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Paul A. Fuhrmann [a]
[a] Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva, Israel

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Linear feedback via polynomial models

PAUL A. FUHRMANN†

This paper is an attempt to study state feedback from the module theoretic point of view. It uses the theory of polynomial models to study state feedback in a way which emphasizes both the module theoretic aspects as well as the state space point of view.

1. Introduction

Beginning with Kalman's work (Kalman *et al.* 1969), module theory has proved to be a natural setting for the study of linear systems. While this approach was very powerful in the study of the realization problem and isomorphism results it took a longer time for module theory to prove its usefulness in problems of design and particularly the problem of feedback.

While the pole placement problem has been solved definitively by Rosenbrock (1970), certain of the matrix manipulations seemed formal and devoid of a system or module theoretic interpretation. One of the first attempts at the study of feedback in a module theoretic setting has been made by Eckberg (1974). Significant results have been obtained by Hautus and Heyman (1978) who made an analysis of feedback equivalence in terms of polynomial modules. In their work feedback was studied solely from the input–output point of view. This obscures the fact that state feedback is essentially a state space phenomenon.

More recent is the work of Münzner and Prätzel-Wolters (1978) which applies Forney's theory of minimal bases (Byrnes and Gauger 1977) as well as module theory to the study of feedback. This important paper contains also a derivation of the Brunovsky (1970) canonical form as well as a proof of Rosenbrock's theorem derived within the module theoretic framework.

This paper is in the spirit of previous publications (Fuhrmann 1976, 1977, 1978) in which an attempt has been made to reconcile state space theory, polynomial modules and the theory of polynomial system matrices.

Basically, the reconciliation of the various existing theories of multivariable linear system that has been attempted previously (Fuhrmann 1976, 1977, 1978) hinges on the idea of functional (polynomial) models for linear transformations and linear systems. This approach has the added advantage that it generalizes, *modulo* the expected and unexpected technical difficulties, to certain infinite dimensional situations. This unification is the theme of a forthcoming monograph (Fuhrmann 1979). These functional models will be used here once again in the study of feedback. The idea is to replace a given pair (A, B) by an isomorphic functional model and to find in this model the proper representation of the feedback group. While this representation

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† Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva, Israel.

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remains elusive so far, certain elementary operations on the model, and hence on the corresponding polynomial system matrix, can be shown to arise out of state feedback. These suffice for the derivation of Brunovsky's canonical form and hence generate the entire feedback group.

The situation at hand is analogous to the problem of deciding when two $n \times n$ matrices A and A_1 with elements in a field F are similar. A classical theorem, going back to Frobenius states that A and A_1 are similar if and only if the polynomial matrices $\lambda I - A$ and $\lambda I - A_1$ are equivalent. This of course is equivalent to bringing $\lambda I - A$ to $\lambda I - A_1$ by a finite sequence of elementary row and column operations. The question of similarity is answered by reducing $\lambda I - A$ and $\lambda I - A_1$ to their Smith canonical form by applying the invariant factor algorithm. For more on the problem of similarity one can refer to Byrnes and Gauger (1977). This paper adopts a similar philosophy as far as the problem of feedback equivalence is concerned, the difficulties arising out of the more complicated structure of the feedback group.

My interest in the problems discussed in this paper has been aroused by Sanjoy K. Mitter in various conservations from which I greatly benefited.

2. Preliminaries

We recall a few notions introduced earlier (Fuhrmann 1976, 1977, 1978).

Let V be a finite dimensional vector space over an arbitrary field F. We denote by $V((\lambda^{-1}))$ the set of all truncated Laurent series with coefficients in V, that is the set of all formal sums of the form $\sum_{n>k} v_{-n}\lambda^{-n}$ with $v_n \in V$ and $k \in Z$. $V((\lambda^{-1}))$ is a module over either of the rings $F((\lambda^{-1}))$ and $F[\lambda]$ the ring of polynomials over F. $V[\lambda]$ is the $F[\lambda]$-submodule of $V((\lambda^{-1}))$ consisting of vector polynomials.

Consider as $F[\lambda]$-modules we have the following short exact sequence of module homomorphisms

$$0 \rightarrow V[\lambda] \xrightarrow{j} V((\lambda^{-1})) \xrightarrow{\pi_-} V((\lambda^{-1}))/V[\lambda] \rightarrow 0 \tag{2.1}$$

where j is the embedding of $V[\lambda]$ into $V((\lambda^{-1}))$ and π_- is the canonical projection of $V((\lambda^{-1}))$ onto the quotient module $V((\lambda^{-1}))/V[\lambda]$. Since $V((\lambda^{-1}))$ has a natural direct sum decomposition.

$$V((\lambda^{-1})) = V[\lambda] \oplus \lambda^{-1} V[[\lambda^{-1}]] \tag{2.2}$$

where $\lambda^{-1} V[[\lambda^{-1}]]$ is the set of all formal power series in λ^{-1} with vanishing constant term. Thus $V((\lambda^{-1}))/V[\lambda]$ can be identified with $\lambda^{-1} V[[\lambda^{-1}]]$ and we will use this identification in the sequel. The projection complementary to π_-, that is the projection of $V((\lambda^{-1}))$ onto $V[\lambda]$ will be denoted by π_+.

