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On observability subspaces

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The paper presents an in depth study of topics in geometric control pertaining to observer theory from a functional point of view. We give characterizations of several classes of subspaces, including observability, almost observability and reconstructibility subspaces. We solve completely the problem of spectral assignability for observer dynamics by generalizing Rosenbrock's pole placement theorem. These results are then applied to observer theory.

1. Introduction

The object of this paper is to study in depth some of the basic objects of geometric control, in the sense of Basile and Marro (1973) and of Wonham and Morse, see Wonham (1979). The principal motivation for us stems from observer theory. This explains the reason that we focus in this paper on the set of conditioned invariant subspaces and the subset of observability subspaces. Indeed, it has been known for a long time that the existence of various classes of observers has, among other, characterizations in terms of geometric control objects. Some of the main references for this are Kawaji (1980), Schumacher (1980), Fuhrmann and Helmke (2001) and Trumpf (2002).

The approaches to the study of observers are as many as are approaches to the study of linear systems. Thus we can consider the problem of constructing observers for partial states, that is linear functions of the state, from the point of view of state space, polynomial system matrices, functional equations, module theory or behaviours, to list the main possibilities.

Lately, there has been renewed interest in a more detailed study of observers which resulted in new results from several different perspectives. One is to be found in Fuhrmann and Helmke (2001) where a detailed analysis of conditioned invariant subspaces and their parametrization is carried out and certain aspects of observer theory are analysed in more detail. The methods are mostly based on the theory of polynomial models introduced in Fuhrmann (1976) and developed further in many subsequent papers. Another source is the thesis (Trumpf 2002), of one of the authors of the present paper that deals also with singular observers, mostly from a state space point of view. Finally, one should mention the behavioural approach to observers. This direction of study has been initiated in Valcher and Willems (1999a) and is important because of the conceptual clarity that it brings to the study of observers. The connections between the classical, state space based, approach to observers and the behavioural approach will be published separately. However, a preliminary version of these results can be found in Fuhrmann (2003a).

The principal results of this paper are given in § 3 and are related to spectral assignment for observers. By that we mean finding a constructive method for observer design that allows as much control as possible on the error dynamics of the observer. To solve this problem, one needs to understand the constraints the system and the choice of observed

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variables impose on the error dynamics. As mentioned above, there exist geometric characterizations for this set of problems, given in terms of conditioned invariant, detectability and observability subspaces. We will use the shift realization to transform the problem to a functional setting. Then we use a functional characterization of conditioned invariant subspaces, obtained in Fuhrmann (1981b), and extend it to cover the case of observability subspaces, a problem that was left open for a long time. The related characterizations are given in Theorem 5 and Corollary 1. Having this characterizations, we proceed to relate the spectral assignment problem to the problem of parametrizing all friends of a given conditioned invariant subspace. In Theorem 6 we show how this is equivalent to a polynomial matrix extension problem. Finally, in Theorem 7, we prove a constructive extension of Rosenbrock's generalized pole placement theorem to the case of quotient spaces.

Section 4 is technical and is devoted to a brief analysis of the reversion operator, an operator used later to illuminate the relation between almost observability and outer reconstructibility subspaces that play a role in the analysis of singular and dead beat observers. In the case of polynomial Brunovsky form, an interesting duality theory is brought to light, and in that context, almost observability and outer reconstructible subspaces turn out to be related by duality.

The concept of an almost observability subspace is the dual to that of an almost controllability subspace, introduced in Willems (1980). The definition of these subspaces, over the real and complex fields, is analytic. However, they have nice algebraic characterizations. Since the present paper deals mostly with discrete time systems over arbitrary fields, we take one of equivalent algebraic characterizations as our definition and continue the study from there. This characterization of almost observability subspaces involves the solution of a Sylvester type equation, much as conditioned invariant subspaces have such a characterization, see Fuhrmann and Helmke (2001).

In § 5, we will develop functional techniques applicable to the study of the classes of singular as well as dead beat observers. As shown in Trumpf (2002), singular observers are related to the class of almost observability subspaces associated with an observable pair (C, A). Since, up to isomorphism, a pair (C, A) is completely determined, via the shift realization, by the denominator in any left coprime factorization $D^{-1}\Phi = C(zI - A)^{-1}$, it seems worthwhile to characterize the set of almost observability subspaces directly in terms of the non-singular polynomial matrix $D(z)$. Such an approach was undertaken in Fuhrmann and Willems (1980), characterizing the class of controlled invariant subspaces and later extended, in Fuhrmann (1981), to the characterization of conditioned invariant subspaces. The resulting, elegant, characterizations were based on module theoretic considerations. It turns out that the functional characterization of almost observability subspaces involves full column rank monomic polynomial matrices, i.e. matrices all whose invariant factors are monomials. There is another class of subspaces in which the functional, or module theoretic, characterization involves monomic polynomial matrices. This is the subset of conditioned invariant subspaces that consists of outer reconstructible subspaces. These spaces are analogs of outer detectable subspaces, which were studied, by Schumacher (1981) and Willems and Commault (1981). The terminology is consistent with the use of reconstructibility as in Valcher and Willems (1999b), in connection with the study of dead beat observers. Outer reconstructible subspaces can be considered to be outer detectable subspaces when the set of stable polynomials consists of monomials. This is consistent with the intuition that, over an arbitrary field, with the discrete topology, asymptotic stability of a sequence means that it is eventually zero. One suspects that there should be a relation between almost observability and outer reconstructible subspaces. To analyse this, we define and study the reversion operator in the case of polynomial Brunovsky form. An interesting duality theory is brought to light, and in that context, almost observability and outer reconstructible subspaces turn out to be related by duality. The details of this appear in Theorem 8.

It has been shown, by state space methods, that a subspace of the state space $\mathcal{X}$ is an observability subspace if and only if it is simultaneously conditioned invariant as well as an almost observability subspace, for the details of this see Willems (1982). Based on the module theoretic characterizations of these subspaces, we give in Theorem 9 a module theoretic proof of this. We return, in Theorem 10, to the problem of spectral assignability, this time using state space methods. We prove a pole placement result using a solution to two Sylvester equations. This result, though of intrinsic interest, is slightly weaker than that obtained in Theorem 7 where also invariant factors were taken into account.

Finally, in § 6, we summarize the application of the previous results to the characterization of various classes of observers. We conclude with a short summary indicating a few directions worth exploring.

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2. Preliminaries

In this section we will present several results that will be of use later on.

Since observers are naturally defined on quotient spaces, we find it important to analyse when a quotient space splits into a direct sum. Strangely, this is omitted from most linear algebra texts. We will say that a subspace $\mathcal{V} \subset \mathcal{X}$ is the transversal intersection of the subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ if the following conditions hold

$$\left.\begin{aligned} \mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}, \\ \mathcal{V}_1 + \mathcal{V}_2 = \mathcal{X}. \end{aligned}\right\} \tag{1}$$

Lemma 1: *Let $\mathcal{X}$ be a vector space and let $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{V}_2$ be subspaces of $\mathcal{X}$ with $\mathcal{V} \subset \mathcal{V}_1 \cap \mathcal{V}_2$. Then we have the direct sum decomposition*

$$\mathcal{X}/\mathcal{V} = \mathcal{V}_1/\mathcal{V} \oplus \mathcal{V}_2/\mathcal{V} \tag{2}$$

if and only if $\mathcal{V}$ is the transversal intersection of $\mathcal{V}_1$ and $\mathcal{V}_2$.

