad \forall f \in H_+^2 \tag{169}$$

We take now special choices for f, namely

$$f = \frac{\xi}{s+\bar{\omega}}, \qquad \xi \in \operatorname{Im}\pi_1, \quad \operatorname{Re}\omega > 0$$

Note that for any H_+^2 function h, we have $\langle h, \xi/(s+\bar{\omega})\rangle_{H_+^2} = h(\omega)\xi^*$. It is easy to check that

$$\frac{\xi}{s+\bar{\omega}} = \frac{\xi - \xi Q(\omega)^* Q(s)}{s+\bar{\omega}} + \frac{\xi Q(\omega)^* Q(s)}{s+\bar{\omega}}$$

is an orthogonal decomposition relative to $H_+^2 = H_r(Q) \oplus H_+^2 Q$. In fact,

$$\mathcal{K}(s,\omega) := \frac{I - Q(\omega)^* Q(s)}{s+\bar{\omega}}$$

is the reproducing kernel for $H_r(Q)$. Inequality (169) translates therefore into

$$\frac{\|\xi\|^2 - \xi Q''_\alpha(\omega)^* Q''_\alpha(\omega)\xi^*}{\omega+\bar{\omega}} \geq \frac{\|\xi\|^2 - \xi Q''_\beta(\omega)^* Q''_\beta(\omega)\xi^*}{\omega+\bar{\omega}},$$

or, since $\omega + \bar{\omega} > 0$

$$\xi Q''_\beta(\omega)^* Q''_\beta(\omega)\xi^* \geq \xi Q''_\alpha(\omega)^* Q''_\alpha(\omega)\xi^*$$

This is equivalent to

$$\pi_1 Q''_\beta(\omega)^* Q''_\beta(\omega)\pi_1 \geq \pi_1 Q''_\alpha(\omega)^* Q''_\alpha(\omega)\pi_1$$

Conversely, assume (163) holds. Computing backwards, it is easy to see that this entails

$$\Pi_{\mathbb{I}}\mathcal{K}_\alpha(\omega,\omega)\Pi_{\mathbb{I}} \geq \Pi_{\mathbb{I}}\mathcal{K}_\beta(\omega,\omega)\Pi_{\mathbb{I}} \tag{170}$$

where

$$\mathcal{K}_i(s,\omega) := \frac{I - Q''_i(\omega)^* Q''_i(s)}{s+\bar{\omega}}, \quad \text{for } i = \alpha, \beta$$

We assume first that

$$H_r(Q''_\alpha)\Pi_{\mathbb{I}} = H_r(Q''_\beta)\Pi_{\mathbb{I}} \tag{171}$$

and denote the common dimension of these spaces by r. Then we can chose in these spaces basis matrices F_α and F_β, which we consider as a column of r rows of H^2_+, such that $F_\alpha\Pi_{\mathbb{I}} = F_\beta\Pi_{\mathbb{I}} = F_+$. This can be achieved by taking any basis F_α in $H_r(Q''_\alpha)$ and constructing F_β as $(F_\alpha\Pi_{\mathbb{I}})(H_r(Q''_\beta)|\Pi_{\mathbb{I}})^{-1}$, noting that, by Proposition 15, the map $H_r(Q''_\beta)|\Pi_{\mathbb{I}}$ is injective. Then it is well known, see Dym (1989), that

$$\mathcal{K}_i(s,\omega) = F_i(\omega)^* P_i^{-1} F_i(s), \qquad i = \alpha, \beta$$

where P_i is the Gram matrix of the basis F_i defined, for $i = \alpha$, as $(P_\alpha)_{j\cdot k} = \langle F_{j,\alpha}, F_{k,\alpha}\rangle$ where j, $k = 1, \ldots, r$ and $F_{j,\alpha}$ is the jth row of the matrix F_α. Therefore the relation (170) can be written as

$$\Pi_{\mathbb{I}} F_\alpha(\omega)^* P_\alpha^{-1} F_\alpha(\omega)\Pi_{\mathbb{I}} \geq \Pi_{\mathbb{I}} F_\beta(\omega)^* P_\beta^{-1} F_\beta(\omega)\Pi_{\mathbb{I}}$$

which, in view of our choice of the basis, becomes

$$F_+(\omega)^* P_\alpha^{-1} F_+(\omega) \geq F_+(\omega)^* P_\beta^{-1} F_+(\omega) \tag{172}$$

We claim that (172) entails

$$P_\alpha^{-1} \geq P_\beta^{-1} \tag{173}$$

To see this, we need to show that

$$F(\omega)^* P F(\omega) \geq 0 \qquad \omega \in \mathbb{I} \tag{174}$$

implies $P \geq 0$. Assume $\hat{u}(\omega)$ is a row vector in H^2_+ of suitable dimension. Then denoting by $\mathcal{F}$ the Fourier transform, it is well known that

$$\mathcal{F}^{-1}\left[F(\omega)\hat{u}(\omega)\right](t) = \int_{-\infty}^{t} e^{A(t-s)} B u(s)\, ds = x(t)$$

where u is the inverse transform of $\hat{u}$ and (A, B) denotes a controllable pair such that $F(\omega) = (\omega I - A)^{-1} B$. Similarly,

$$\mathcal{F}^{-1}\left[\hat{u}(\omega)^* F(\omega)^* P G_+(\omega)\hat{u}(\omega)\right](t) = \int_{-\infty} x(s-t)^* P x(s)\, ds$$

which evaluated at zero yields, taking into consideration inequality (174),

$$0 \leq \int_{-\infty}^{0} x(s)^* P x(s)\, ds \tag{175}$$

Suppose now $\xi^* P \xi < 0$ for some $\xi \in \mathbb{C}^m$. Then, since $\xi^* e^{A^* s} P e^{As}\xi$ is continuous in s, there exists a $\tau < 0$ such that the integral $\int_\tau^0 \xi^* e^{A^* s} P e^{As}\xi\, ds$ is strictly negative. On the other hand, in view of controllability, we can find a control u_0 s.t.

$x(\tau) = \int_{-\infty}^{\tau} e^{A(\tau - s)} Bu(s)\, ds = e^{-A\tau}\xi$. Moreover, we can choose this control so that $\int_{-\infty}^{\tau} x^*(s)Px(s)\, ds$ is arbitrarily small, in particular smaller than $-\int_{\tau}^{0} \xi^* e^{A^* s} P e^{As}\xi ds$. Furthermore, we set $u_0(s) = 0$ for $\tau \le s \le 0$. Therefore we have constructed a control such that $\int_{-\infty}^{0} x^*(s)Px(s)\, ds < 0$. As this contradicts (175), we can conclude that $P \ge 0$.

If we go now back to (172), we immediately see that the above argument yields (173). To conclude, let now $g \in H_+^2 \pi_+$. Then, by a well-known property of reproducing kernels, see Dym (1989),

$$gP_{H_r(Q_i)} = \langle g(\omega), \mathcal{K}(\cdot, \omega)\rangle = \langle g, F_i\rangle P_i^{-1} F_i$$

and

$$\|gP_{H_r(Q_i)}\|^2 = \langle g, F_i\rangle P_I^{-1}\langle F_i, g\rangle$$

Since $\langle g, F_\alpha\rangle = \langle g, F_\beta\rangle$ we see that $P_\alpha^{-1} \ge P_\beta^{-1}$ implies $\|gP_{H_r(Q_\alpha)}\|^2 \ge \|gP_{H_r(Q_\beta)}\|^2$. This shows the implication under the restrictive assumption (171).

If the assumption (171) is not satisfied, then, necessarily, we must have $H_r(Q_\alpha)\pi_+ \supset H_r(Q_\beta)\pi_+$. Indeed, inequality (162) implies that

$$Q_\beta(-\bar{s}_j)\pi_+ \xi^* = 0 \implies Q_\alpha(-\bar{s}_j)\pi_+ \xi^* = 0 \tag{176}$$

and, more generally, if the superscript indicates the kth derivative,

$$Q_\beta^{(k)}(-\bar{s}_j)\pi_+ \xi^* = 0 \implies Q_\alpha^{(k)}(-\bar{s}_j)\pi_+ \xi^* = 0 \tag{177}$$

Therefore, we conclude that if

$$\frac{\xi}{(s - s_j)^{k+1}}\pi_+ \in H_r(Q_\beta)\pi_+$$

then

$$\frac{\xi}{(s - s_j)^{k+1}}\pi_+ \in H_r(Q_\alpha)\pi_+$$

which is our claim. Thus we can decompose $H_r(Q_\alpha)\pi_+$ as

$$H_r(Q_\alpha)\pi_+ = H_r(Q_\beta)\pi_+ \oplus \left[H_r(Q_\alpha)\pi_+ \ominus H_r(Q_\beta)\pi_+\right]$$

This means that if $\hat{F}_\beta$ is a basis in $H_r(Q_\beta)\pi_+$, we can find a basis $\hat{F}_\alpha$ of $H_r(Q_\alpha)\pi_+$ of the form

$$\hat{F}_\alpha = \begin{pmatrix} \hat{F}_\beta \\ x \end{pmatrix}$$

Let now F_i be the inverse image in $H_r(Q_i)$ under π_+ of $\hat{F}_i$, for $i = \alpha, \beta$. Construct the reproducing kernels $\mathcal{K}_i(s, \omega) - F_i(\omega)^* P_i^{-1} F_i(s)$. Then

$$\pi_+ \mathcal{K}_\alpha \pi_+ = \hat{F}_\alpha(\omega)^* P_\alpha^{-1} \hat{F}_\alpha(s)$$

and

$$\pi_+ \mathcal{K}_\beta \pi_+ = \hat{F}_\beta(\omega)^* P_\beta^{-1} \hat{F}_\beta(s) = \hat{F}_\alpha(\omega)^* \begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & 0 \end{pmatrix} \hat{F}_\alpha(s)$$

so that, again in view of the ordering on the matrices following from (172) we obtain

$$P_\alpha^{-1} \geq \begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

The conclusion of the backward implication then follows as above and the equivalence is thus proved.