$V[\lambda]$, being a submodule of $V((\lambda^{-1}))$ has clearly an $F[\lambda]$-module structure. In its turn $\lambda^{-1} V[[\lambda^{-1}]]$ has an induced $F[\lambda]$-module structure which is given by

$$p \cdot v = \pi_-(pv) \quad \text{for} \quad p \in F[\lambda], \quad v \in \lambda^{-1} V[[\lambda^{-1}]] \tag{2.3}$$

Given a non-singular element $D \in (V, V)_F[\lambda]$, that is we assume det D is a non-zero polynomial, we can use the two projections π_+ and π_- to construct two projection operators acting in $V[\lambda]$ and $\lambda^{-1}V[[\lambda^{-1}]]$ respectively by defining

$$\pi_D f = D\pi_- D^{-1} f \quad \text{for} \quad f \in V[\lambda] \tag{2.4}$$

and

$$\pi^D g = \pi_- D^{-1}\pi_+ D g \quad \text{for} \quad g \in \lambda^{-1} V[[\lambda^{-1}]] \tag{2.5}$$

The following theorem summarizes the basic properties of π_D and π^D.

Theorem 2.1

(*a*) π_D and π^D are projection maps in $V[\lambda]$ and $\lambda^{-1}V[[\lambda^{-1}]]$ respectively.

(*b*) Ker $\pi_D = D V[\lambda]$ is a submodule of $V[\lambda]$. Moreover, K_D defined by

$$K_D = \text{Range } \pi_D \tag{2.6}$$

with the $F[\lambda]$-module structure given by

$$p \cdot f = \pi_D(pf) \quad \text{for} \quad p \in F[\lambda], \quad f \in K_D \tag{2.7}$$

is a finitely generated torsion module isomorphic to $V[\lambda]/D V[\lambda]$.

(*c*) $L_D = \text{Range } \pi^D$ is a finitely generated torsion submodule of $\lambda^{-1}V[[\lambda^{-1}]]$.

(*d*) K_D and L_D are isomorphic $F[\lambda]$-modules, the isomorphism $\rho_D : L_D \to K_D$ and its inverse $\rho_D^{-1} : K_D \to L_D$ given by

$$\rho_D g = Dg \tag{2.8}$$

and

$$\rho_D^{-1} f = \pi_- D^{-1} f \tag{2.9}$$

respectively.

Introduce now the shift operator S in $V((\lambda^{-1}))$ by

$$(Sv)(\lambda) = \lambda v(\lambda) \tag{2.10}$$

The submodule $V[\lambda]$ is invariant under S, thus the restriction of S to $V[\lambda]$ makes sense and we define $S_+ : V[\lambda] \to V[\lambda]$ by

$$S_+ = S \mid V[\lambda] \tag{2.11}$$

In $\lambda^{-1}V[[\lambda^{-1}]]$ which has an induced quotient module structure we define S_- by

$$S_- g = \pi_- S g, \quad g \in \lambda^{-1} V[[\lambda^{-1}]] \tag{2.12}$$

Now L_D is a submodule of $\lambda^{-1}V[[\lambda^{-1}]]$ and hence is S_-invariant. Thus we define

$$S^D = S_- | L_D \tag{2.13}$$

or equivalently

$$S^D g = \pi_- \lambda \cdot g \quad \text{for} \quad g \in L_D \tag{2.14}$$

Now K_D is not a submodule of $V[\lambda]$ but inherits a quotient module structure and hence we let

$$S_D f = \pi_D \lambda \cdot f \quad \text{for} \quad f \in K_D \tag{2.15}$$

Not surprisingly the operators S_D and S^D are related.

Theorem 2.2.

S_D and S^D are isomorphic and the diagram

$$\begin{array}{ccc} L_D & \xrightarrow{\rho_D} & K_D \\ {\scriptstyle S^D}\downarrow & & \downarrow{\scriptstyle S_D} \\ L_D & \xrightarrow{\rho_D} & K_D \end{array}$$

is commutative.

3. The feedback group

Let (A, B) be a reachable pair. As usual we assume $A \in (V, V)_F$ and $B \in (U, V)_F$, where U and V are finite dimensional vector spaces over the field F. U and V are the space of input (or control) values whereas V is the state space (A, B) in the standard notation for the dynamical equation

$$x_{\nu+1} = Ax_\nu + Bu_\nu \tag{3.1}$$

and reachability is equivalent to

$$\bigcap_{\nu \geqslant 0} \text{Ker } B^* A^{*i} = \{0\} \tag{3.2}$$

For background material and definitions we refer to Kalman *et al.* (1969).

If we assume dim $U = m$ and dim $V = n$ then by a choice of bases in U and V they may be identified with F^m and F^n respectively. In that case A and B are represented by $n \times n$ and $n \times m$ matrices respectively.

If we augment (3.1) by the identity readout map, i.e. by

$$y_\nu = x_\nu \tag{3.3}$$

then the transfer function of the triple (A, B, I) is given by

$$T(\lambda) = (\lambda I - A)^{-1} B \tag{3.4}$$

We now pass on to the analysis of *feedback*. Suppose we change (3.1) by specifying

$$u_\nu = Kx_\nu + w_\nu \tag{3.5}$$

with $w_\nu \in U$ as the new input.