Proof: Assume conditions (1) hold. Let $[x]_{\mathcal{V}}$ denote the equivalence class of x modulo $\mathcal{V}$, i.e. $[x]_{\mathcal{V}} = x + \mathcal{V} = \{y | y - x \in \mathcal{V}\}$. The equality $\mathcal{X}/\mathcal{V} = \mathcal{V}_1/\mathcal{V} + \mathcal{V}_2/\mathcal{V}$ follows from $\mathcal{V}_1 + \mathcal{V}_2 = \mathcal{X}$. To show that this is a direct sum, assume $[x]_{\mathcal{V}} \in \mathcal{V}_1/\mathcal{V} \cap \mathcal{V}_2/\mathcal{V}$, i.e. there exist $v_i \in \mathcal{V}_i$ such that $x - v_1 \in \mathcal{V}$ and $x - v_2 \in \mathcal{V}$. This shows that $x \in \mathcal{V}_1$ and $x \in \mathcal{V}_2$, i.e. $x \in \mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}$. So $[x]_{\mathcal{V}} = [0]_{\mathcal{V}}$.

Conversely, assume we have the direct sum representation (2). The equality $\mathcal{X}/\mathcal{V} = \mathcal{V}_1/\mathcal{V} + \mathcal{V}_2/\mathcal{V}$ implies that, for every $x \in \mathcal{X}$, we have $x - v = (v_1 + v') + (v_2 + v'')$, with $v_i \in \mathcal{V}_i$ and $v, v', v'' \in \mathcal{V}$. This shows that $\mathcal{V}_1 + \mathcal{V}_2 = \mathcal{X}$. Since we assume that $\mathcal{V}_1/\mathcal{V} \cap \mathcal{V}_2/\mathcal{V} = 0$, we conclude that if $w \in \mathcal{V}_1 \cap \mathcal{V}_2$, then $[w]_{\mathcal{V}} = [v_1]_{\mathcal{V}} = [v_2]_{\mathcal{V}}$ and this implies $[w]_{\mathcal{V}} = 0$ or $w \in \mathcal{V}$, i.e. $\mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}$. □

Realization theory is one of the cornerstones of linear system theory. The polynomial model approach to linear systems, initiated in Fuhrmann (1976), and in particular the shift realization have proved to be a very powerful tool in the study of systems. The shift realization was mostly applied to the realization of proper rational functions, i.e. rational functions having no singularity at infinity. The same techniques can be applied to the realization and analysis of polynomial matrices. Some previous work in this direction can be found in Wimmer (1979, 1981).

We will say that a triple (J, N, L), with N nilpotent, is a realization of a polynomial matrix $P(z)$ if we can write

$$P(z) = P(0) + J(zN - I)^{-1}L, \tag{3}$$

or, with $P(z) = \sum_{i=0}^{s} P_i z^i$, that

$$P_i = -JN^i L, \quad i = 0, \ldots, s. \tag{4}$$

In the standard theory, great emphasis was given to various rings and modules. In particular, given a field $\mathbb{F}$, $\mathbb{F}((z^{-1}))$ denotes the field of truncated Laurent series. By $\mathbb{F}((z^{-1}))^m$ we denote the space of all m-vectors with $\mathbb{F}((z^{-1}))$ entries. We will identify $\mathbb{F}((z^{-1}))^m$ with $\mathbb{F}^m((z^{-1}))$, the space of all truncated Laurent series with $\mathbb{F}^m$ coefficients. The space $\mathbb{F}((z^{-1}))^m$ has the following direct sum representation

$$\mathbb{F}((z^{-1}))^m = \mathbb{F}[z]^m \oplus z^{-1}\mathbb{F}[[z^{-1}]]^m, \tag{5}$$

where $z^{-1}\mathbb{F}[[z^{-1}]]^m$ is the space of formal power series in z^{-1} vanishing at infinity. We denote by π_- the projection of $F^m((z^{-1}))$ onto $z^{-1}F^m[[z^{-1}]]$ corresponding to the previous direct sum decomposition, and by π_+ the complementary projection.

Since our interest in this paper is focused on conditioned invariant subspaces, almost observability and observability subspaces, we find it convenient, even necessary, to consider other module structures.

We note that $\mathbb{F}((z^{-1}))$ is itself a field and $\mathbb{F}[z, z^{-1}]$, the space of all polynomials in z and z^{-1} is a subring. It is well known, see Pernebo (1978) and Vidyasagar (1985), that it is actually an integral domain. $\mathbb{F}[z]$ and $\mathbb{F}[z^{-1}]$ are subrings of both $\mathbb{F}[z, z^{-1}]$ and $\mathbb{F}((z^{-1}))$ and both are principal ideal domains.

We will be interested in the $\mathbb{F}[z^{-1}]$-module structure of $\mathbb{F}^p[z, z^{-1}]$. Obviously, we have the direct sum representation

$$\mathbb{F}[z, z^{-1}]^p = \mathbb{F}[z]^p \oplus z^{-1}\mathbb{F}[z^{-1}]^p \tag{6}$$

which is the counterpart of (5). The projections of $\mathbb{F}[z, z^{-1}]^p$ on $\mathbb{F}[z]^p$ and $z^{-1}\mathbb{F}[z^{-1}]^p$ respectively are the restrictions of the projections π_+, π_- to $\mathbb{F}[z, z^{-1}]^p$ and will be denoted by the same letters.

The next computational lemma is recorded for later use, in particular in the proof of Proposition 5.4.

Lemma 2: *Let $\phi(z) = \sum_{j=-l}^{k} \phi_j z^j \in \mathbb{F}[z, z^{-1}]^p$. Then we have*

$$\pi_+\phi = z^{-1}[\pi_- z^{-1}\phi(z^{-1})](z^{-1}). \tag{7}$$

Proof: We clearly have $\pi_+\phi = \sum_{j=0}^{k} \phi_j z^j$. On the other hand $\phi(z^{-1}) = \sum_{j=-l}^{k} \phi_j z^{-j}$ and hence $z^{-1}\phi(z^{-1}) = \sum_{j=-l}^{k} \phi_j z^{-j-1}$ which, in turn, implies $\pi_- z^{-1}\phi(z^{-1}) = \sum_{j=0}^{k} \phi_j z^{-j-1}$. So $[\pi_- z^{-1}\phi(z^{-1})](z^{-1}) = \sum_{j=0}^{k} \phi_j z^{j+1}$ from which (7) follows. □

$\mathbb{F}[z,z^{-1}]^p$ is a rank p module over the ring $\mathbb{F}[z,z^{-1}]$, but, at the same time, it has natural module structures over the rings $\mathbb{F}[z]$ and $\mathbb{F}[z^{-1}]$. With respect to the ring $\mathbb{F}[z^{-1}]$, $z^{-1}\mathbb{F}[z^{-1}]^p$ is a submodule of $\mathbb{F}[z,z^{-1}]^p$. $\mathbb{F}[z]^p \subset \mathbb{F}[z,z^{-1}]^p$ is not an $\mathbb{F}[z^{-1}]$-submodule, however it has a naturally induced module structure defined by

$$\sigma_+ f = z^{-1}\cdot f = \pi_+ z^{-1} f = \frac{f(z)-f(0)}{z}, \quad f \in \mathbb{F}[z]^p. \tag{8}$$

We will refer to σ_+ as the downward shift operator.