(3) $\Leftrightarrow$(5)
Assume first that

$$\pi_+ \bar{H}_c((Q_\alpha^{\prime\prime})^*) = \pi_+ \bar{H}_c\left[(Q_\beta^{\prime\prime})^*\right] \tag{178}$$

This is a coinvariant subspace of $\bar{H}_c(Q_+^*)$, hence of the form $\bar{H}_c(Q^*)$ for some left factor $\bar{Q}^*$ of Q_+^*. The common dimension of these subspaces will be denoted by r. Let $\bar{F}$ be a basis matrix of $\bar{H}_c(Q^*)$. Since, by Proposition 15, the projections $\pi_+ \big| \hat{H}_c\left[(\bar{Q}_i^{\prime\prime})^*\right]$ are injective, we can find basis matrices $\bar{F}_\alpha$ and $\bar{F}_\beta$ in $\bar{H}_c\left[(Q_\alpha^{\prime\prime})^*\right]$ and $\bar{H}_c\left[(Q_\beta^{\prime\prime})^*\right]$ respectively such that $\bar{F} = \pi_+ \bar{F}_\alpha = \pi_+ \bar{F}_\beta$. Let P_α, P_β be the Gram matrices of these bases. Thus the projection of $g \in \bar{H}_c(Q_+^*)$ on $\bar{H}_c\left[(Q_i^{\prime\prime})^*\right]$ for $i = \alpha, \beta$, can be written as

$$P_{\bar{H}_c(Q_i^{-1})} g = \bar{F}_i P_i^{-1} \langle \bar{F}_i, g \rangle \qquad i = \alpha, \beta$$

and hence

$$\left\| P_{\bar{H}_c(Q_i^{-1})} g \right\|^2 = \langle g, \bar{F}_i \rangle P_i^{-1} \langle \bar{F}_i g \rangle \qquad i = \alpha, \beta \tag{179}$$

In view of our choice of $\bar{F}_i$, it is, for all $g \in \pi_+ H_-^2$ and thus *a fortiori* for all $g \in H_c(Q_+^{-1})$

$$\langle g, \bar{F}_\alpha \rangle = \langle g, \bar{F}_\beta \rangle = \langle g, \bar{F}_+ \rangle$$

Therefore, we can write:

$$\left\| P_{\bar{H}_c(Q_i^{-1})} \right\|^2 = \langle g, \bar{F}_+ \rangle P_i^{-1} \langle \bar{F}_+, g \rangle \qquad i = \alpha, \beta$$

Since $\bar{F}$ is a basis matrix, we have $\left\{ \langle g, \bar{F}_i \rangle g \in H_c(Q^*) \right\} = \mathbf{C}^r$. Therefore, we can conclude that

$$\left\| P_{\bar{H}_c((Q_\alpha^{\prime\prime})^*)} g \right\| \geq \left\| P_{\bar{H}_c((Q_\beta^{\prime\prime})^*)} g \right\| \qquad \forall g \in \bar{H}_c(Q_+^*)$$

if and only if

$$P_\alpha^{-1} \geq P_\beta^{-1} \tag{180}$$

We have shown that (164) is equivalent to (180). We show now that (180) is equivalent to (162). First, we show that if Q is inner in the right half plane and $\bar{F}$ is a basis matrix for $H_c(Q^*)$ with Gram matrix P, then we have, for $\mathrm{Re}\, s > 0$,

$$\frac{Q^{-1}(s) Q^{-1}(s)^* - I}{s + \bar{s}} = \bar{F}(s) P^{-1} \bar{F}(s)^* \tag{181}$$

In fact, by a standard result [see Dym (1989)], the reproducing kernel of a coinvariant subspace $H_c(Q)$ can be written as:

$$\frac{I - Q(s) Q(\omega)^*}{s + \bar{\omega}} = F(s) P^{-1} F(\omega)^* \tag{182}$$

where F is any basis matrix of $H_c(Q)$ and P its Gram matrix.

Let now $\bar{F} = \bar{Q}^{-1}F$ be a left coprime Douglas–Shapiro–Shields factorization of $\bar{F}$. Then $\bar{F}$ is a basis matrix of $\bar{H}_c(Q^*)$ if and only if F is a basis matrix of $H_c(Q)$. Therefore multiplying (182) on the left by $Q^{-1}(s)$, on the right by $Q^{-1}(\omega)^*$ and taking $\omega = s$ we get (181). We can now write the following chain of equivalences in $\mathrm{Re}\, s > 0$.

$$\pi_q Q_\alpha^{-1}(s)Q_\alpha^{-1}(s)^*\pi_q \ge \pi_q Q_\beta^{-1}(s)Q_\beta^{-1}(s)^*\pi_q$$

$$\Leftrightarrow$$

$$\pi_q \frac{Q_\alpha^{-1}(s)Q_\alpha^{-1}(s)^* - I}{s+\bar{s}}\pi_q \ge \pi_q \frac{Q_\beta^{-1}(s)Q_\beta^{-1}(s)^* - I}{s+\bar{s}}\pi_q$$

$$\Leftrightarrow$$

$$\pi_q \bar{F}_\alpha(s)P_\alpha^{-1}\bar{F}_\alpha(s)^*\pi_q \ge \pi_q \bar{F}_\beta(s)P_\beta^{-1}\bar{F}_\beta(s)^*\pi_q$$

$$\Leftrightarrow$$

$$\bar{F}_+(s)P_\alpha^{-1}\bar{F}_+(s)^* \ge \bar{F}_+(s)P_\beta^{-1}\bar{F}_+(s)^*$$

The last line is clearly implied by $P_\alpha^{-1} \ge P_\beta^{-1}$. The opposite implication is seen as above. If the assumption (178) is not satisfied, we proceed as in the proof of the previous equivalence.

(6) $\Leftrightarrow$(5)
Follows from the fact that the map $J : H_+^2 \to H_-^2$ defined by

$$(gJ)(s) := g(-\bar{s})^*$$

where H_+^2 and H_-^2 are row and column spaces respectively, is a unitary map and hence preserves ortogonality. In particular, given any inner function Q, we have $H_+^2 = H_r(Q) \oplus H_+^2 Q$ which, since $(gQ)J = Q^*(gJ)$, implies the direct sum decomposition $H_-^2 = H_c(Q^*) \oplus Q^* H_-^2$. Moreover, we have for $g \in H_+^2$,

$$(gP_{H_r(Q)})J = P_{\bar{H}_c(Q^*)}(gJ)$$

Finally, we note that $H_r(Q_+)J = \bar{H}_c(Q_+^*)$, and this completes the proof of the claimed equivalence.

(8) $\Leftrightarrow$(6)
We assume (167) holds. Since

$$H_r(K_i)Q_i^e \oplus H_r(Q_i^e) = H_r(K_iQ_i^e) = H_r(\bar{Q}_i^e K_+^e) = H_r(\bar{Q}_i^e)K_i^e \oplus H_r(K_+^e)$$

we have

$$\|f\|^2 = \|fP_{H_r(K_i)Q_i^e}\|^2 + \|fP_{H_r(Q_i^e)}\|^2 \qquad \forall f \in H_r(K_+^e)$$

Thus (167) holds if and only if

$$\|fP_{H_r(Q_\alpha^e)}\| \ge fP_{H_r(Q_\beta^e)}\| \qquad \forall f \in H_r(K_+^e) \tag{183}$$

We claim that the last inequality implies

$$\|gP_{H_r(Q_\alpha^e)}\| \ge \|gP_{H_r(Q_\beta^e)}\| \qquad \forall g \in H_r(Q_+^e) \tag{184}$$

To see this, note that the factorization

$$Q_i'Q_i'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

implies the inclusion

$$H_r(Q_i'') \subset H_r(Q_+^e) \oplus H_r(R^e) \tag{185}$$

We also note that $H_r(Q_+^e) \perp H_r(R^e)$. Therefore, using the inclusion (185), we have for $f \in H_r(K_+^e)$

$$\begin{aligned} fP_{H_r(Q_i'')} &= fP_{H_r\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}} P_{H_r(Q_i'')} \\ &= f\left[P_{H_r(Q_+^e)} \oplus P_{H_r(R^e)}\right] P_{H_r(Q_i'')} \\ &= fP_{H_r(Q_+^e)} P_{H_r(Q_i'')} \end{aligned}$$