Substituting (3.5) back into (3.1) shows that this amounts to transforming the pair (A, B) into the pair $(A + BK, B)$. We call (3.5) a state feedback and say that $(A + BK, B)$ has been obtained from (A, B) by state feedback. Clearly the applications of feedback form a commutative group. If we enlarge the group to the one generated by similarities in U and V as well as state feedbacks we obtain the non-commutative *feedback group* $\mathscr{F}$. Thus an element of $\mathscr{F}$ is a triple of maps (R, K, P) with $R \in (V, V)_F$ and $P \in (U, U)_F$ non-singular and $K \in (U, V)_F$. The feedback group acts on a pair (A, B) by

$$(A, B) \xrightarrow{(R, K, P)} (R^{-1}AR + R^{-1}BK, R^{-1}BP) \tag{3.6}$$

This implies that the group composition law is

$$(R, K, P) \circ (R_1, K_1, P_1) = (RR_1, PK_1 + KR_1, PP_1) \tag{3.7}$$

This composition law is clearly associative as it can be expressed in terms of matrix multiplications (Brockett 1977) as follows

$$\begin{pmatrix} R & 0 \\ K & P \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ K_1 & P_1 \end{pmatrix} = \begin{pmatrix} RR_1 & 0 \\ KR_1 + PK_1 & PP_1 \end{pmatrix} \tag{3.8}$$

This also clearly shows that

$$(R, K, P)^{-1} = (R^{-1}, -P^{-1}KR^{-1}, P^{-1}) \tag{3.9}$$

which shows that $\mathscr{F}$ is a bona fide group.

It is clear from the matrix representation of the feedback group that every element of $\mathscr{F}$ is the product of three types of elements :

(i) similarity, or change of basis, in the state space, i.e. elements of the form $(R, 0, I)$ with R invertible ;

(ii) similarity, or change of basis, in the input space, i.e. elements of the form $(I, 0, P)$ with P invertible ; and finally

(iii) pure feedbacks, i.e. elements of the form (I, K, I).

Indeed we clearly have

$$(R, K, P) = (R, 0, I) \circ (I, K, I) \circ (I, 0, P) \tag{3.10}$$

The feedback group $\mathscr{F}$ induces a natural equivalence relation in the set of reachable pairs (A, B) with state space and input space V and U respectively. Thus (A, B) and (A_1, B_1) are *feedback equivalent* if there is an element (R, K, P) of $\mathscr{F}$ which transforms (A, B) into (A_1, B_1). It is easily checked that this is indeed an equivalence relation. The equivalence classes are called the orbits of the group $\mathscr{F}$ and we are interested in characterizing the orbit invariants, and the specification of an element in the orbit, a canonical form, which exhibits the orbit invariants.

For the method that follows we will have to enlarge the notion of feedback equivalence. Thus if (A_1, B_1) is another reachable pair with state space and input space V_1 and U_1, respectively, we say that (A_1, B_1) is feedback equivalent to (A, B) if there exist invertible maps $P : U_1 \to U$ and $R : V_1 \to V$ such that (RA_1R^{-1}, RB_1P^{-1}) is feedback equivalent to (A, B).

The feedback group has been introduced through a state space formalism. However, many aspects of linear system theory are easier to handle if we operate with polynomial system matrices. We saw in Fuhrmann (1977) how with each polynomial system matrix corresponding to a factorization of a transfer function there is associated a special state space realization which is based on the use of the canonical models introduced in Fuhrmann (1976).

Our main object will be the analysis of the feedback group action in terms of these canonical models and their associated polynomial system matrices.

Thus given a reachable pair (A, B) then $\lambda I - A$ and B are left coprime polynomial matrices. Since with every left coprime factorization of a rational function, there is an associated right coprime factorization we can write

$$(\lambda I - A)^{-1}B = H(\lambda)D(\lambda)^{-1} \tag{3.11}$$

where $(H, D)_R = I$. Furthermore, H and D in the coprime factorization (3.11) are determined up to a common unimodular factor on the right.

It follows from the results of Fuhrmann (1976, 1977) that the pair (S_D, π_D) is a reachable pair, with state space K_D, which is isomorphic to (A, B). With (S_D, π_D) we associate the polynomial matrix $(D \quad I)$ which is just the upper row of Rosenbrock's polynomial system matrix (Rosenbrock 1970).

To analyse the similarity transformations in the state space the following results from Fuhrmann (1976) will be needed.

Theorem 3.1

A linear map $X : K_D \to K_{D_1}$ satisfies the intertwining relation

$$XS_D = S_{D_1}X \tag{3.12}$$

if and only if it is of the form

$$Xf = \pi_{D_1} \Xi f \tag{3.13}$$

for some Ξ and $\Xi_1 \in (V, V_1)_F[\lambda]$ which satisfy

$$\Xi D = D_1 \Xi \tag{3.14}$$

The map X of (3.13) is injective if and only if the coprimeness condition $(D, \Xi_1)_R = I$ holds and surjective if and only if $(\Xi, D_1)_L = I$.

The following theorem is due to Hautus and Heymann (1978) who give a completely different proof. This theorem allows us to study feedback in the context of polynomial system matrices.

Theorem 3.2

Let (A, B), with $A \in (V, V)_F$ and $B \in (U, V)_F$, be a reachable pair and let $H(\lambda)D(\lambda)^{-1}$ be a right coprime factorization of $(\lambda I - A)^{-1}B$. Then a necessary and sufficient condition for a reachable pair (A_1, B_1) to be feedback equivalent to (A, B) is that

$$(\lambda I - A_1)^{-1}B_1 = RH(\lambda)(D(\lambda) + Q(\lambda))^{-1}P^{-1} \tag{3.15}$$

for some $Q \in (U, U)_F[\lambda]$ for which QD^{-1} is strictly proper and invertible maps R and P in $(V, V)_F$ and $(U, U)_F$ respectively.