We recall, see Fuhrmann (2002), that the backward shift $\sigma\colon z^{-1}\mathbb{F}[[z^{-1}]]^p \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^p$ is defined by

$$\sigma h = \pi_- z h \quad h \in z^{-1}\mathbb{F}[[z^{-1}]]^p. \tag{9}$$

Clearly, $z^{-1}\mathbb{F}[z^{-1}]^p \subset z^{-1}\mathbb{F}[[z^{-1}]]^p$ is an $\mathbb{F}[z]$-submodule. We still use the letter σ for the restriction of the backward shift operator to $z^{-1}\mathbb{F}[z^{-1}]^p$. For a polynomial matrix $S(z) \in \mathbb{F}^{p\times k}[z]$, we denote by $S(\sigma)\colon z^{-1}\mathbb{F}[z^{-1}]^k \longrightarrow z^{-1}\mathbb{F}[z^{-1}]^p$ the map defined by

$$S(\sigma)h = \pi_- S h \quad h \in z^{-1}\mathbb{F}[z^{-1}]^k. \tag{10}$$

A polynomial matrix $S(z) \in \mathbb{F}[z]^{p\times k}$ will be called monomic if all its nonzero invariant factors, $\delta_1,\dots,\delta_p$ are monomials, i.e. $\delta_i = z^{\nu_i}$, with ν_i non-negative. Since the determinant of a square polynomial matrix is the product of its invariant factors, a square polynomial matrix S is monomic if and only if $\det S(z) = z^n$ with $n = \sum_{i=1}^k \nu_i$.

Lemma 3: *Given a non-singular polynomial matrix $S(z) \in \mathbb{F}[z]^{p\times p}$ and $f \in \mathbb{F}[z]^p$, then $S^{-1}f \in \mathbb{F}[z,z^{-1}]^p$ for all $f \in \mathbb{F}[z]^p$ if and only if it is monomic.*

Proof: That $S(z)$ being monomic is a sufficient condition is trivial. To prove the converse, assume $S^{-1}f \in \mathbb{F}[z,z^{-1}]^p$ for all $f \in \mathbb{F}[z]^p$. Choosing constant unit vectors $e_1,\dots,e_p$, we get $S^{-1} \in \mathbb{F}[z,z^{-1}]^{p\times p}$. Let U, V be unimodular polynomial matrices such that

$$USV = \Delta = \begin{pmatrix} \delta_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \delta_p \end{pmatrix}.$$

Therefore $S^{-1} \in \mathbb{F}[z,z^{-1}]^{p\times p}$ iff $V^{-1}S^{-1}U^{-1} \in \mathbb{F}[z,z^{-1}]^{p\times p}$ iff for all i, $\delta_i^{-1} \in \mathbb{F}[z,z^{-1}]$. The last condition is equivalent to the existence of non-negative integers ν_i such that $z^{\nu_i}\delta_i^{-1} \in \mathbb{F}[z]$. This forces the δ_i to be monomials, i.e. $S(z)$ is monomic. □

Factorization theory is a most powerful tool for the study of linear systems. In fact, one can easily argue that linear, time invariant system theory is equivalent to factorization theory of rational, including polynomial, matrix functions. It is well known, see Fuhrmann (1976), that in the polynomial model space X_D, a subspace $\mathcal{V} \subset X_D$ is S_D-invariant if and only if $\mathcal{V} = D_1 X_{D_2}$ for a factorization $D = D_1 D_2$ into non-singular factors.

Proposition 1: *Let $D(z) \in \mathbb{F}[z]^{p\times p}$ be non-singular and let*

$$D = E_1 F_1 = E_2 F_2 \tag{11}$$

be two factorizations of D into non-singular factors. Then

$$E_1 X_{F_1} + E_2 X_{F_2} = E X_F, \tag{12}$$

where E is a greatest common left divisor of E_1, E_2 and

$$E_1 X_{F_1} \cap E_2 X_{F_2} = \overline{E} X_{\overline{F}}, \tag{13}$$

where $\overline{E}$ is a least common right multiple of E_1, E_2.

A special case of the previous proposition is the following, see Fuhrmann and Willems (1980). This result is essentially equivalent to the spectral decomposition of a linear transformation.

Proposition 2: *Let $D(z) \in \mathbb{F}[z]^{p\times p}$ be non-singular and let $d(z) = \det D(z)$. For any factorization $d = d_1 d_2$ into coprime factors, there exist essentially unique factorizations*

$$D(z) = D_1(z) D_2(z) = \overline{D}_2(z)\overline{D}_1(z) \tag{14}$$

satisfying $d_i(z) = \det D_i(z) = \det \overline{D}_i(z)$, $i = 1, 2$.

The following is a version of the shift realization as proved in Fuhrmann (1976).

Theorem 1: *Let $G = VT^{-1}U + W$ be a representation of a proper, $p\times m$ rational function. In the state space X_T a system is defined by*

$$\begin{cases} Af = S_T f & f \in X_T \\ B\xi = \pi_T U\xi, & \xi \in \mathbb{F}^m \\ Cf = (VT^{-1}f)_{-1} & f \in X_T \\ D = G(\infty). \end{cases} \tag{15}$$

Then this is a realization of G. This realization is observable if and only if V and T are right coprime and it is reachable if and only if T and U are left coprime. We will call (15) *the shift realization and denote it by $\Sigma(VT^{-1}U + W)$.*

Note that in the case $G = D^{-1}N$, the pair (C, A) defined by realization (15), depends only on D, and we will denote it by (C_D, A_D). Note that in this case

$$\begin{aligned} A_D f &= zf - D(z)\xi_f \\ C_D f &= (D^{-1}f)_{-1} = \xi_f. \end{aligned} \tag{16}$$

In Fuhrmann (1981), a duality theory was developed for the study of polynomial and rational models. Later, in Fuhrmann (2002, 2003b), it was extended to the study of discrete time behaviours. For an extension to multidimensional systems, see Oberst (1990). We start with the introduction of a non-degenerate bilinear form defined by

$$[f, g] = \sum_{j=-\infty}^{\infty} [f_j, g_{-j-1}] = \sum_{i=1}^{m} \left(f^{(i)}, g^{(i)}\right)_{-1} \tag{17}$$

on $\mathbb{F}((z^{-1}))^m \times \mathbb{F}((z^{-1}))^m$. For a non-singular $D \in \mathbb{F}[z]^{m\times m}$, we have with the pairing (17),

$$(\mathbb{F}[z]^m / D\mathbb{F}[z]^m)^* \simeq (D\mathbb{F}[z]^m)^{\perp} = X^{\tilde{D}}, \tag{18}$$

and as $X_D^* \simeq (\mathbb{F}[z]^m / D\mathbb{F}[z]^m)^*$, we have $X_D^* \simeq X^{\tilde{D}}$. Here, as throughout the paper, $\tilde{A}$ denotes the transpose of A.