Now, by Corollary 3, we have $H_r(K_+^e)P_{H_r(Q_+^e)} = H_r(Q_+^e)$ and hence (183) implies (184). Conversely, assume (165) holds. By (185), we have for every $f \in H_r(K_+^e)$

$$\begin{aligned} fP_{H_r(Q_i'')} &= fP_{H_r(Q_+^e) \oplus H_r(R^e)} P_{H_r(Q_i'')} \\ &= f\left[P_{H_r(Q_+^e)} + P_{H_r(R^e)}\right] P_{H_r(Q_i'')} \\ &= fP_{H_r(Q_+^e)} P_{H_r(Q_i'')} \end{aligned}$$

as, clearly, $H_r(K_+^e)\big|P_{H_r(R^e)} = 0$. Now, inequality (165) implies, once again using Corollary 3, that for every $f \in H_r(K_+^e)$

$$\left\|fP_{H_r(K_1)Q_1''}\right\| = \left\|fP_{H_r(Q_+^e)}P_{H_r(K_1)Q_1''}\right\| \geq \left\|fP_{H_r(Q_+^e)}P_{H_r(K_2)Q_2''}\right\| = \left\|fP_{H_r(K_2)Q_2''}\right\|$$

(6) $\Leftrightarrow$(7)

Assume inequality (165) holds. This implies $\operatorname{Ker}\left(H_r(Q_+^e)\big|P_{H_r(Q_\alpha'')}\right) \subset \operatorname{Ker}\left(H_r(Q_+^e)\big|P_{H_r(Q_\beta'')}\right)$. Therefore there exists a map $Y: H_r(Q_\alpha'') \to H_r(Q_\beta'')$ for which

$$H_r(Q_+^e)\big|P_{H_r(Q_\alpha'')}\, Y = H_r(Q_+^e)\big|P_{H_r(Q_\beta'')}$$

i.e. figure 10 commutes.

Such a map Y is unique if and only $H_r(Q_+^e)\big|P_{H_r(Q_\alpha'')}$ is surjective. This, by Proposition 15, is indeed the case. Inequality (165) shows that $Y = T_\alpha^{-1}T_\beta$ is contractive.

$$\begin{tikzcd}
H_r(Q''_\alpha) \arrow[dd, "Y"'] & \\
& H_r(Q^e_+) \arrow[ul, "T_\alpha = H_r(Q^e_+)|P_{H_r(Q''_\alpha)}"'] \arrow[dl, "T_\beta = H_r(Q^e_+)|P_{H_r(Q''_\beta)}"] \\
H_r(Q''_\beta) &
\end{tikzcd}$$

Figure 10.

Conversely, assume there exists a unique contraction $Y : H_r(Q_\alpha^e) \rightarrow H_r(Q_\beta^e)$ satisfying $T_\alpha Y = T_\beta$. For any $f \in H_r(Q_\alpha^e)$ we have $\|fT_\alpha^{-1}T_\beta\| \leq \|f\|$. Since $H_r(Q_+^e)|P_{H_r(Q_\alpha^e)}$ is surjective, we can write $f = gP_{H_r(Q_\alpha^e)} = gT_\alpha$, with $g \in H_r(Q_+)$. This implies (165).

(8) $\Leftrightarrow$(9)
The proof is similar to the equivalence (6) $\Leftrightarrow$(7). Thus, assume (167) holds. This implies $\operatorname{Ker} H_r(K_+^e)|P_{H_r(K_\beta)Q_\beta^e} \subset \operatorname{Ker} H_r(K_+^e)|P_{H_r(K_\alpha)Q_\alpha^e}$. Hence there exists a map $Z : H_r(K_\beta)Q_\beta^e \rightarrow H_r(K_\alpha)Q_\alpha^e$ for which $R_\beta Z = R_\alpha$. Z is uniquely determined if and only if R_β is surjective, which, by Proposition 18, is the case. Now from the inequality $\|fR_\alpha\| \leq \|fR_\beta\|$, setting $f = gR_\beta^{-1}$, it follows that $\|gR_\beta^{-1}R_\alpha\| \leq \|f\|$, i.e. Z is contractive.

Conversely, assume there exists a unique contraction $Z : H_r(K_\beta)Q_\beta^e \rightarrow H_r(K_\alpha)Q_\alpha^e$ satisfying $R_\beta Z = R_\alpha$. Thus, for all $f \in H_r(K_\alpha)Q_\alpha^e$, we have $\|fR_\beta^{-1}R_\alpha\| \leq \|f\|$. Setting $f = gR_\beta$ we obtain $\|gR_\alpha\| \leq \|gR_\beta\|$. □

Remark: The equivalence between (1) and (8) is well known. For the uniform choice of basis in the state space formulation, see Caines and Delchamps (1980), and also Lindquist and Picci (1985) for the geometric approach from which these formulas are derived. The inequality (163) is very similar to the definition (155). Nevertheless, it is a more general statement about the partial ordering of inner functions, which coincides with the usual one when $\pi_{\mathrm{u}} = I$. There is no need to introduce spectral factors to formulate this result. It seems to us that this is the actual key to the understanding of the order structure. We also point out that inequality (165) implies that the angle between the subspaces $H_r(Q_+^e)$ and $H_r(Q_\alpha^e)$ is smaller than the angle between $H_r(Q_+^e)$ and $H_r(Q_\beta^e)$. Similarly inequality (167) implies that the angle between the subspaces $H_r(K_+^e)$ and $H_r(K_\beta)Q_\beta^e$ is smaller than the angle between $H_r(K_+^e)$ and $H_r(K_\alpha)Q_\alpha^e$. These implications are one sided, but we refrain from a further discussion of this. Finally, one might expect that the equivalence of conditions (6) and (8) might be proved directly from the direct sum decompositions (127), however we failed to do so.

Corollary 12: *With the previously defined partial order in $\mathcal{W}^m$, W_-^e and W_+^e are the minimal and maximal elements respectively.*

Proof: Clearly, in formula (165), the maximal element is Q_+ and the minimal one is I, which are the inner functions corresponding to W_+ and W_-, respectively. □

This previous corollary explains the terminology of extremal factors for $W_\pm^e$, $\bar{W}_\pm^e$.

We turn now to the geometric structure of the set of minimal Markovian splitting subspaces. The reader familiar with stochastic realization theory will recognize the connection to the geometric constructs used in the stochastic domain. In particular, some of the spaces pertain to the zero dynamics associated with a given spectral factor, i.e. to the maximal output nulling inner (anti)-stabilizable subspaces and the input containing outer (anti)-detectable subspaces associated to any given minimal realization. This study was initialized in Lindquist *et al.* (1995). A further analysis will be given in Gombani and Fuhrmann (1998) and Fuhrmann (1998).

To facilitate the reading we give here a table of the basic constructs used. At the risk of repeating ourselves, we point out again that the various spaces in table 1 are a result of a normalization, where with W_+ we associate the state space $H_r(K_+^e)$.

Space	W_-	W	W_+
X	$H_r(K_-)Q_+$	$H_r(K)Q''$	$H_r(K_+)$
H_-	$H_+^2 \ominus Q_+$	$H_+^2 \ominus Q_+$	$H_+^2 \ominus Q_+$
H^+	$H_-^2 \ominus K_+$	$H_-^2 \ominus K_+$	$H_-^2 \ominus kK+$
S	$H_+^2 Q_+$	$H_+^2 Q''$	H_+^2
$\bar{S}$	$H_-^2 K_- Q_+$	$H_-^2 K Q''$	$H_-^2 K_+$
$X \cap H^-$	$H_r(K_-)Q_+$	$\left[H_r(K) \cap H_r(K_-)Q'\right]Q''$	$H_r(K_+) \cap H_r(K_-)Q_+$
$\dim(X \cap H^-)$	$\delta(K_-)$	$\delta(K) - \delta(Q')$	$\delta(K_+) - \delta(Q_+)$
$X \cap H^+$	$H_r(K_+) \cap H_r(K_-)Q_+$	$H_r(K)Q'' \cap H_r(K_+)$	$H_r(K_+)$
$\dim(X \cap H^+)$	$\delta(K_-) - \delta(Q_-)$	$\delta(K) - \delta(\bar{Q}'')$	$\delta(K_+)$
$(H^-)^{\perp} P_X$	$\{0\}$	$\left[H_r(Q')P_{H_r(K)}\right]Q''$	$H(Q_+)P_{H_r(K_+)}$
$\dim\left[(H^-)^{\perp} P_X\right]$	0	$\delta(Q')$	$\delta(Q_+)$
$(H^+)^{\perp} P_X$	$H(Q_-)K_+ P_{H_r(K_-)Q_+}$	$\left[H_r(\bar{Q}'')K_+\right]P_{H_r(K)Q''}$	$\{0\}$
$\dim\left[(H^+)^{\perp}\right] P_X$	$\delta(Q_-)$	$\delta(\bar{Q}'')$	0

Table 1.