Proof

Assume $T(\lambda) = (\lambda I - A)^{-1}B = H(\lambda)^{-1}D(\lambda)$ are coprime factorizations, and let (A_1, B_1) be feedback equivalent to (A, B). Thus there exist invertible maps $R : V \to V$ and $P : U \to U$ such that $A_1 = R(A + BK)R^{-1}$ and $B_1 = RBP^{-1}$.

Hence

$$(\lambda I - A_1)^{-1}B_1 = (R(\lambda I - A - BK)^{-1}R^{-1})^{-1}RBP = R(\lambda I - A - BK)^{-1}BP^{-1}$$

Now

$$\begin{aligned}(\lambda I - A - BK)^{-1}B &= [(\lambda I - A)(I - (\lambda I - A)^{-1}BK)]^{-1}B \\ &= (I - (\lambda I - A)^{-1}BK)^{-1}(\lambda I - A)^{-1}B \\ &= (I - T(\lambda)K)^{-1}T(\lambda)\end{aligned}$$

But from the equality

$$T(\lambda)(I - KT(\lambda)) = (I - T(\lambda)K)T(\lambda)$$

it follows that

$$(I - T(\lambda)K)^{-1}T(\lambda) = T(\lambda)(I - KT(\lambda))^{-1}$$

Hence it follows that

$$\begin{aligned}T_f(\lambda) = (\lambda I - A_1)^{-1}B_1 &= RT(\lambda)(I - KT(\lambda))^{-1}P^{-1} \\ &= RH(\lambda)D(\lambda)^{-1}(I - KH(\lambda)D(\lambda)^{-1})^{-1}P^{-1} \\ &= RH(\lambda)(D(\lambda) - KH(\lambda))^{-1}P^{-1}\end{aligned}$$

If we put $Q(\lambda) = -KH(\lambda)$ then clearly $T_f(\lambda) = RH(\lambda)(D(\lambda) + Q(\lambda))^{-1}P^{-1}$ and $QD^{-1} = -KT$ is strictly proper.

This proves the necessity part of the theorem. The proof of sufficiency is delayed till we compile some additional information of independent interest.

4. On multivariable control canonical forms

It is widely known that the control canonical form plays a central role in the proof of the pole shifting theorem in the single input case. Thus it might be natural to expect that a suitable generalization will play a role in the study of the general case of state feedback. In this section we make an effort of obtaining such a generalization which leads naturally to the study of a class of Toeplitz operators.

We saw previously the role the $F[\lambda]$-modules K_D and L_D played in the study of finite dimensional linear systems. From the computational point of view it is important to find a suitable parametrization of these modules and thus we proceed to do.

Let D be a non-singular element of $(U, U)_F[\lambda]$, have the representation

$$D(\lambda) = D_0 + D_1\lambda + \dots + D_s\lambda^s \tag{4.1}$$

Since we have the direct sum decomposition

$$\lambda^{-1}U[[\lambda^{-1}]] = U_s[\lambda^{-1}] \oplus \frac{1}{\lambda^{s+1}}\, U[[\lambda^{-1}]]$$

where

$$U_s[\lambda^{-1}] = \pi_-\frac{1}{\lambda^s}\, U[\lambda] = \left\{ \sum_{j=1}^{s} \frac{\xi_j}{\lambda^j} \middle|, \quad \xi_j \in U \right\}$$

it clearly follows that for every

$$y \in \frac{1}{\lambda^{s+1}} U[[\lambda^{-1}]]$$

we have, with

$$y(\lambda) = \frac{1}{\lambda^{s+1}} y'(\lambda) \quad \text{and} \quad y' \in U[[\lambda^{-1}]]$$

that

$$\pi_+ D(\lambda) y(\lambda) = \pi_+ \frac{D(\lambda)}{\lambda^{s+1}} y'(\lambda) = 0 \tag{4.2}$$

Thus to obtain all the vectors in L_D it suffices to consider the linear combinations of the projections onto L_D of the vectors $\left\{ \frac{\xi}{\lambda^j} \middle| , \quad 1 \leqslant j \leqslant s, \quad \xi \in U \right\}$.

For $1 \leqslant j \leqslant s$

$$\pi_+ \frac{D\xi}{\lambda^j} = (D_j + \ldots + D_s \lambda^{s-j})\xi \tag{4.3}$$

Let us define now $s+1$ polynomials in $(U, U)_F[\lambda]$ by

$$E_j(\lambda) = \begin{cases} 0, & j = 0 \\ D_j + D_{j+1}\lambda + \ldots + D_s \lambda^{s-j}, & 1 \leqslant j \leqslant s \end{cases} \tag{4.4}$$

Equation (4.2) can be rewritten now as

$$\pi_+ \frac{D\xi}{\lambda^j} = E_j(\lambda)\xi, \quad 1 \leqslant j \leqslant s \tag{4.5}$$

and so

$$\pi^D \frac{\xi}{\lambda^j} = \pi_- D^{-1} E_j \xi \tag{4.6}$$

So for L_D we have the representation

$$L_D = \left\{ \sum_{j=1}^{s} \pi_- D^{-1} E_j \xi_j \middle| , \quad \xi_j \in U, \quad 1 \leqslant j \leqslant s \right\} \tag{4.7}$$

Multiplication by D maps L_D onto K_D and, recalling the definition of the projection π_D, we obtain

$$K_D = \{ \sum \pi_D E_j \xi_j |, \quad \xi_j \in U, \quad 1 \leqslant j \leqslant s \} \tag{4.8}$$

We shall call representation (4.8) of K_D the *control representation* of K_D.