Lemma 4: *Let G be a rational, full row rank $k \times l$ matrix. Assume all right Wiener-Hopf factorization indices are non-positive. Let $G^\sharp$ be any rational right inverse of G. Then all left Wiener-Hopf factorization indices of $G^\sharp$ are non-negative.*

Proof: By assumption, we have the right Wiener-Hopf factorization $G = U\begin{pmatrix}\Delta^{-1} & 0\end{pmatrix}\Gamma$, with U unimodular, Γ biproper and $\Delta(z) = \operatorname{diag}(z^{\nu_1}, \ldots, z^{\nu_k})$ with $\nu_i \geq 0$. Since $I = GG^\sharp = U(\Delta^{-1} \quad 0)\Gamma G^\sharp$, we get $(\Delta^{-1} \quad 0) \times \Gamma G^\sharp U = I$ and hence

$$\Gamma G^\sharp U = \begin{pmatrix} \Delta \\ \Omega \end{pmatrix}$$

with Ω rational. Applying left elementary operations over the ring $\mathbb{F}[[z^{-1}]]$, we may assume that Ω is polynomial. This shows that all column indices of $\left(\begin{smallmatrix} \Delta \\ \Omega \end{smallmatrix}\right)$ are nonnegative and hence so are the left Wiener-Hopf factorization indices of $G^\sharp$. □

3. On conditioned invariant and observability subspaces

Geometric control was developed in the state space setting by Basile and Marro (1973) and Wonham and Morse, see Wonham (1979), as a design tool to solve a wide range of control synthesis problems. The basic objects of geometric control are controlled and conditioned invariant subspaces. With them, more intricate objects like (output nulling) reachability and (input containing) observability subspaces, as well as and many others, were introduced and studied.

Definition 1:

1. A subspace $\mathcal{V}$ is controlled invariant for a pair (A, B), if and only if there exists a map K for which $\mathcal{V}$ is $(A - BK)$-invariant. Such a map K will be called a friend of $\mathcal{V}$. The set of all friends of a controlled invariant subspace $\mathcal{V}$ will be denoted by $\mathcal{F}(\mathcal{V})$. A controlled invariant subspace $\mathcal{V}$ will be called an reachability subspace if for each monic polynomial q of degree equal to $\dim \mathcal{V}$, there exists a friend $K \in \mathcal{F}(\mathcal{V})$ such that q is the characteristic polynomial of $(A - BK)|\mathcal{V}$.
2. A subspace $\mathcal{V}$ is conditioned invariant for a pair (C, A), if and only if there exists a map J for which $\mathcal{V}$ is $(A - JC)$-invariant. Such a map J will be called a friend of $\mathcal{V}$. The set of all friends of a conditioned invariant subspace $\mathcal{V}$ will be denoted by $\mathcal{G}(\mathcal{V})$. A conditioned invariant subspace $\mathcal{V}$ will be called an observability subspace if for each monic polynomial q of degree equal to $\operatorname{codim} \mathcal{V}$, there exists a friend $J \in \mathcal{G}(\mathcal{V})$ such that q is the characteristic polynomial of $(A - JC)|_{\mathcal{X}/\mathcal{V}}$, the map induced on the quotient space $X/\mathcal{V}$ by $A - JC$.

For a pair (C, A), a conditioned invariant subspace $\mathcal{V} \subset \mathcal{X}$ is called tight if it satisfies

$$\mathcal{V} + \operatorname{Ker} C = \mathcal{X}. \tag{19}$$

There are several alternative, but equivalent, definitions for controlled and conditioned invariant subspaces. It is well known that the class of controlled invariant subspaces is closed under sums and the class of conditioned invariant subspaces is closed under intersections. Thus for each subspace $\mathcal{L} \subset \mathcal{X}$, there exists a largest controlled invariant subspace contained in $\mathcal{L}$ and a smallest conditioned invariant subspace containing it. These are denoted by $\mathcal{V}^*(\mathcal{L})$ and $\mathcal{V}_*(\mathcal{L})$ respectively.

Given a pair (A, B), we will say that two controlled invariant subspaces $\mathcal{V}_1, \mathcal{V}_2$ are compatible if $\mathcal{F}(\mathcal{V}_1) \cap \mathcal{F}(\mathcal{V}_2) \neq \emptyset$. Similarly, given a pair (C, A), we will say that two conditioned invariant subspaces $\mathcal{V}_1, \mathcal{V}_2$ are compatible if $\mathcal{G}(\mathcal{V}_1) \cap \mathcal{G}(\mathcal{V}_2) \neq \emptyset$.

Lemma 5:

1. *Let $\mathcal{V}_1, \mathcal{V}_2$ be controlled invariant subspaces. Then $\mathcal{V}_1, \mathcal{V}_2$ are compatible if and only if $\mathcal{V}_1 \cap \mathcal{V}_2$ is a controlled invariant subspace.*

2. *Let $\mathcal{V}_1, \mathcal{V}_2$ be conditioned invariant subspaces. Then $\mathcal{V}_1, \mathcal{V}_2$ are compatible if and only if $\mathcal{V}_1 + \mathcal{V}_2$ is a conditioned invariant subspace.*

Proof:

1. Clearly, if the subspaces $\mathcal{V}_i$ are compatibe controlled invariant subspaces, then there exists a feedback map K such that $(A - BK)\mathcal{V}_i \subset \mathcal{V}_i$. This implies $(A - BK)(\mathcal{V}_1 \cap \mathcal{V}_2) \subset (\mathcal{V}_1 \cap \mathcal{V}_2)$, i.e. $\mathcal{V}_1 \cap \mathcal{V}_2$ is controlled invariant.

   To prove the converse, assume that $\mathcal{V}_1 \cap \mathcal{V}_2$ is a controlled invariant subspace. Let $\{e_1, \ldots, e_r\}$ be a basis for $\mathcal{V}_1 \cap \mathcal{V}_2$. We extend it to a basis $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_q, e_{q+1}, \ldots, e_s\}$ of $\mathcal{V}_1 + \mathcal{V}_2$ so that $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_q\}$ is a basis of $\mathcal{V}_1$ and $\{e_1, \ldots, e_r, e_{q+1}, \ldots, e_s\}$ is a basis of $\mathcal{V}_1$. For $i = 1, \ldots, q$, we have $Ae_i = v_i + B\eta_i$ with $v_i \in \mathcal{V}_1$. For $i = q+1, \ldots, s$, we have $Ae_i = v_i + B\eta_i$ with $v_i \in \mathcal{V}_2$. We define $Ke_i = \eta_i$ and extend the definition of K arbitrarily to a basis of the whole space and by linearity to a state feedback map. By construction $K \in \mathcal{F}(\mathcal{V}_1) \cap \mathcal{F}(\mathcal{V}_2)$, i.e. the two subspaces are compatible.
2. Follows from the first part by duality. □

Note that the first statement is an exercise in Wonham (1979).

Given the observable pair (C, A) in the state space X, a subspace $\mathcal{V} \subset X$ is conditioned invariant if for some $J \in \mathcal{G}(\mathcal{V})$, we have $(A - JC)\mathcal{V} \subset \mathcal{V}$. We are interested in the dynamics of the induced map $(A - JC)|_{X/\mathcal{V}}$ and in particular on how much control we have on the spectral property of the induced map. The approach we adopt is functional in nature. If $D(z)^{-1}\Phi(z)$ is a left coprime factorization of the state to output transfer function $C(zI - A)^{-1}$, then the pair (C, A) is isomorphic to the pair (C_D, A_D) obtained from the shift realization (15) corresponding to the left coprime factorization $D(z)^{-1}\Phi(z) = C(zI - A)^{-1}$. It is well known, see Hautus and Heymann (1978), or Fuhrmann and Willems (1980), that the columns of Φ constitute a basis for the polynomial model X_D. Moreover, the map $\phi\colon X \longrightarrow X_D$ defined by

$$\phi(\xi) = \Phi(z)\xi \tag{20}$$

is an isomorphism that intertwines the pairs (C, A) and (C_D, A_D).