Theorem 10: *Let W be a minimal, stable spectral factor. Let $W = \bar{W}K$ be its DSS factorization over H_-^{∞}. Let the inner functions Q', Q'', $\bar{Q}'$, $\bar{Q}''$ be determined by Proposition* 13*, and let Q_+^e, Q_-^e, K_+^e, K_-^e be the inner functions for which diagram* 2 *commutes.*

Definite the spaces X, S and $\bar{S}$ as in Proposition 18*, i.e.*

$$\left.\begin{aligned} X &= H_r(K)Q'' \\ S &= H_+^2 Q'' \\ \bar{S} &= H_-^2 KQ'' \end{aligned}\right\} \tag{186}$$

and the subspaces H^- and H^+ by

$$\left.\begin{aligned} H^- &= H_+^2 \ominus Q_+ \\ H^+ &= H_-^2 \ominus K_+ \end{aligned}\right\} \tag{187}$$

Then:

(1) *We have*

$$\left.\begin{aligned} X \cap H^- &= \left[H_r(K) \cap H_r(K_-)Q'\right]Q'' \\ X \cap H^+ &= H_r(K)Q'' \cap H_r(K_+^e) \end{aligned}\right\} \tag{188}$$

(2) *If $X,\ Y \subset H^2_+$ and $Y \subset X$ denote by $Y^{\perp}_X$ the orthogonal complement of Y in X. We have*

$$\left.\begin{aligned} (X \cap H^-)^{\perp}_X &= (H^-)^{\perp}_X = (H^-)^{\perp} P_X = \left[H_r(Q')P_{H_r(K)}\right]Q'' \\ (X \cap H^+)^{\perp}_X &= (H^+)^{\perp} P_X = (H_r(\bar{Q}'')K^e_+)P_{H_r(K)Q''} \end{aligned}\right\} \tag{189}$$

(3) *We have*

$$\left.\begin{aligned} \dim\,(H^+)^{\perp} P_X &= \delta(\bar{Q}'') \\ \dim\,(H^-)^{\perp} P_X &= \delta(Q') \end{aligned}\right\} \tag{190}$$

(4) *We have*

$$\left.\begin{aligned} \dim\,(X \cap H^-) &= \delta(K) - \delta(Q') \\ \dim\,(X \cap H^+) &= \delta(K) - \delta(\bar{Q}'') \end{aligned}\right\} \tag{191}$$

Proof:

(1) In view of equation (75) and the fact, see Lemma 7, that $Q_+\pi_- = \pi_- Q_+$, we have the equality $K^e_- Q^e_+ \pi_- = \bar{Q}' K Q'' \pi_-$. Thus we compute

$$\begin{aligned} X \cap H^- &= H_r(K)Q'' \cap H^2_+ \pi_- Q_+ = H_r(K)Q'' \cap H^2_+ Q^e_+ \pi_- \\ &= \bar{H}_r(K^*)KQ'' \cap H^2_+ (K^e_-)^* K^e_- Q^e_+ \pi_- \\ &= \bar{H}_r(K^*)KQ'' \cap \left[H^2_+ + \bar{H}_r((K^e_-)^*)\right] K^e_- Q^e_+ \pi_- \\ &= \left[H_r(K^*)(\bar{Q}')^* \cap H^2_+ (K^e_-)^*\right] K^e_- Q^e_+ \pi_- \\ &= \left[H_r(K^*)(\bar{Q}')^* \cap (H^2_+ \oplus H_r((K^e_-)^*))\right] K^e_- Q^e_+ \pi_- \\ &= \left[H_r(K^*)(\bar{Q}')^* \cap H_r((K^e_-)^*)\right] K^e_- Q^e_+ \pi_- \\ &= H_r(K)Q'' \cap H_r(K^e_-)\pi_- Q^3_+ \\ &= \left[H_r(K) \cap H_r(K^e_-)Q'\right]Q'' \end{aligned}$$

where in the last equality we have used the fact that $H_r(K^e_-)\pi_- = H_r(K^e_-)$ and therefore the projection can be eliminated. Similarly

$$\begin{aligned} X \cap H^+ &= H_r(K)Q'' \cap H^2_- \pi_- K_+ = H_r(K)Q'' \cap H^2_- K^e_+ \\ &= H_r(K)Q'' \cap \left[H^2_- \oplus H_r(K_+)\right]\pi_- \\ &= H_r(K)Q'' \cap H_r(K^e_+) \end{aligned}$$

(2) We compute first the orthogonal complement of $X \cap H^-$ in L^2. We use the decomposition $(H_r(K)Q'')^{\perp} = H^2_+ KQ'' \oplus H_r(Q'') \oplus H^2_-$ and $(H^2_+ Q^e_+)^{\perp} = H^2_- Q^e_+$, as well as the equality $H^2_- Q^e_+ = H^2_- \oplus H_r(Q^e_+)$ to get

$$\begin{aligned}(X \cap H^-)^{\perp} &= (H_r(K)Q'' \cap H_r(K_-^e)Q_+^e)^{\perp}\\ &= (H_r(K)Q'')^{\perp} + (H_r(K_-^e)Q_+^e)^{\perp}\\ &= \left[H_+^2 KQ'' \oplus H_-^2 Q''\right] + \left[H_+^2 K_-^e Q_+^e \oplus H_-^2 Q_+^e\right]\\ &= \left[H_+^2 KQ'' \oplus H_-^2 Q''\right] + \left[H_+^2 \bar{Q}' KQ'' \oplus H_r(R^e) \oplus H_-^2 Q_+^e\right]\\ &= \left[H_+^2 KQ'' \oplus H_-^2 Q''\right] + \left[H_+^2 \bar{Q}' KQ2 \oplus H_-^2 \oplus H_r(Q')Q'' \oplus H_r(Q'')\right]\end{aligned}$$

Projecting orthogonally on $H_r(K)Q''$ we obtain the equality

$$(X \cap H^-)_X^{\perp} = H_r(Q')Q''P_{H_r(K)Q''} = \left[H_r(Q')P_{H_r(K)}\right]Q''$$

Similarly, we compute the L^2 orthogonal complement of $X \cap H^+$.

$$\begin{aligned}(X \cap H^+)^{\perp} &= H_+^2 KQ'' + H_r(Q'') + H_-^2 + H_+^2 K_+ + H^2\\ &= H_+^2 KQ'' + H_r(Q'') + H_-^2 + (H_+^2 \bar{Q}'' + H_r(\bar{Q}''))K_+\\ &= H_+^2 KQ'' + H_r(Q'') + H_-^2 + H_+^2 \bar{Q}'' K_+ + H_r(\bar{Q}'')K_+\\ &= H_+^2 KQ'' + H_r(Q'') + H_-^2 + H_r(\bar{Q}'')K_+\end{aligned}$$

Projecting orthogonally on $X = H_r(K)Q''$ we get

$$(X \cap H^+)_X^{\perp} = (H_r(\bar{Q}'')K_+^e)P_{H_r(K)Q''}$$

as all other components are orthogonal to X.

(3) We need to show that $\dim (H_r(\bar{Q}'')K_+)P_{H_r(K)Q''} = \delta(\bar{Q}'')$. For this it suffices to show that the map $H_r(\bar{Q}'')K_+ \big| P_{H_r(K)Q''}$ is injective. But this follows from Lemma 5. Similarly for $H_r(Q')Q'' \big| P_{H_r(K)Q''}$.

(4) It follows from (3), since $\dim (X \cap H_-) = \dim X - \dim (X \cap H_-)_X^{\perp} = \delta(K) - \delta(Q'')$. Similarly for $\dim (X \cap H_-)$. □

Remark: It should be noted that in the Lindquist–Picci framework the multiplicity of K_+ could be strictly less than the rank of Φ. This implies that H^- is not necessarily contained in the past of the state process of the realization. Since we do assume that the multiplicity of K_+ is m_0, this can never happen. As we said, this is a rather special situation and we will not dwell on it here.

The set $\mathcal{W}^{m_0}$ inherits the partial order from $\mathcal{W}^m$ and, as W_-^e, $W_+^e \in \mathcal{W}^{m_0}$, they are the minimal and maximal elements respectively of $\mathcal{W}^{m_0}$ with respect to this partial order. The set $\mathcal{W}^m$ and its relation to $\mathcal{W}^{m_0}$ will be studied via the arithmetic properties of inner functions.

Given a minimal, stable $p \times m$ spectral factor W, we say, again following Lindquist and Picci (1991), that a pair of internal spectral factors W_{0-}, $W_{0+} \in \mathcal{W}^{m_0}$ is a *tightest internal bound* for W if

(1) We have

$$W_{0-} \le W \le W_{0+}$$

(2) Given $W_1, W_2 \in \mathcal{W}^{m_0}$ for which $W_1 \leq W \leq W_2$ we have

$$W_1 \leq W_{0-}$$

$$W_{0+} \leq W_2$$

We can state now:

Proposition 20: *Let W be an arbitrary minimal, stable $p \times m$ spectral factor W. Then*

(1) *A tightest internal bound for W exists.*

(2) *The tightest internal bound can be characterized as follows. Let Q', Q'' be the inner functions determined by Proposition* 13. *Let*

$$Q' = Q'_1 Q'_2$$

$$Q'' = Q''_2 Q''_1$$

be the internal–external and external internal factorizations of Q' and Q'' respectively. Then

$$W_{0_-} = W^e_- Q'_1$$

$$W_{0_+} = W^e_+ (Q''_1)^*$$

Proof: Follows from Theorem 4. □

Proposition 20 gave an arithmetic characterization of the tightest internal bound for a given minimal, stable spectral factor. We proceed next to derive a geometric characterization for it.