The usefulness of the control representation (4.8) of K_D becomes apparent in the study of the operator S_D. Indeed we have the following result.

Theorem 4.1

Let $S_D : K_D \to K_D$ be defined by (2.15) and let E_j be defined by (4.4). Then

$$S_D \pi_D E_j(\lambda)\xi = \pi_D E_{j-1}(\lambda)\xi - \pi_D D_{j-1}\xi \tag{4.9}$$

Proof

$$S_D\pi_D E_j(\lambda)\xi = \pi_D\lambda \cdot \pi_D E_j(\lambda)\xi = \pi_D\lambda E_j(\lambda)\xi = \pi_D(E_{j-1}(\lambda) - D_{j-1})\xi$$
$$= \pi_D E_{j-1}(\lambda)\xi = \pi_D D_{j-1}\xi$$

For $j=1$ we have of course

$$S_D\pi_D E(\lambda)\xi = -\pi_D D_0\xi \tag{4.10}$$

In order to obtain some feeling for the preceding theorem let us specialize to the case of a degree s monic polynomial D. Thus

$$D(\lambda) = D_0 + \ldots + D_{s-1}\lambda^{s-1} + I\lambda^s$$

In this case

$$\rho_D\pi^D \frac{\xi}{\lambda^j} = \pi_D E_j(\lambda)\xi = E_j(\lambda)\xi$$

This implies that

$$S_D E_j(\lambda)\xi = E_{j-1}(\lambda)\xi - D_{j-1}\xi \tag{4.11}$$

Since K_D coincides with all vector polynomials of degree $s-1$ then each such vector polynomial $u(\lambda)$ can be uniquely expressed in the form

$$u(\lambda) = \sum_{j=1}^{s} E_j(\lambda)\xi_j$$

If we map bijectively K_D onto U^s by mapping $\sum_{j=1}^{s} E_j(\lambda)\xi_j$ into $\begin{pmatrix}\xi_1\\ \vdots \\ \xi_s\end{pmatrix}$ we obtain for S_D the block matrix representation

$$\begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & I \\ -D_0 & -D_1 & -D_2 & \ldots & -D_{n-1} \end{bmatrix} \tag{4.12}$$

But this is just the classical control canonical form for S_D.

That D be monic is not necessary for $\pi_D E_j\xi = E_j\xi$ to hold. In fact we have the following.

Lemma 4.2

$D \in (U, U)_F[\lambda]$. If D^{-1} is proper then

$$\pi_D E_j\xi = E_j\xi \quad \text{for all } \xi \in U \quad \text{and} \quad 1 \leqslant j \leqslant s \tag{4.13}$$

Proof

$$\pi_D E_j\xi = D\pi_- D^{-1}E_j\xi = D\pi_- D^{-1}\frac{(D(\lambda) - (D_0 + \ldots + D_{j-1}\lambda^{j-1}))}{\lambda^j}\xi$$
$$= D\pi_-\left(\frac{\xi}{\lambda^j} - D(\lambda)^{-1}\frac{(D_0 + \ldots + D_{j-1}\lambda^{j-1})}{\lambda^j}\xi\right)$$

Now

$$\frac{\xi}{\lambda^j} \quad \text{and} \quad \frac{D_0+\ldots+D_{j-1}\lambda^{j-1}}{\lambda^j}$$

are strictly proper whereas $D(\lambda)^{-1}$ is proper by assumption. Thus also

$$D(\lambda)^{-1}\frac{(D_0+\ldots+D_{j-1}\lambda^{j-1})\xi}{\lambda^j}$$

is strictly proper and hence

$$\pi_-\left(\frac{\xi}{\lambda^j}-D(\lambda)^{-1}\frac{(D_0+\ldots+D_{j-1}\lambda^{j-1})}{\lambda^j}\,\xi\right)=\frac{\xi}{\lambda^j}-D(\lambda)^{-1}\frac{(D_0+\ldots+D_{j-1}\lambda^{j-1})\xi}{\lambda^j}$$

and this implies (4.13).

The next theorem is a key result in the study of state feedback and provides the key to the proof of the sufficiency part of Theorem 3.2.

Theorem 4.3

Let Q, $D\in(U,U)_F[\lambda]$ with D non-singular and QD^{-1} strictly proper. Let $D_1=D+Q$ and let E_j and E'_j be the polynomials associated with D and D_1 respectively that are defined by (4.4). Then the map X defined by

$$Xf=\pi_+D_1D^{-1}f \quad \text{for} \quad f\in K_D \tag{4.14}$$

is an invertible map of K_D onto K_{D_1} that satisfies

$$X\pi_D E_j\xi=\pi_{D_1}E'_j\,\xi \tag{4.15}$$

Proof

Assume $D(\lambda)=D_0+D_1\lambda+\ldots+D_s\lambda^s$. Since $D_1D^{-1}=I+QD^{-1}$ with QD^{-1} strictly proper it follows that $D_1(\lambda)=D'_0+D'_1\,\lambda+\ldots+D'_s\,\lambda^s$. Let D_1D^{-1} have the expansion

$$D_1(\lambda)D(\lambda)^{-1}=I+\frac{\Gamma_1}{\lambda}\frac{\Gamma_2}{\lambda^2}+\ldots \tag{4.16}$$

or

$$D_1(\lambda)=\left(I+\frac{\Gamma_1}{\lambda}+\frac{\Gamma_2}{\lambda^2}+\ldots\right)D(\lambda) \tag{4.17}$$