It has been shown in Fuhrmann (1981) that for the coprime factorizations $G(z) = C(zI - A)^{-1} = D^{-1}N$, a pair (C_{D_1}, A_{D_1}) is output injection equivalent to (C_D, A_D) if and only if all the left Wiener-Hopf indices of $D_1^{-1}D$ are zero. Since a right unimodular factor applied to D corresponds, in state space terms, to a similarity, we may assume without loss of generality that $D_1^{-1}D$ is normalized biproper. Invariant subspaces for S_{D_1} are parametrized by factorizations of D_1. Hence, as S_D-invariant subspaces of X_D correspond to factorizations of D, we have

Proposition 3: *Let $D \in \mathbb{F}[z]^{p\times p}$ be non-singular. A subspace of X_D is conditioned invariant for the pair (C_D, A_D) if and only if*

$$\mathcal{V} = E_1 X_{F_1} \tag{21}$$

for some polynomial matrix $D_1 \in \mathbb{F}[z]^{p\times p}$ admitting the factorization $D_1 = E_1F_1$ into non-singular factors, and for which all the left Wiener-Hopf indices of $D_1^{-1}D$ are zero.

It is easily checked that the representation (21) is equivalent to $\mathcal{V} = X_D \cap E_1\mathbb{F}[z]^p$, where all the left Wiener-Hopf indices of $D^{-1}E_1$ are non-negative.

The previous analysis leads to the following, see Fuhrmann (1981) for the details.

Theorem 2: *With respect to the realization* (15) *in the state space X_D, a subspace $\mathcal{V} \subset X_D$ is conditioned invariant if and only if*

$$\mathcal{V} = X_D \cap \mathcal{M} \tag{22}$$

for some submodule $\mathcal{M} \subset \mathbb{F}[z]^p$.

The characterization given in Theorem 2 is as clean as one can get. However, some information is lost when stated in this form. The main problem with this characterization is the fact that in general the representation is non-unique. As an example, consider the case of a scalar, monic polynomial d. A submodule $\mathcal{M}$ of $\mathbb{F}[z]$ is an ideal and hence has a representation $\mathcal{M} = h\mathbb{F}[z]$ for an essentially unique polynomial h. In particular, for the zero subspace $\{0\}$ we have the representation $\{0\} = X_d \cap h\mathbb{F}[z]$ whenever $\deg h \geq \deg d$. However, if $\mathcal{V} \subset X_d$ is not the zero subspace, then h in the representation $\mathcal{V} = X_d \cap h\mathbb{F}[z]$ is unique up to a non-zero constant factor. In the matrix case, the degree conditions are replaced by conditions on the Wiener-Hopf factorization indices.

In order to overcome the nonuniqueness issue, we look for a submodule of $\mathbb{F}[z]^p$ that is uniquely determined by $\mathcal{V}$. This can be done and in this we follow Hinrichsen *et al.* (1981), see also the discussion in Fuhrmann and Helmke (2001) from which the following is quoted.

Proposition 4: *Let $D(z)$ be a non-singular $p \times p$ polynomial matrix. Let $\mathcal{V} \subset X_D$ be a conditioned invariant subspaces.*

1. *Let $\langle \mathcal{V} \rangle$ be the submodule of $\mathbb{F}[z]^p$ generated by $\mathcal{V}$, that is the smallest submodule of $\mathbb{F}[z]^p$ that contains $\mathcal{V}$. Then*

$$\mathcal{V} = X_D \cap \langle \mathcal{V} \rangle. \tag{23}$$

2. *If $E \subset X_D$ is an arbitrary subspace, then $X_D \cap \langle E \rangle$ is the smallest conditioned invariant subspace of X_D that contains E.*
3. *A subspace $\mathcal{V} \subset X_D$ is a conditioned invariant subspace if and only if it has a representation of the form*

$$\mathcal{V} = X_D \cap H(z)\mathbb{F}[z]^k, \tag{24}$$

where $H(z)$ is a full column rank $p \times k$ polynomial matrix whose columns are in $\mathcal{V}$. $H(z)$ is uniquely determined up to a right $k \times k$ unimodular factor.

Any full column rank polynomial matrix H has a factorization of the form $H = H_1 H_0$ with H_1 right prime and H_0 non-singular. We call such a factorization an external/internal factorization. An external/internal factorization is essentially unique, i.e. unique up to a right unimodular factor for H_1 and its inverse a left factor of H_0.

Proposition 4 is the key to the parametrization of all conditioned invariant subspaces of a given observable pair (C, A), that can be taken, without loss of generality, to be in dual Brunovsky form. Again, the basic results are those of Hinrichsen *et al.* (1981) with extensions given in Fuhrmann and Helmke (2001). As a result of the above, all information, up to similarity, on the conditioned invariant subspace is, in principle, derivable from the polynomial matrices $D(z)$ and $H(z)$. In particular, because of our interest in observers, we will emphasize the characterization of observability subspaces.

Let us proceed with a short digression aimed at clarifying the connection of observers to geometric control. Given the linear system

$$\left.\begin{aligned} x_{j+1} &= Ax_j + Bu_j \\ y_j &= Cx_j \\ z_j &= Kx_j \end{aligned}\right\} \tag{25}$$

in the state space $\mathcal{X}$. Here y_j is the measured output vector and z_j the vector of variables to be estimated. A tracking observer can be constructed if and only if there exists a conditioned invariant subspace $\mathcal{V} \subset \operatorname{Ker} K$. In that case, a natural state space for the tracking observer can be taken to be $\mathcal{X}/\mathcal{V}$ with the module structure given by the induced map $(A - JC)|_{\mathcal{X}/\mathcal{V}}$ for $J \in \mathcal{G}(\mathcal{V})$. This module structure determines the error dynamics. Clearly, there is always a conditioned invariant subspace $\mathcal{V} \subset \operatorname{Ker} K$ and that is the zero subspace. If we choose to have our construction of an observer to be based on the zero subspace, then the observer state space has the same dimension as the system state space, which means that the dimension may be bigger than necessary. To decrease the dimension of the observer state space as much as possible, we have to look for maximal dimensioned conditioned invariant subspaces included in $\operatorname{Ker} K$. Such subspaces exist. However, since the set of conditioned invariant subspaces is not closed under sums, maximal dimensional conditioned invariant subspaces included in $\operatorname{Ker} K$ are generally not unique.

In the polynomial model context, the problem of non-uniqueness relates to the nonuniqueness of a representation (22). This leaves open the question of how much control do we have on the module structure of $\mathcal{X}/\mathcal{V}$. Let us consider the two extreme cases. On the one hand we have the case of $\mathcal{V}$ being a tight conditioned invariant subspace, a case where there is a unique module structure on the quotient space $\mathcal{X}/\mathcal{V}$. At the other extreme, we have $\mathcal{V}$ being an observability subspace, a case in which we have full control of the error dynamics, constrained only by dimensionality. Obviously, in general, we have to deal with intermediate cases. The clue for us is Lemma 1, which shows when a quotient space decomposes into a direct sum. Thus, clearly, if we can show that every conditioned invariant subspace $\mathcal{V}$ is the transversal intersection $\mathcal{V} = \mathcal{O} \cap \mathcal{T}$ of an observability subspace $\mathcal{O}$ and a tight conditioned invariant subspace $\mathcal{T}$, then we have a decomposition of the error dynamics into a fixed part given by $\mathcal{T}/\mathcal{V}$ and a freely assignable part given by $\mathcal{O}/\mathcal{V}$. In this connection, see Willems (1982).