Proposition 21: *Given a minimal, stable spectral factor W, let W_{0-}, W_{0+} be the tightest internal bound. Let S_{0-}, $\bar{S}_{0-}$ and S_{0_+}, $\bar{S}_{0_+}$ be defined, as in Proposition* 18, *for W_{0_-} and W_{0_+} respectively. Let $H_0 = L^2\pi_1$, i.e. it is the subspace of L^2 of all those functions whose last $m - m_0$ coordinates vanish. Then, with the notation of Theorem* 4,

(1) *We have $S_{0-} = S \cap H_0$, i.e.*

$$H^2_+ Q Q''_1 = H^2_+ Q'' \cap L^2 \pi_1 \tag{192}$$

(2) *We have $S_{0+} = S P_{H_0}$, i.e.*

$$H^2_+ \pi_1 Q''_1 = H^2_+ Q'' \pi_1 \tag{193}$$

(3) *We have $\bar{S}_{0-} = \bar{S} P_{H_0}$, i.e.*

$$H^2_- K_{0-} Q Q''_1 = H^2_- K Q'' \pi_1 \tag{194}$$

(4) *We gave $\bar{S}_{0+} = \bar{S} \cap H_0$, i.e.*

$$H^2_- K_{0+} Q''_1 = H^2_- K Q'' \cap L^2 \pi_1 \tag{195}$$

(5) *We have*

$$S_{0-} \cap \bar{S}_{0-} = H_r(K_{0-})QQ_1'' \tag{196}$$

and

$$S_{0+} \cap \bar{S}_{0+} = H_r(K_{0+}Q_1'' \tag{197}$$

(6) *We have*

$$X \cap H_0 = (H_r(K_{0+}) \cap H_+^2 Q)Q_1'' \tag{198}$$

and

$$\dim Z \cap H_0 = \delta(K) - \delta(Q_2'') \tag{199}$$

Proof:

(1) We proved in Proposition 15 that $H_r(Q'')\pi_1 = H_r(QQ_1'')$. This means that

$$H_r(Q'')\pi_1 + H_+^2\pi_2 = H_r(QQ_1'') + H_+^2\pi_2$$

Taking orthogonal complements in H_+^2, we obtain

$$H_+^2 Q'' \cap L^2\pi_1 = H_+^2 Q'' \cap H^2\pi_1 = H_r(QQ_1'')$$

(2) We have

$$H_+^2 Q''\pi_1 = H_+^2 Q_2'' Q_1''\pi_1$$

$$= H_+^2 Q_2''\pi_1 Q_1'' \subset H_+^2\pi_1 Q_1''$$

We will show that $H_+^2 Q_2''\pi_1 = H_+^2\pi_1$, which will prove the assertion. In fact, $H_+^2 Q_2''\pi_1$ is an H_+^∞-invariant subspace of $H_+^2\pi_1$, hence of the form $H_+^2\pi_1 T$ for some inner function T. Clearly, T has a representation of the form

$$\begin{pmatrix} T_0 & 0 \\ 0 & I \end{pmatrix}$$

and is a right factor of Q_2'', and so also of $Q'Q_2''$. Thus TQ_1'' is a right factor of Q_+, however this contradicts the fact that the greatest common right divisor of Q'' and Q_+^e is Q_1''. Thus T is necessarily constant unitary.

(3) Referring to figure 5, we have the equality $K_{0-}Q_2' = Q_2'K$. We compute

$$H_-^2 KQ''\pi_1 = H_-^2(\bar{Q}_2')^* K_{0-} Q_2' Q''\pi_1$$

$$= H_-^2(\bar{Q}_2')^* K_{0-} Q_2' Q_2'' Q_1''\pi_1$$

$$= H_-^2(\bar{Q}_2')^* K_{0-} \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} Q_1''\pi_1$$

$$= H_-^2(\bar{Q}_2')^*\pi_1 K_{0-} QQ_1''$$

$$= H_-^2\pi_1 K_{0-} QQ_1''$$

We have used the fact that $H^2_-(\bar{Q}_2)^*\pi_1$ is an H^∞_--invariant subspace of H^2_-, hence of the form $H^2_-\pi_1 T^*$ for some inner function T. Such a function would be a common left factor of $\bar{Q}_2\bar{Q}''$ which is necessarily trivial.

(4) Referring to figure 5, we have $KQ_2' = \bar{Q}_2' K_{0+}$. We compute

$$
\begin{aligned}
H^2_- KQ'' \cap L^2\pi_1 &= H^2_- KQ_2'Q_1' \cap L^2\pi_1 \\
&= H^2_- \bar{Q}_2' K_{0+} Q_1' \cap L^2\pi_1 \\
&= (H^2_- \bar{Q}_2' \cap L^2\pi_1) K_{0+} Q_1' \\
&= H^2_- K_{0+} Q_1'
\end{aligned}
$$

Using arguments as above, we used the fact that $H^2_- \bar{Q}_2' \cap L^2\pi_1 = H^2_- \cap L^2\pi_1$.

(5) By trivial computations.

(6) Since $X = S \cap \bar{S}$, we have

$$
\begin{aligned}
X \cap H_0 &= (S \cap \bar{S}) \cap H_0 = (S \cap H_0) \cap (\bar{S} \cap H_0) \\
&= S_{0-} \cap S_{0+} = H^2_+ QQ_1' \cap H^2_- K_{0+} Q_1' \\
&= (H_r(K_{0+}) \cap H^2_+ Q) Q_1'
\end{aligned}
$$

By Proposition 17, the Toeplitz operator $\mathcal{T}^r_{K_{0+}Q^*}$ is injective, hence, by Theorem 1, we have $H^2_+ K_{0+} \cap H_r(Q) = 0$. We compute

$$
\begin{aligned}
\dim X \cap H_0 &= \dim H_r(K_{0+}) \cap H^2_+ Q = \dim \operatorname{Ker} \mathcal{T}^r_{QK^*_{0+}} \\
&= \operatorname{codim} (H^2_+ K_{0+} + H_r(Q)) \\
&= \delta(K_{0+}) - \delta(Q) \\
&= \delta(K) - \delta(Q_2')
\end{aligned}
$$

This means that the dimension of $X \cap H_0$ is smaller than the dimension of X exactly by the number of external zeros of W. □

Remarks: We would like to stress that the above results are much harder to prove if instead of the extended scattering pairs we try to use those defined in Proposition 19. A glance at the formulas appearing in Theorem 10 will convince the initiated reader that there are very close links here to geometric control theory. The study of these links is beyond the scope of this paper and will be relegated to subsequent publications.

We focus now on the analysis of the lattice structure of the sets $\mathcal{W}^m$ and $\mathcal{X}^m$. Unhappily, we cannot prove that $\mathcal{W}^m$ is a complete lattice. In fact, in view of the fact that the set of all non-negative matrices is not a complete lattice, see Ex. 9 on p. 142 of Halmos (1958), it is doubtful whether $\mathcal{W}^m$ is a complete lattice. However, some results are obtainable, in special cases. The analysis uses the arithmetic of inner functions. Of course, this arithmetic is reflected in the geometry of splitting sub-

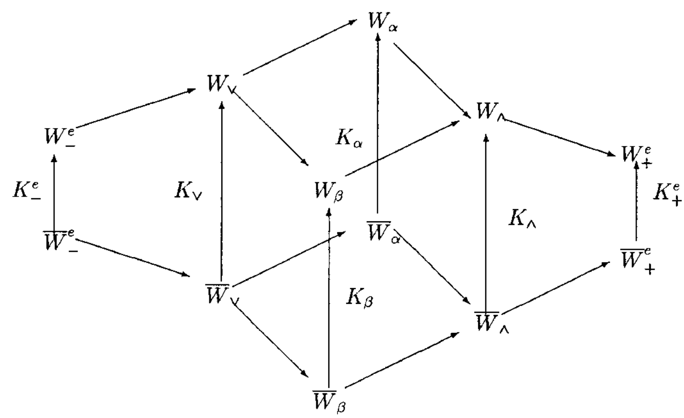

Figure 11.

spaces. We begin by proving the following proposition. In following the proof, it may be convenient to refer to figure 5.

Proposition 22: *Denote by $\mathcal{W}^m_R$ the set of all minimal, stable spectral factors of size $p \times m$ for which, with the notation of Theorem 4, we have*

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

Let W_α, $W_\beta \in \mathcal{W}^m_R$. Let S_{W_α}, S_{W_β} be defined by (129)*. Then we have*

(1)

$$S_{W_\alpha} \cap S_{W_\beta} = H^2_+(Q''_\alpha \vee_L Q''_\beta) \tag{200}$$

where $Q''_\alpha \vee_L Q''_\beta$ is the least common left inner multiple of Q''_α and Q''_β.