By equating coefficients we obtain

$$\left.\begin{aligned} D'_s&=D_s\\ D'_{s-1}&=D_{s-1}+\Gamma_1D_s\\ &\;\vdots\\ D'_0&=D_0+\Gamma_1D_1+\ldots+\Gamma_sD_s\end{aligned}\right\} \tag{4.18}$$

or in block matrix form

$$\begin{bmatrix} D'_0 \\ \vdots \\ \vdots \\ \vdots \\ D'_s \end{bmatrix} = \begin{bmatrix} I & \Gamma_1 & \dots & & \Gamma_s \\ 0 & I & \Gamma_1 & \dots & \Gamma_{s-1} \\ & & & I & \Gamma_1 \\ 0 & & & & I \end{bmatrix} \begin{bmatrix} D_0 \\ \vdots \\ \vdots \\ \vdots \\ D_s \end{bmatrix} \tag{4.19}$$

Now

$$\begin{aligned} X\pi_D E_j\xi &= \pi_+ D_1 D^{-1}\pi_D E_j\xi = \pi_+ D_1 D^{-1} D\pi_- D^{-1} E_j\xi = \pi_+ D_1 \pi_- D^{-1} E_j \xi \\ &= \pi_{D_1}\pi_+ D_1 D^{-1} E_j\xi = \pi_{D_1}\pi_+ \left\{\left(I + \frac{\Gamma_1}{\lambda} + \frac{\Gamma_2}{\lambda^2} + \dots\right) E_j(\lambda)\xi\right\} \\ &= \pi_{D_1}\pi_+ \left\{\left(I + \frac{\Gamma_1}{\lambda} + \frac{\Gamma_2}{\lambda^2} + \dots\right)(D_j + D_{j+1}\lambda + \dots + D_s\lambda^{s-j})\xi\right\} \\ &= \pi_{D_1}\{D_s\lambda^{s-j} + (D_{s-1} + \Gamma_1 D_s)\lambda^{s-j-1} + \dots + (D_j + \Gamma_1 D_{j+1} \\ &\qquad\qquad + \dots + \Gamma_{s-j} D_s)\}\xi \\ &= \pi_{D_1}(D'_s + D'_{s-1} + \dots + D'_j\ \lambda^{s-j})\xi = \pi_{D_1} E'_j\ \xi \end{aligned}$$

This shows, by the control representations of K_D and K_{D_1} that X is a map of K_D onto K_{D_1}. If we define Y on K_{D_1} by

$$Yg = \pi_+ D D_1^{-1} g, \quad g \in K_{D_1} \tag{4.20}$$

then it is easily checked that for $f \in K_D$

$$\begin{aligned} YXf &= \pi_+ D D_1^{-1}\ \pi_+ D_1 D^{-1} f = \pi_+ D D_1^{-1}(I - \pi_-) D_1 D^{-1} f \\ &= \pi_+ D D_1^{-1}\ D_1 D^{-1} f - \pi_+ D D_1^{-1}\ \pi_- D_1 D^{-1} f \\ &= f - \pi_+ D D_1^{-1}\ \pi_- D_1 D^{-1} f = f \end{aligned}$$

as $DD_1^{-1}\ \pi_- D_1 D^{-1} f \in \lambda^{-1} U[[\lambda^{-1}]]$.

We conclude that X is also injective, hence invertible. Necessarily $X^{-1} = Y$.

The map X defined by (4.14) relates also the projections π_D and π_{D_1} in a simple way.

Lemma 4.4

Let D and D_1 be as in Theorem 4.3 and let $X : K_D \to K_{D_1}$ be defined by (4.14). Then for every $P \in (U, U)_F$ and $\xi \in U$ we have

$$X\pi_D P\xi = \pi_{D_1} P\xi \tag{4.21}$$

Proof

$$\begin{aligned} X\pi_D P\xi &= \pi_+ D_1 D^{-1}\pi_D P\xi = \pi_+ D_1 D^{-1} D\pi_- D^{-1} P\xi \\ &= \pi_+ D_1 \pi_- D^{-1} P = \pi_{D_1}\pi_+ D_1 D^{-1} P\xi = \pi_{D_1} P\xi \end{aligned}$$

As a corollary to Theorem 4.3 we can state the following result.

Theorem 4.5

With the notation of Theorem 4.3 the operator $\bar{X} : U[\lambda]\to U[\lambda]$ defined by

$$\bar{X}f=\pi_+D_1D^{-1}f \quad \text{for} \quad f\in U[\lambda] \tag{4.22}$$

is an invertible map in $U[\lambda]$.

Proof

We clearly have the direct sum decompositions

$$U[\lambda]=K_D\oplus DU[\lambda]=K_{D_1}\oplus D_1U[\lambda]$$

We saw that X maps K_D bijectively onto K_{D_1}. Moreover, it clearly maps $DU[\lambda]$ bijectively onto $D_1U[\lambda]$ and hence is invertible.