The principal reason for studying observability subspaces in the context of observer theory is that the dynamics of the observer is derived from the induced module structure on the quotient module of the state space modulo the observability subspace. Thus for this class of subspaces, the dynamics of the observer is freely assignable. We would like to understand if only the characteristic polynomial is assignable, modulo the degree constraint, or we have some control also on the fine strucure of the induced map, i.e. what are the constraints on the assignment of the invariant factors. Moreover, we would like to have a constructive way to implement the spectral assignment. The difficulty stems from the fact that our proof of Theorem 3 is based on the Morse relations (28). Thus we are left with the question of how to implement the spectral assignment, via output injection, on the quotient spaces $\mathcal{X}/\mathcal{V}_*$ and $\mathcal{X}/\mathcal{O}_*$. This problem, to which we refer as the outer spectral assignability problem, is the dual to the problem of implementability of spectral assignment, by state feedback, in $\mathcal{V}^*$ and $\mathcal{R}^*$.

This has been treated in great detail in Fuhrmann (2005). Duality theory allows us to lift these results to the context of input containing subspaces. However this lifting by duality is not straightforward and the full treatment of duality will be given in Fuhrmann (2006). In this paper we choose to derive all results pertaining to outer spectral assignability directly. In fact, some results are more easily derived directly rather than via duality considerations.

A comparison of the characterizations (21) and (22), given in Propositions 3 and 4 respectively, indicates that given a representation of a conditioned invariant subspace of the form (24), we might expect that there exists a non-singular polynomial matrix extension $T = \begin{pmatrix} H & \overline{H} \end{pmatrix}$ of H such that $D^{-1}T$ is proper and we have

$$\mathcal{V} = X_D \cap H(z)\mathbb{F}[z]^k = X_D \cap T(z)\mathbb{F}[z]^p. \tag{26}$$

Naturally, in general, we don't expect such an extension to be unique. A full analysis of this issue and its relation to kernel representations of conditioned invariant subspaces can be found in Fuhrmann and Helmke (2001). The analysis of the extension procedure is central to the understanding of the error dynamics of observers, the analysis of the amount of freedom we have in the choice of observer dynamics and in particular to the construction procedures for such observers.

Given a triple (C, A, B) in the state space $\mathcal{X}$ and a subspace $\mathcal{V} \subset \mathcal{X}$. We denote by $\mathcal{V}^*(\mathcal{V}), \mathcal{R}^*(\mathcal{V}), \mathcal{V}_*(\mathcal{V}), \mathcal{O}_*(\mathcal{V})$ the maximal controlled invariant subspace contained in $\mathcal{V}$, the maximal controllability subspace contained in $\mathcal{V}$, the minimal conditioned invariant subspace containing $\mathcal{V}$ and the minimal observability subspace containing $\mathcal{V}$ respectively. If $\mathcal{V} = \operatorname{Ker} C$ then we just write $\mathcal{V}^* = \mathcal{V}^*(\operatorname{Ker} C)$ and $\mathcal{R}^* = \mathcal{R}^*(\operatorname{Ker} C)$. Similarly, we write $\mathcal{V}_* = \mathcal{V}_*(\operatorname{Im} B)$ and $\mathcal{O}_* = \mathcal{O}_*(\operatorname{Im} B)$. These subspaces are the most important objects in geometric control and there exist state space algorithms to compute them. Our interest is, given a matrix fraction representation $G = T^{-1}V$ of a (strictly) proper rational function, to give explicit formulas for these subspaces with respect to the shift realization in the state space X_T. The initial result in this direction was the characterization of $\mathcal{V}^*$ given in Emre and Hautus (1980), see also Fuhrmann and Willems (1980). The following theorem generalizes these results as well those of Fuhrmann (1981). For a more detailed, state space analysis, see Aling and Schumacher (1984).

Theorem 3: *Let $G = T^{-1}V$, with $T \in \mathbb{F}[z]^{p\times p}$ non-singular, be strictly proper and let (C_D, A_D, B_D) be the associated shift realization, given by* (15), *in the state space X_T. Then we have the following characterizations, namely*

$$\begin{aligned} \mathcal{O}_* &= X_V + X_T \cap V\mathbb{F}[z]^k \\ \mathcal{V}^* &= X_V \\ \mathcal{V}_* &= X_T \cap V\mathbb{F}[z]^k \\ \mathcal{R}^* &= X_V \cap V\mathbb{F}[z]^k. \end{aligned} \tag{27}$$

Moreover, we have the Morse relations, see Morse (1973),

$$\begin{aligned} \mathcal{R}^* &= \mathcal{V}^* \cap \mathcal{V}_* \\ \mathcal{O}_* &= \mathcal{V}^* + \mathcal{V}_* \end{aligned} \tag{28}$$

as well as the following isomorphism

$$\mathcal{O}_*/\mathcal{V}_* \simeq \mathcal{V}^*/\mathcal{R}^*. \tag{29}$$

The inclusions are summarized in figure 1.

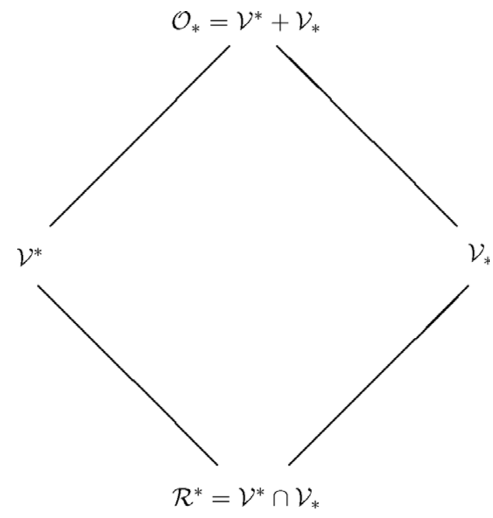

Figure 1.

Proof: That $\mathcal{V}^* = X_V$ was proved in Emre and Hautus (1980) and also in Fuhrmann and Willems (1980).

That $\mathcal{V}_* = X_T \cap V\mathbb{F}[z]^k$ can be proved by rather intricate duality considerations. However a shockingly short, direct proof is available. Since $\mathcal{V}_*$ is in particular a conditioned invariant subspace of X_T, it has, by Theorem 2, a representation of the form $\mathcal{V}_* = X_T \cap \mathcal{M}$ for some submodule $\mathcal{M} \subset \mathbb{F}[z]^p$. Since $\mathcal{V}_*$ is input containing, we must have $\{V(z)\xi | \xi \in \mathbb{F}^m\} \subset \mathcal{M}$. Since $\mathcal{M}$ is a submodule, we have $V\mathbb{F}[z]^m \subset \mathcal{M}$. By minimality, we must have the equality $V\mathbb{F}[z]^m = \mathcal{M}$.

The other two equalities follow from the Morse relations (28), see Morse (1973). A characterization of $\mathcal{R}^*$ was given in Fuhrmann (2001). Direct characterization of $\mathcal{O}_*$ in terms of right primeness is available and will be given below. It can be related to the above mentioned characterization of $\mathcal{R}^*$ by way of an intricate duality. The full exposition of this theme is beyond the scope of the present paper, see Fuhrmann (2006). $\square$

The importance of the previous characterizations is that they can be immediately applied to the study of arbitrary controlled and conditioned invariant subspaces. For this, the following theorem is important. It allows us to use polynomial characterizations of the previous objects for the characterization of arbitrary conditioned invariant and observability subspaces.