(2)

$$S_{W_\alpha} + S_{W_\beta} = H^2_+(Q''_\alpha \wedge_R Q''_\beta) \tag{201}$$

where $Q''_\alpha \wedge_R Q''_\beta$ is the greatest common right inner divisor of Q''_α and Q''_β.

(3)

$$\bar{S}_{W_\alpha} \cap \bar{S}_{W_\beta} = H^2_-(\bar{Q}''_\alpha \wedge_R \bar{Q}''_\beta)K^e_+ \tag{202}$$

(4)

$$\bar{S}_{W_\alpha} + \bar{S}_{W_\beta} = H^2_-(\bar{Q}''_\alpha \vee_L \bar{Q}''_\beta)K^e_+ \tag{203}$$

(5) (a) *There exists a unique inner function $K_\vee$ satisfying*

$$K_\vee(Q''_\alpha \vee_L Q''_\beta) = (\bar{Q}''_\alpha \vee_L \bar{Q}''_\beta)K^e_+ \tag{204}$$

(b) *There exists a unique inner function $K_\wedge$ satisfying*

$$K_\wedge(Q''_\alpha \wedge_R Q''_\beta) = (\bar{Q}''_\alpha \wedge_R \bar{Q}''_\beta)K^e_+ \tag{205}$$

(6) *There exist inner functions Z, $\bar{Z}$ for which*

$$K_\vee Z = \bar{Z} K_\wedge \tag{206}$$

(7) *Define rational functions $W_\vee$, $W_\wedge$ by*

$$\left.\begin{aligned} W_\wedge &= W_+^e (Q''_\alpha \vee_R Q''_\beta)^* \\ W_\vee &= W_+^e (Q''_\alpha \vee_L Q''_\beta)^* \end{aligned}\right\} \tag{207}$$

Then $W_\vee$, $W_\wedge$ are minimal, stable spectral factors.

(8) *$\bar{W}_\vee$, $\bar{W}_\wedge$ defined by*

$$\left.\begin{aligned} \bar{W}_\vee &= W_\vee K_\vee^* \\ \bar{W}_\wedge &= W_\wedge K_\wedge^* \end{aligned}\right\} \tag{208}$$

are antistable spectral factors.

(9) *For Z, $\bar{Z}$, we have*

$$\left.\begin{aligned} W_\vee Z &= W_\wedge \\ \bar{W}_\vee \bar{Z} &= \bar{W}_\wedge \end{aligned}\right\} \tag{209}$$

(10) *We have*

$$(S_{W_\alpha} \cap S_{W_\beta}) \cap (\bar{S}_{W_\alpha} + \bar{S}_{W_\beta}) = H_r(K_\vee)(Q''_\alpha \vee_L Q''_\beta) \tag{210}$$

and

$$(S_{W_\alpha} + S_{W_\beta}) \cap (\bar{S}_{W_\alpha} \cap \bar{S}_{W_\beta}) = H_r(K_\wedge)(Q''_\alpha \wedge_R Q''_\beta) \tag{211}$$

So $(S_{W_\alpha} \cap S_{W_\beta}, \bar{S}_{W_\alpha} + \bar{S}_{W_\beta})$ is a scattering pair for $W_\vee$.
Similarly, $(S_{W_\alpha} + S_{W_\beta}, \bar{S}_{W_\alpha} \cap \bar{S}_{W_\beta})$ is a scattering pair for $W_\wedge$.

Proof:

(1) We compute

$$S_{W_\alpha} \cap S_{W_\beta} = H_+^2 Q''_\alpha \cap H_+^2 Q''_\beta = H_+^2 (Q''_\alpha \vee_L Q''_\beta)$$

(2) Similarly,

$$S_{W_\alpha} + S_{W_\beta} = H_+^2 Q''_\alpha + H_+^2 Q''_\beta = H_+^2 (Q''_\alpha \wedge_R Q''_\beta)$$

(3) The algebra becomes a bit more complicated in the complementary computations of $\bar{S}_{W_\alpha} \cap \bar{S}_{W_\beta}$ and $\bar{S}_{W_\alpha} + \bar{S}_{W_\beta}$. We have

$$K_\alpha Q''_\alpha = \bar{Q}''_\alpha K_+^e$$

$$K_\beta Q''_\beta = \bar{Q}''_\beta K_+^e$$

Therefore

$$
\begin{aligned}
\bar S_{W_\alpha} \cap \bar S_{W_\beta} &= H^2_- K_\alpha Q^e_\alpha \cap H^2_- K_\beta Q^e_\beta = H^2_- \bar Q^e_\alpha K^e_+ \cap H^2_- \bar Q^e_\beta K^e_+ \\
&= \left[H^2_- \bar Q^e_\alpha \cap H^2_- \bar Q^e_\beta \right] K^e_+ = \left[(H^2_- \oplus H(\bar Q^e_\alpha)) \cap (H^2_- \oplus H(\bar Q^e_\beta)) \right] K^e_+ \\
&= \left[H^2_- \oplus (H(\bar Q^e_\alpha) \cap H(\bar Q^e_\beta)) \right] K^e_+ = \left[H^2_- \oplus H(\bar Q^e_\alpha \wedge_R \bar Q^e_\beta) \right] K^e_+ \\
&= H^2_- (\bar Q^e_\alpha \wedge_R \bar Q^e_\beta) K^e_+
\end{aligned}
$$

(4) Similarly,

$$
\begin{aligned}
\bar S_{W_\alpha} + \bar S_{W_\beta} &= H^2_- K_\alpha Q^e_\alpha + H^2_- K_\beta Q^e_\beta = H^2_- \bar Q^e_\alpha K^e_+ + H^2_- \bar Q^e_\beta K^e_+ \\
&= \left[H^2_- \bar Q^e_\alpha + H^2_- \bar Q^e_\beta \right] K^e_+ = \left[H^2_- (\bar Q^e_\alpha \vee_L \bar Q^e_\beta) \right] K^e_+ \\
&= H^2_- K_\vee (Q^e_\alpha \vee_L Q^e_\beta)
\end{aligned}
$$

(5) (a) By (69), we have

$$
\left.
\begin{aligned}
K_\alpha Q^e_\alpha &= \bar Q^e_\alpha K^e_+ \\
K_\beta Q^e_\beta &= \bar Q^e_\beta K^e_+
\end{aligned}
\right\} \tag{212}
$$

Then we have

$$
T = K_\alpha Q^e_\alpha \vee_L K_\beta Q^e_\beta = \bar Q^e_\alpha K^e_+ \vee_L \bar Q^e_\beta K^e_+ = (\bar Q^e_\alpha \vee_L \bar Q^e_\beta) K^e_+
$$

Thus there exist inner functions Y_α, Y_β for which

$$
T = Y_\alpha Q^e_\alpha = Y_\beta Q^e_\beta
$$

Since $Q^e_\alpha \vee_L Q^e_\beta$ divides T on the right, it follows that there exists an inner function $K_\vee$ for which (204) holds.

(b) Similarly,

$$
S = K_\alpha Q^e_\alpha \wedge_R K_\beta Q^e_\beta = \bar Q^e_\alpha K^e_+ \wedge_R \bar Q^e_\beta K^e_+ = (\bar Q^e_\alpha \wedge_R \bar Q^e_\beta) K^e_+
$$

Clearly, $Q^e_\alpha \wedge_R Q^e_\beta$ divides S on the right, hence it follows that there exists an inner function $K_\wedge$ for which (205) holds.

(6) Clearly $Q^e_\alpha \wedge_R Q^e_\beta$ divides $Q^e_\alpha \vee_L Q^e_\beta$ on the right. Thus there exists an inner function Z satisfying

$$
Q^e_\alpha \vee_L Q^e_\beta = Z(Q^e_\alpha \vee_R Q^e_\beta) \tag{213}
$$

Analogously, there exists an inner function $\bar Z$ satisfying

$$
\bar Q^e_\alpha \vee_L \bar Q^e_\beta = \bar Z(Q^e_\alpha \wedge_R \bar Q^e_\beta) \tag{214}
$$

Next, we compute

$$
\begin{aligned}
K_\vee Z(Q^e_\alpha \wedge_R Q^e_\beta) &= K_\vee (Q^e_\alpha \vee_L Q^e_\beta) = (\bar Q^e_\alpha \vee_L \bar Q^e_\beta) K^e_+ \\
&= (\bar Q^e_\alpha \vee_L \bar Q^e_\beta)(Q^e_\alpha \wedge_R \bar Q^e_\beta)^* (\bar Q^e_\alpha \wedge_R \bar Q^e_\beta) K^e_+ \\
&= \bar Z(\bar Q^e_\alpha \wedge_R \bar Q^e_\beta) K^e_+ \\
&= \bar Z K_\wedge (Q^e_\alpha \wedge_R Q^e_\beta)
\end{aligned}
$$

Hence equality (206) follows.

(7) As $(Q_\alpha^{\#} \wedge_R Q_\beta^{\#})^*$ is unitary on the imaginary axis, it follows that $W_\wedge$ is a spectral factor. The same holds for $W_\vee$. We have $W_+^e = W_\alpha Q_\alpha^{\#} = W_\beta Q_\beta^{\#}$. This clearly implies that $W_\wedge = W_+^e (Q_\alpha^{\#} \wedge_R Q_\beta^{\#})^* \in H_+^\infty$. The minimality of $W_\wedge$ follows from Theorem 2.