For the following we need the simple lemma which we state without proof.

Lemma 4.6

Let V_0, V_1 and V_2 be finite dimensional vector spaces over the field F and let $A : V_1\to V_0$ and $B : V_2\to V_0$ be linear transformations. Then there exists a linear transformation $C : V_1\to V_2$ such that

$$A=BC \tag{4.23}$$

if and only if

$$\text{Range } A\subset\text{Range } B \tag{4.24}$$

We are now in a position to complete the proof of Theorem 3.2.

Proof of Theorem 3.2 (Sufficiency part)

It suffices to show that $H(\lambda)(D(\lambda)+Q(\lambda))^{-1}=(\lambda I-A_1)^{-1}B_1$ for a pair (A_1, B_1) which is feedback equivalent to (A, B). In that case $RH(\lambda)(D(\lambda)+Q(\lambda))^{-1}P^{-1}$ is associated with (RA_1R^{-1}, RB_1P^{-1}).

Let $D_1=D+Q$, then by Theorem 4.3 the map $X : K_D\to K_{D_1}$ defined by

$$Xf=\pi_+D_1D^{-1}f, \quad f\in K_D \tag{4.25}$$

is invertible and its inverse $Y=X^{-1}$ is given by

$$Yg=\pi_+DD_1^{-1}g, \quad g\in K_{D_1} \tag{4.26}$$

The realization procedure developed by Fuhrmann (1977) associates with the factorizations HD^{-1} and HD_1^{-1} realizations in which the input and state operators are (S_D, π_D) and (S_{D_1}, π_{D_1}) respectively, both of which are reachable. Since H and D are right coprime the realization of HD^{-1} will also be observable which is not necessarily so for HD_1^{-1} as H and D_1 are not always right coprime.

Thus it suffices to show that (S_D, π_D) and (S_{D_1}, π_{D_1}) are feedback equivalent or that for some invertible map $Y : K_{D_1}\to K_D$ and $K : K_D\to U$ we have

$$S_D-YS_{D_1}Y^{-1}=BK \tag{4.27}$$

where $B : U\to K_D$ is given by $B\xi=\pi_D\xi$ for $\xi\in U$. Clearly (4.27) is equivalent to

$$S_DY-YS_{D_1}=BKY=BK_1 \tag{4.28}$$

and hence, considering Lemma 4.6 it suffices to show that

$$\text{Range}\,(S_D Y - Y S_{D_1}) \subset \text{Range}\, B \tag{4.29}$$

and this we proceed to do.

From the control representation of K_{D_1} and K_D we know that they are spanned by vectors of the form $\pi_{D_1} E'_j\, \xi$ and $\pi_D E_j \xi$ respectively. Therefore it suffices to show (4.29) for vectors of this form. Now using (4.15), (4.21) and (4.9) we have

$$\begin{aligned}(S_D Y - Y S_{D_1})\pi_{D_1} E'_j\, \xi &= S_D Y \pi_{D_1} E'_j\, \xi - Y S_{D_1} \pi_{D_1} E'_j\, \xi \\ S_D \pi_D E_j \xi - Y(\pi_{D_1} E'_{j-1}\, \xi - \pi_{D_1} D'_{j-1}\, \xi) &= (\pi_D E_{j-1} \xi - \pi_D D_{j-1} \xi) \\ &\quad - (\pi_D E_{j-1} \xi - \pi_D D_{j-1} \xi) = -\pi_D (D_{j-1} - D'_{j-1}) \xi\end{aligned}$$

which proves the assertion.

5. Brunovsky's canonical form

In this section we apply the information accumulated previously to the study of feedback in terms of polynomial matrices.

Since many of our transformations can be presented as simple matrix operations it will be convenient to identify the vector spaces U and V with F^m and F^n respectively. Thus A and B will denote $n \times n$ and $n \times m$ matrices respectively over F.

Without loss of generality we will assume B to be injective, which is equivalent to rank $B = m$, that is B is assumed to be of full column rank. We noted already that if $H(\lambda)D(\lambda)^{-1}$ is a right coprime factorization of $(\lambda I - A)^{-1} B$ then (S_D, π_D) is isomorphic to (A, B). If $X : K_D \to K_{D_1}$ is an invertible map intertwining S_D and S_{D_1} then it has a representation of the form (3.13) with (3.14) holding together with the coprimeness conditions $(\Xi, D_1)_L = I$ and$(\Xi_1, D)_R = I$.

Let us consider the special cases

$$D_1 = ND \tag{5.1}$$

and

$$D_1 = DM \tag{5.2}$$

where N and M are unimodular matrices. In that case the corresponding coprimeness conditions are automatically satisfied.

In the first case we have (S_D, π_D) transformed into $(S_{D_1}, X\pi_D)$ and since, using (5.1),

$$X \pi_D \xi = \pi_{D_1} N \pi_D \xi = \pi_{D_1} N \xi$$

it follows that (S_D, π_D) is similar to $(S_{D_1}, \pi_{D_1} N)$ which is associated with the polynomial matrix $(ND \quad N)$. From (5.2) it follows that $D = D_1 M^{-1}$ and so the map $X : K_D \to K_{D_1}$ given by $Xf = \pi_{D_1} f$ is invertible and $X \pi_D \xi = \pi_{D_1} \pi_D \xi = \pi_{D_1} \xi$ and the pair (S_{D_1}, π_{D_1}) is associated with the polynomial matrix $(DM \quad I)$. Furthermore, an invertible $m \times m$ matrix P transforms $(S_D, \pi_D L)$ into $(S_D, \pi_D LP)$ and hence $(D \quad L)$ into $(D \quad LP)$. Theorem 3.2 implies that (S_D, π_D) and (S_{D+Q}, π_{D+Q}) are feedback equivalent. Thus in terms of polynomial matrices $(D \quad I)$ transforms into $(D + Q \quad I)$ with the assumption that QD^{-1} is strictly proper. To summarize we have the following.