Theorem 4: *Given $D(z) \in \mathbb{F}[z]^{p\times p}$ non-singular. Let (C_D, A_D) be the observable pair, in the state space X_D, defined by the shift realization. Let $\mathcal{V} \subset X_D$ be a conditioned invariant subspace having the representation*

$$\mathcal{V} = X_D \cap H(z)\mathbb{F}[z]^k = X_D \cap \langle \mathcal{V} \rangle, \tag{30}$$

where $H(z)$ is an, essentially unique, basis matrix for $\langle\mathcal{V}\rangle$, the submodule of $\mathbb{F}[z]^p$ generated by $\mathcal{V}$, whose columns are in $\mathcal{V}$. Let *$H = H_1H_0$ be an external/internal factorization for which H_1 is right prime and H_0 is non-singular.*

1. *With respect to the shift realization associated with $G = D^{-1}H$, we have*

$$\left.\begin{aligned} &\mathcal{O}_* = X_H + X_D \cap H\mathbb{F}[z]^k = X_D \cap H_1\mathbb{F}^k[z] \\ &\mathcal{V}^* = X_H = H_1X_{H_0} \\ &\mathcal{V}_* = \mathcal{V} = X_D \cap H\mathbb{F}[z]^k = X_D \cap H_1H_0\mathbb{F}[z]^k \\ &\mathcal{R}^* = X_H \cap H\mathbb{F}[z]^k = H_1X_{H_0} \cap (X_D \cap H_1H_0\mathbb{F}[z]^k) = \{0\}. \end{aligned}\right\} \tag{31}$$

2. *We have the following isomorphism*

$$\mathcal{O}_*/\mathcal{V}_* = X_D \cap H_1\mathbb{F}[z]^k/(X_D \cap H_1H_0\mathbb{F}[z]^k) \simeq X_{H_0}. \tag{32}$$

3. *The following dimension formula holds*

$$\dim X_D \cap H\mathbb{F}[z]^k = \dim X_D \cap H_1\mathbb{F}[z]^k - \deg\det H_0. \tag{33}$$

Proof:

1. Since $H(z)$ is a basis matrix for $\langle\mathcal{V}\rangle$ whose columns are contained in $\mathcal{V}$, $G = D^{-1}H$ is strictly proper. Then $\mathcal{V}_* = \mathcal{V} = X_D \cap H\mathbb{F}[z]^k = X_D \cap H_1H_0\mathbb{F}[z]^k$ is an immediate consequence of Theorem 3.

   Next we prove $\mathcal{V}^* = H_1X_{H_0}$. The inclusion $H_1X_{H_0} \subset X_{H_1H_0} = X_H$ is immediate. To prove the converse, let $f \in \mathcal{V}^* = X_H$, i.e. $f = H_1H_0h$ for some $h \in z^{-1}\mathbb{F}[z^{-1}]^k$. As H_1 is right prime, it has a polynomial left inverse. Necessarily H_0h is a polynomial, that is $H_0h \in X_{H_0}$ and $f \in H_1X_{H_0}$. Thus $X_H \subset H_1X_{H_0}$ and the two inclusions imply the equality $X_H = H_1X_{H_0}$.

   The strict properness of $D^{-1}H$ implies the inclusion $H_1X_{H_0} \subset X_D$.

   We proceed to compute, using the injectivity of the multiplication by H_1,

$$\begin{aligned}\mathcal{R}^* &= \mathcal{V}^* \cap \mathcal{V}_* = H_1X_{H_0} \cap (X_D \cap H_1H_0\mathbb{F}[z]^k) \\ &= H_1X_{H_0} \cap H_1H_0\mathbb{F}[z]^k = H_1(X_{H_0} \cap H_0\mathbb{F}[z]^k) = \{0\},\end{aligned}$$

   i.e. $\mathcal{R}^* = \{0\}$.

   Finally, we compute

$$\mathcal{O}_* = \mathcal{V}^* + \mathcal{V}_* = H_1X_{H_0} + X_D \cap H_1H_0\mathbb{F}[z]^k.$$

   Since $H_1X_{H_0} \subset X_D$, we have $H_1X_{H_0} = X_D \cap H_1X_{H_0}$, and hence

$$\begin{aligned}\mathcal{O}_* &= X_D \cap H_1X_{H_0} + X_D \cap H_1H_0\mathbb{F}[z]^k \\ &\subset X_D \cap (H_1X_{H_0} + H_1H_0\mathbb{F}[z]^k) = X_D \cap H_1\mathbb{F}[z]^k.\end{aligned}$$

   To prove the converse inclusion, we have

$$H_1\mathbb{F}[z]^k = H_1(X_{H_0} + H_0\mathbb{F}[z]^k) = H_1X_{H_0} + H_1H_0\mathbb{F}[z]^k.$$

   Assume next that $f \in X_D \cap H_1\mathbb{F}[z]^k$, then $f = H_1g$ with $D^{-1}H_1g$ strictly proper. Write $g = g_1 + H_0g_2$, with $g_1 \in X_{H_0}$, which implies $f = H_1g_1 + H_1H_0g_2$. Now $D^{-1}H_1g_1 = D^{-1}H_1H_0h$ for some $h \in z^{-1}H_1\mathbb{F}[[z^{-1}]]^k$. $D^{-1}H$ is proper, so it follows that $H_1g_1 \in X_D \cap H_1X_{H_0}$ and we get the inclusion

$$X_D \cap H_1\mathbb{F}[z]^k \subset X_D \cap H_1X_{H_0} + X_D \cap H_1H_0\mathbb{F}[z]^k.$$

   The two inclusions lead to the equality $\mathcal{O}_* = X_D \cap H_1\mathbb{F}[z]^k$.
2. The isomorphism $\mathcal{O}_*/\mathcal{V}_* \simeq \mathcal{V}^*/\mathcal{R}^*$ implies $X_D \cap H_1\mathbb{F}[z]^k/(X_D \cap H_1H_0\mathbb{F}^k[z]) \simeq H_1X_{H_0}/\{0\}$. However, by the injectivity of multiplication by H_1, we have $H_1X_{H_0} \simeq X_{H_0}$ and hence the isomorphism $X_D \cap H_1\mathbb{F}[z]^k/(X_D \cap H_1H_0\mathbb{F}[z]^k) \simeq X_{H_0}$.
3. Follows from the isomorphism (32).

Corollary 1: *Under the assumptions of Theorem* 4, *let $\mathcal{V} = X_D \cap H\mathbb{F}[z]^k$ be a conditioned invariant subspace with H of full column rank* (*and with its columns in $\mathcal{V}$*). *Let $H = H_1H_0$ be an external/internal factorization.*

Then $\mathcal{O} = X_D \cap H_1\mathbb{F}[z]^k$ *is the smallest observability subspace containing* $\mathcal{V}$, *namely* $\mathcal{O} = \mathcal{O}_*(\mathcal{V})$.

Proof: According to (31), $\mathcal{O} = X_D \cap H_1\mathbb{F}[z]^k$ and hence is an observability subspace. As $H_1\mathbb{F}[z]^k \supset H_1H_0\mathbb{F}[z]^k = H\mathbb{F}[z]^k$, it contains $\mathcal{V}$.