Similarly, by our assumption that $W_\alpha, W_\beta \in \mathcal{W}_R^m$, we have that both $Q_\alpha^{\#}$ and $Q_\beta^{\#}$ are right factors of the inner function

$$\begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

Hence $Q_\alpha^{\#} \vee_L Q_\beta^{\#}$ is also a right factor. This implies $W_+^e (Q_\alpha^{\#} \vee_L Q_\beta^{\#})^* \in H_+^\infty$. Invoking Theorem 2 once again, we conclude that $W_\vee$ is a minimal, stable spectral factor.

(8) We compute

$$\begin{aligned} \bar{W}_\wedge &= W_\wedge K_\wedge^* = W_+^e (Q_\alpha^{\#} \wedge_R Q_\beta^{\#})^* K_\wedge^* \\ &= W_+^e (K_+^e)^* (\bar{Q}_\alpha^{\#} \wedge_R \bar{Q}_\beta^{\#})^* \\ &= \bar{W}_+^e (\bar{Q}_\alpha^{\#} \wedge_R \bar{Q}_\beta^{\#})^* \in H_-^\infty \end{aligned}$$

(9) Using equality (213), we compute

$$\begin{aligned} W_\vee Z &= W_+^e (Q_\alpha^{\#} \vee_L Q_\beta^{\#})^* (Q_\alpha^{\#} \vee_L Q_\beta^{\#})(Q_\alpha^{\#} \wedge_R Q_\beta^{\#})^* \\ &= W_+^e (Q_\alpha^{\#} \wedge_R Q_\beta^{\#})^* = W_\wedge \end{aligned}$$

Similarly, using equality (214), we compute

$$\begin{aligned} \bar{W}_\vee \bar{Z} &= W_\vee K_\vee^* \bar{Z} = W_\vee Z K_\wedge^* \\ &= W_+^e (Q_\alpha^{\#} \vee_L Q_\beta^{\#})^* (Q_\alpha^{\#} \vee_L Q_\beta^{\#})(Q_\alpha^{\#} \wedge_R Q_\beta^{\#})^* K_\wedge^* \\ &= W_\wedge K_\wedge^* = \bar{W}_\wedge \end{aligned}$$

(10) We compute

$$\begin{aligned} (S_{W_\alpha} \cap S_{W_\beta}) \cap (\bar{S}_{W_\alpha} + \bar{S}_{W_\beta}) &= H_+^2 (Q_\alpha^{\#} \vee_L Q_\beta^{\#}) \cap H_-^2 K_\vee (Q_\alpha^{\#} \vee_L Q_\beta^{\#}) \\ &= (H_+^2 \cap H_-^2 K_\vee)(Q_\alpha^{\#} \vee_L Q_\beta^{\#}) \\ &= H_r(K_\vee)(Q_\alpha^{\#} \vee_L Q_\beta^{\#}) \end{aligned}$$

Similarly, we compute

$$\begin{aligned} (S_{W_\alpha} + S_{W_\beta}) \cap (\bar{S}_{W_\alpha} \cap \bar{S}_{w_\beta}) &= H_+^2 (Q_\alpha^{\#} \vee_R Q_\beta^{\#}) \cap H_-^2 (\bar{Q}_\alpha^{\#} \wedge_R \bar{Q}_\beta^{\#}) K_+^e \\ &= H_+^2 (Q_\alpha^{\#} \wedge_R Q_\beta^{\#}) \cap H_-^2 K_\wedge (Q_\alpha^{\#} \wedge_R Q_\beta^{\#}) \\ &= (H_+^2 \cap H_-^2 K_\wedge)(Q_\alpha^{\#} \wedge_R Q_\beta^{\#}) \\ &= H_r(K_\wedge)(Q_\alpha^{\#} \wedge_R Q_\beta^{\#}) \end{aligned}$$

□

Corollary 13: *The set $\mathcal{W}_0$ of all minimal, stable, internal spectral factors is a complete lattice.*

Proof: We apply the previous theorem for the case $R = I$. □

The following is our version of Theorem 6.8 in Lindquist and Picci (1991).

Theorem 11:

(1) *Let $X_i \in \mathcal{X}^{m_0}$, $i = 1, 2$. Then the following statements are equivalent:*
   (a) $X_1 \le X_2$
   (b) $S_1 \subset S_2$
   (c) $\bar{S}_2 \subset \bar{S}_1$
(2) *Let $X_1 \in \mathcal{X}^{m_0}$, $X_2 \in \mathcal{X}^{m}$. Then the following statements are equivalent:*
   (a) $X_1 \le X_2$
   (b) $S_1 \subset S_2$
   (c) $\bar{S}_2 \pi_1 \subset \bar{S}_1$
(3) *Let $X_2 \in \mathcal{X}^{m_0}$, $X_1 \in \mathcal{X}^{m}$. Then the following statements are equivalent:*
   (a) $X_1 \le X_2$
   (b) $S_1 \pi_1 \subset S_2$
   (c) $\bar{S}_2 \subset \bar{S}_1$

Proof:

(1) Assume $X_1 \le X_2$. By Theorem 8, this is equivalent to $W_1 Q = W_2$ or to $Q_1'' = Q Q_2''$. We compute

$$S_{W_1} = H_+^2 Q_1'' = H_+^2 Q Q_2'' \subset H_+^2 Q_2'' = S_{W_2}$$

Conversely, assume $S_{W_1} \subset S_{W_2}$, i.e. $H_+^2 Q_1'' \subset H_+^2 Q_2''$. This inclusion implies the existence of an inner function Q such that $Q_1'' = Q Q_2''$. Therefore

$$W_2 = W_+^e (Q_2'')^* = W_+^e (Q_1'')^* Q = W_1 Q$$

that is $W_1 \le W_2$ and hence $X_1 \le X_2$.

Assume $W_1 < W_2$. Since both factors are internal, this means that there exists an inner function Q for which $W_2 = W_1 Q$. In turn we have, using the standard notation, that $K_1 Q = \bar{Q} K_2$. We proceed to compute

$$\bar{S}_2 = H_-^2 K_2 Q_2'' = H_-^2 \bar{Q}^* K_1 Q Q_2''$$

$$= H_-^2 \bar{Q}^* K_1 Q_1'' \subset H_-^2 K_1 Q_1'' = \bar{S}_1$$

Conversely, assume $\bar{S}_2 \subset \bar{S}_1$, i.e. $H_-^2 K_2 Q_2'' \subset H_-^2 K_1 Q_1''$. This means the existence of an inner function $\underline{Q}$ for which $\underline{Q} K_2 Q_2'' = K_1 Q_1''$. Since $K_i Q_i'' = \bar{Q}_i'' K_+^e$, we have $\overline{Q Q_2''} = \bar{Q}_1''$. This means that $W_1 \le W_2$ and hence that $W_1 \le W_2$. This is equivalent to $X_1 \le X_2$.

(2) Assume $X_1 \le X_2$ and X_1 internal. Let X_{2-} be the greatest internal lower bound for X_2. By Proposition 21 we have $S_{2-} = S_2 \cap H_0$. Thus, by part 1,

$$S_1 \subset S_{2-} = S_2 \cap H_0 \subset S_2$$

Conversely, assume $S_1 \subset S_2$. This implies

$$S_1 = S_1 \cap H_0 \subset S_2 \cap H_0 = S_{2-}$$

or $X_1 \leq X_{2-} \leq X_2$.

Assuming $X_1 \leq X_2$ we obtain as before $X_1 \leq X_{2-}$. Since X_1, X_{2-} are both internal, we can apply part 1, to get $\bar{S}_{2-} \subset \bar{S}_1$. Since, by Proposition 21, we have $\bar{S}_{2-} = \bar{S}_2 \pi_1$, it follows that $\bar{S}_2 \pi_1 \subset \bar{S}_1$.

Conversely, assuming the last inclusion, we get $X_1 \leq X_{2-} \leq X_2$.

(3) This follows from the previous part by symmetry, or can be proved directly in an analogous way. □

To get some feeling about external factors, we work out a very simple example.