Theorem 5.1

Let D, Q, N and M be $m \times m$ polynomial matrices with D non-singular, QD^{-1} strictly proper and N and M unimodular and let P be an invertible constant $m \times m$ matrix. Then $(D \quad I)$ and $(N(D+Q)M \quad NP)$ are associated with feedback equivalent pairs.

Of course we recall that left (right) multiplication by a unimodular matrix is equivalent to a finite series of elementary row (column) operations. Our aim is to use the freedom of Theorems 3.1 and 3.2 to reduce $(D \quad I)$ to canonical form. To this end we introduce column properness. Let $D(\lambda)$ be an $m \times m$ non-singular polynomial matrix with its columns given by $D^{(1)}(\lambda), \ldots, D^{(m)}(\lambda)$. We define the degree of $D^{(i)}(\lambda)$, $\deg D^{(i)}(\lambda)$, to be the degree of the highest degree element in $D^{(i)}(\lambda)$. $D(\lambda)$ is called *column proper* if $\deg(\det D(\lambda) = \sum_{i=1}^{m} \deg D^{(i)}(\lambda)$. We quote the following result of Wolovich (1974, Theorem 2.5.7).

Theorem 5.2

Let D be an $m \times m$ non-singular polynomial matrix. Then there exists a unimodular polynomial matrix M such that DM is column proper. If $D^{(i)}$ are the columns of DM we may assume without loss of generality that, with $\kappa_i = \deg D^{(i)}$, $\kappa_1 \geqslant \kappa_2 \geqslant \ldots \geqslant \kappa_m \geqslant 0$.

This theorem yields immediately as a corollary the following result of Brunovsky (1970).

Theorem 5.3

Let D be a $m \times m$ non-singular polynomial matrix with $n = \deg\det D$. Then there exist uniquely determined numbers $\kappa_1 \geqslant \kappa_2 \geqslant \ldots \geqslant \kappa_m \geqslant 0$ with $\sum_{i=1}^{m} \kappa_i = n$ such that $(D \quad I)$ and $(\Delta \quad I)$, $\Delta = \operatorname{diag}(\lambda^{\kappa_1}, \ldots, \lambda^{\kappa_m})$, are associated with feedback equivalent pairs.

Proof

By Theorems 5.1 and 5.2 we may assume without loss of generality that D is column proper with column degrees $\kappa_1 \geqslant \ldots \geqslant \kappa_m$ satisfying $\sum_{i=1}^{m} \kappa_i = n$. By left multiplication with a constant matrix P, the product of constant elementary matrices, we can bring $(D \quad I)$ to $(PD \quad P)$ and PD has the form $PD = \Delta + Q$ with the column degrees of Q less than the corresponding column degrees of Δ. By the similarity P^{-1} in the input space we transform $(\Delta + Q \quad P)$ into $(\Delta + Q \quad I)$. Finally, state feedback transforms $(\Delta + Q \quad I)$ into $(\Delta \quad I)$.

Now, since Δ is diagonal, we have $K = K_{\lambda^{\kappa_1}} \oplus \ldots \oplus K_{\lambda^{\kappa_m}}$. Let $e_1, \ldots, e_m$ be the standard basis in F^m, then the vectors $\{\lambda^i e_j \mid j = 1, \ldots, m, \; i = 0, \ldots, \kappa_{j-1}\}$ are a basis for K_Δ. Relative to these bases in F^m and K_Δ the pair (S_Δ, π_Δ) has the matrix representation

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_m \end{pmatrix} \tag{5.3}$$

with

$$A_j = \begin{pmatrix} 0 & & & \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix}_{\kappa_j \times \kappa_j}, \quad B_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{\kappa_j \times 1} \tag{5.4}$$

and this is the *Brunovsky canonical form.*

To prove uniqueness of the numbers $\kappa_1, \ldots, \kappa_m$ assume (S_Δ, π_Δ) and $(S_{\Delta_1}, \pi_{\Delta_1})$ are feedback equivalent with $\Delta_1 = \text{diag}(\lambda^{\delta_1}, \ldots, \lambda^{\delta_m})$. By Theorem 3.2 we have $\Delta_1 = P(\Delta + Q)$ for some constant invertible matrix P. This in turn implies that $\Delta_1 \Delta^{-1} = P(I + Q\Delta^{-1})$ is proper. Since $\Delta_1 \Delta^{-1} = \text{diag}(\lambda^{\delta_1 - \kappa_1}, \ldots, \lambda^{\delta_m - \kappa_m})$ we have $\kappa_j \geqslant \delta_j$ for $j = 1, \ldots, m$. Equality now follows by symmetry considerations.

The numbers $\kappa_1, \ldots, \kappa_m$ will be called the *reachability indices* of D. Obviously the reachability indices of D are identical to the commonly defined reachability indices of the pair (S_D, π_D).

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