Example: Let us take the minimum phase spectral factor

$$W_- = \frac{z+1}{z+3}$$

and the corresponding maximum phase spectral factor

$$W_+ = \frac{1-z}{z+3}$$

Given a pair of complex numbers α, β, satisfying $|\alpha|^2 + |\beta|^2 = 1$, then

$$W = \frac{1}{z+3}\left[\alpha(1+z)\beta(1-z)\right] \tag{215}$$

is a 1×2 spectral factor. We set

$$W_-^e = \begin{pmatrix} \frac{z+1}{z+3} & 0 \end{pmatrix}$$

It is easily computed that

$$Q' = \begin{pmatrix} \alpha & \beta\frac{1-z}{1+z} \\ -\bar{\beta} & \bar{\alpha}\frac{1-z}{1+z} \end{pmatrix}, \qquad Q'' = \begin{pmatrix} \bar{\alpha}\frac{1-z}{1+z} & -\beta\frac{1-z}{1+z} \\ \bar{\beta} & \alpha \end{pmatrix}$$

$$Q'Q'' = \begin{pmatrix} \frac{1-z}{1+z} & 0 \\ 0 & \frac{1-z}{1+z} \end{pmatrix} = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

For the DSS factorization we have

$$\bar{W} = \frac{1}{z-3}\left[\alpha(1+z)\beta(1-z)\right]$$

and

$$K = \frac{z-3}{z+3}$$

For $f \in H_r(K_+^e)$, which up to a constant factor, is given by

$$f = \begin{pmatrix} \frac{1}{z+3} & 0 \end{pmatrix}$$

we compute now,

$$\begin{aligned}
fP_{H(Q'')} &= \begin{pmatrix} \frac{1}{z+3} & 0 \end{pmatrix} (Q'')^* P_- Q'' \\
&= \begin{pmatrix} \frac{1}{z+3} & 0 \end{pmatrix} \begin{pmatrix} \alpha\frac{1+z}{1-z} & \beta \\ -\bar{\beta}\frac{1+z}{1-z} & \bar{\alpha} \end{pmatrix} P_- Q'' \\
&= \begin{pmatrix} \frac{1+z}{(z+3)(1-z)} & \beta\frac{1}{z+3} \end{pmatrix} P_- Q'' \\
&= \begin{pmatrix} \frac{1}{2}\frac{1}{1-z} - \frac{1}{2}\frac{1}{z+3} & \beta\frac{1}{z+3} \end{pmatrix} P_- Q'' \\
&= \begin{pmatrix} \frac{1}{2}\frac{1}{1-z} & 0 \end{pmatrix} \begin{pmatrix} \bar{\alpha}\frac{1-z}{1+z} & -\beta\frac{1-z}{1+z} \\ \bar{\beta} & \alpha \end{pmatrix} \\
&= \begin{pmatrix} \frac{|\alpha|^2}{2}\frac{1}{1+z} & \frac{-\alpha\beta}{2}\frac{1}{1+z} \end{pmatrix}
\end{aligned}$$

This implies

$$\|fP_{H(Q'')}\|^2 = \|f\|^2 \left\{ \frac{|\alpha|^4}{4} + \frac{|\alpha|^2|\beta|^2}{4} \right\} = \|f\|^2 \frac{|\alpha|^2}{4}(|\alpha|^2 + |\beta|^2) = \|f\|^2 \frac{|\alpha|^2}{4}$$

Equivalently $\|fP_{H(Q'')}\| = \|f\|(|\alpha|/2)$. The maximal value is obtained for $|\alpha| = 1$.

Using this we compute

$$\|fP_{H(K_-^e)Q_+^e}\|^2 = \|f\|^2 - \frac{\|f\|^2}{4} = \frac{3\|f\|^2}{4}$$

or $\|fP_{H(K_-^e)Q_+^e}\| = \sqrt{3}\|f\|/2$. Thus the angle between $H(K_-^e)Q_+^e$ and $H_r(K_+^e)$ is, in this case, $\pi/3$.

Note that

$$\lim_{|\alpha| \mapsto 0} \|H_r(K_+^e)|P_{H_r(K)Q''}\| = 1$$

but

$$\lim_{|\alpha| \mapsto 0} Q'' \neq I$$

Next, we show that, except when $\alpha = 0$, we have $H(K_+^e) \cap H_+^2 Q'' = \{0\}$. Indeed, $f \in H(K_+^e) \cap H_+^2 Q''$ if and only if $f \in H(K_+^e)$ and $f(Q'')^* \in H_+^2$. Now

$$\begin{pmatrix} \frac{1}{z+3} & 0 \end{pmatrix} \in H_+^2 Q''$$

if and only if

$$\begin{pmatrix} \frac{1}{z+3} & 0 \end{pmatrix} \begin{pmatrix} \alpha\frac{1+z}{1-z} & \beta \\ -\bar{\beta}\frac{1+z}{1-z} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha\frac{1+z}{(1-z)(z+3)} & \frac{\beta}{z+3} \end{pmatrix} \in H_+^2$$

which is the case if and only if $\alpha = 0$. Therefore, for all factors W of the form (215), we have $H(K_+^e) \cap H_+^2 Q'' = \{0\}$.

7. The frame space

We turn now to the analysis of a space that contains all the zero and pole information of all spectral factors of given size. We define the *frame space* $H_m^\square$ of the spectral functon Φ as the span of the set $\mathcal{X}^m$ of all minimal Markovian splitting subspaces associated with the set $\mathcal{W}^m$ of all minimal, stable $p \times m$ spectral factors.

We start by considering a special case.

Proposition 23: *Assume Φ to be rank m_0. Let Q_-, Q_+, K_-, K_+ be the inner functions as in figure 1. Then the frame space $H_{m_0}^\square$ is given by*

$$H_{m_0}^\square = H_r(K_- Q_+) = H_r(Q_- K_+) \tag{216}$$

Proof: First, we compute

$$H_r(K)Q'' \subset H_r(KQ'') = H_r(\bar{Q}''K_+) \subset H_r(\bar{Q}'\bar{Q}''K_+) = H_r(Q_- K_+)$$

and hence we get the inclusion

$$\bigvee H_r(K)Q'' \subset H_r(Q_- K_+) \tag{217}$$

To prove the converse, we show that $H_r(Q_- K_+)$ is spanned by the two Markovian splitting subspaces associated with W_- and W_+, namely $H_r(K_-)Q_+$ and $H_r(K_+)$ respectively. From Corollary 3 it follows that the projection map $H_r(Q_+)|P_{H_r(K_+)}$ is injective, which means that $H_+^2 K_+ \cap H_r(Q_+) = \{0\}$. Applying Theorem 1, we have $H_+^2 Q_+ + H_r(K_+) = H_+^2$. Now, we have the following direct sum decomposition

$$H_+^2 Q_+ = [H_+^2 K_- \oplus H_r(K_-)]Q_+ + H_+^2 K_- Q_+ \oplus H_r(K_-)Q_+$$

and

$$H_+^2 = H_+^2 K_- Q_+ \oplus H_r(K_- Q_+)$$

Taken together, this implies $H_r(K_-)Q_+ + H_r(K_+) = H_r(K_- Q_+)$, and so we get the inclusion

$$\bigvee H_r(K)Q'' \supset H_r(Q_- K_+)$$

The two inclusions imply the equality (216). □

For the general case we will need the following lemma.

Lemma 8: *Let $\mathcal{R}$ be the set of all $k \times k$ inner functions R having the same set of invariant inner factors $r_1, \dots, r_k$. We assume they are ordered so that $r_i | r_{i-1}$, and we set $r_1 = r$. Then*

$$\bigvee_{R \in \mathcal{R}} H_r(R) = H_r(rI_k) \tag{218}$$

Proof: For each $R \in \mathcal{R}$, the inner function r is its minimal inner function. This of course implies the inclusion $H_r(R) \subset H_r(rI)$ and hence

$$\bigvee_{R \in \mathcal{R}} H_r(R) \subset H_r(rI_k)$$

To see that this inclusion is actually an equality, it suffices to consider the set of all diagonal inner functions of the form $R_\sigma = \text{diag}\,(r_{\sigma 1}, \dots, r_{\sigma k})$ with $\sigma \in S_k$, i.e. a permutation. Clearly we have $R_\sigma \in \mathcal{R}$ and

$$\bigvee_{\sigma \in S_k} H_r(R_\sigma) = H_r(rI_k)$$

and hence (218) follows. $\square$

We pass on now to the general case.

Theorem 12: *Let Φ be a $p \times p$, rank m_0 spectral function. Let r be the scalar minimal inner function of Q_+, or equivalently of Q_-. Then the frame space $H_m^\square$ of Φ is given by*

$$H_m^\square = H_r(K_- Q_+) \oplus H_r(rI_{m-m_0}) \tag{219}$$

Proof: First, we compute

$$\begin{aligned} H_r(K)Q \subset H_r(KQ'') &= H_r(\bar{Q}'' K_+^e) \\ \subset H_r(Q'Q''K_+^e) &= H_r\begin{pmatrix} Q_- K_+ & 0 \\ 0 & R \end{pmatrix} \\ &= H_r(Q_- K_+) \oplus H_r(R) \subset H_r(Q_- K_+) \oplus H_r(rI) \end{aligned}$$

Thus $\bigvee H_r(K)Q'' \subset H_r(Q_- K_+) \oplus H_r(rI)$, i.e.

$$H_m^\square \subset H_r(Q_- K_+) \oplus H_r(rI) = H_{m_0}^\square \oplus H_r(rI_{m-m_0})$$

Obviously, we have $H_{m_0}^\square \oplus \{0\} \subset H_m^\square$, so it suffices to show that $\bigvee H_r(K)Q''\pi_2 = H_r(rI_{m-m_0})$. Choose any R satisfying $R \simeq Q_+$. Let

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

be a balanced factorization which, by Proposition 16, exists. Define $W = W_-^e Q'$ and let $W = \bar{W}K$ be a DSS factorization. Now, by (103), we have $H_r(K)Q''\pi_2 = H_r(R)$, and an application of Lemma 8 completes the proof. $\square$

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