Conversely, let $\mathcal{W}$ be an observability subspace that contains $\mathcal{V}$. Since $\mathcal{V} = \mathcal{V}_*$ it follows that $\mathcal{W}$ is input containing and hence contains $\mathcal{O}_*$, the smallest input containing observability subspace. But then $\mathcal{O} = \mathcal{O}_*$ implies $\mathcal{W} \supset \mathcal{O}$ and indeed $\mathcal{O}$ is the smallest observability subspace containing $\mathcal{V}$. □

We are ready to state the main characterization of observability subspaces in polynomial terms.

Theorem 5: *Given* $D(z) \in \mathbb{F}[z]^{p\times p}$ *non-singular. Let* (C_D, A_D) *be the observable pair, in the state space* X_D, *defined by the shift realization. Let* $\mathcal{O} \subset X_D$ *be conditioned invariant subspace and let* $\langle\mathcal{V}\rangle$ *the submodule of* $\mathbb{F}[z]^p$ *generated by* $\mathcal{V}$. *Assume* $\mathcal{V}$ *has the representation*

$$\mathcal{O} = X_D \cap \langle \mathcal{O} \rangle = X_D \cap H(z)\mathbb{F}[z]^k, \tag{34}$$

where $H(z)$, *the essentially unique, i.e. up to a right unimodular factor, basis matrix for* $\langle\mathcal{O}\rangle$ *whose columns are in* $\mathcal{O}$. *Then* $\mathcal{O} \subset X_D$ *is an observability subspace if and only if* $H(z)$ *is right prime.*

Proof: Let H in (34) be right prime. Applying Corollary 1 with $H_0 = I$ shows that $\mathcal{O}$ is an observability subspace.

Conversely, let $\mathcal{O}$ be an observability subspace and let $H = H_1H_0$ be a factorization for which H_1 is right prime and H_0 is non-singular. Corollary 1 then yields

$$X_D \cap H_1\mathbb{F}[z]^k = \mathcal{O} = X_D \cap H_1H_0\mathbb{F}[z]^k.$$

But then the submodule $\mathcal{M} = H_1\mathbb{F}[z]^k$ of $\mathbb{F}[z]^p$ contains $\mathcal{O}$ and hence also $\langle\mathcal{O}\rangle = H\mathbb{F}[z]^k = H_1H_0\mathbb{F}[z]^k$. Since both H_1 and H have full column rank k this implies $\mathcal{M} = \langle\mathcal{O}\rangle$. But this means that H_1H_0 and H_1 generate the same submodule of $\mathbb{F}[z]^p$ and hence H_0 is unimodular. H_1 being right prime then implies that $H = H_1H_0$ is also right prime. □

In order to gain some intuition, we consider a relatively simple example. We use the parametrization of the set of all conditioned invariant subspaces of X_D, given in Hinrichsen *et al.* (1981) or Fuhrmann and Helmke (2001). In this approach the set of conditioned invariant is decomposed into cells depending on ordered, reduced observability indices.

Example 1: We assume our system to be in dual Brunovsky form with the observability indices given by 3, 2, 1. Polynomially this is expressed by assuming the left denominator matrix is given by

$$D(z) = \begin{pmatrix} z^3 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z \end{pmatrix},$$

i.e. it is in dual polynomial Brunovsky form. The set of such subspaces for which the reduced observability indices are $\lambda = (0, 1, 1)$ is parametrized, by

$$H(z) = \begin{pmatrix} \beta_2 z^2 + \beta_1 z + \beta_0 & \gamma_2 z^2 + \gamma_1 z + \gamma_0 \\ z + \epsilon_0 & \eta_0 \\ 0 & 1 \end{pmatrix},$$

with $\mathcal{V} = X_D \cap H(z)\mathbb{F}[z]^2$. Clearly, $H(z)$ is right prime if and only if $\beta_2\epsilon_0^2 - \beta_1\epsilon_0 + \beta_0 \neq 0$. Now, all appropriate extensions are given by

$$E(z) = \begin{pmatrix} z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 & \beta_2 z^2 + \beta_1 z + \beta_0 & \gamma_2 z^2 + \gamma_1 z + \gamma_0 \\ \delta_0 & z + \epsilon_0 & \eta_0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\alpha_0, \alpha_1, \alpha_2, \delta_0$ free parameters. We have

$$\begin{aligned} \det D(z) &= (z + \epsilon_0)(z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0) \\ &\quad - \delta_0(\beta_2 z^2 + \beta_1 z + \beta_0) \\ &= z^4 + z^3(\epsilon_0 + \alpha_2) + z^2(\epsilon_0\alpha_2 + \alpha_1 - \delta_0\beta_2) \\ &\quad + z(\epsilon_0\alpha_1 + \alpha_0 - \delta_0\beta_1) + (\epsilon_0\alpha_0 - \delta_0\beta_0) \\ &= z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0. \end{aligned}$$

So we need to solve, for arbitrary c_i the following system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \epsilon_0 & 1 & 0 & -\beta_2 \\ 0 & \epsilon_0 & 1 & -\beta_1 \\ 0 & 0 & \epsilon_0 & -\beta_0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} c_3 - \epsilon_0 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix}.$$

Solvability is of course equivalent to the nonvanishing of the determinant, which is easily computed to be $-(\beta_2\epsilon_0^2 - \beta_1\epsilon_0 + \beta_0)$. This is in perfect agreement with Theorem 5. It also indicates that polynomial matrix extension may be the right tool. However, this example also indicates that using this parametrization may be the wrong direction to take as the computations seem prohibitively complex and moreover, not well suited to the problem of invariant factor assignment. Nor do we easily recover the appropriate output injection.

Due to the intersection representation of conditioned invariant subspaces, inclusion relations between polynomial submodules are reflected in inclusion relations between conditioned invariant subspaces. Generally, there are two ways that a submodule $H\mathbb{F}[z]^k \subset \mathbb{F}[z]^p$ can be enlarged. One is via factorization of $H(z)$, the other by addition of generators, i.e. by adding columns to $H(z)$. In the case that $H(z)$ is right prime, only the second option exists. This result is the dual of Theorem 6 in Willems (1997).

Proposition 5: *Let $\mathcal{M} = H\mathbb{F}[z]^k$ and $\mathcal{N} = \overline{H}\mathbb{F}[z]^l$ be submodules of $\mathbb{F}[z]^p$ and we assume $\mathbb{F}[z]^p/\mathcal{M}$ is a torsion free submodule. Then $\mathcal{M} \subset \mathcal{N}$ if and only if there exists a representation*

$$\overline{H} = \begin{pmatrix} H & H' \end{pmatrix} W \tag{35}$$

for some polynomial matrix $H' \in \mathbb{F}[z]^{p\times(l-k)}$ and a unimodular $W \in \mathbb{F}[z]^{l\times l}$.

Proof: If we have a representation of the form (35), then

$$\begin{aligned}\mathcal{M} &= H\mathbb{F}[z]^k \subset H\mathbb{F}[z]^k + H'\mathbb{F}[z]^{(l-k)}\\ &= \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^l = \begin{pmatrix} H & H' \end{pmatrix} W\mathbb{F}[z]^l\\ &= \overline{H}\mathbb{F}[z]^l = \mathcal{N}.\end{aligned}$$

To prove the converse, we note first that $\mathbb{F}[z]^p/\mathcal{M} = \mathbb{F}[z]^p/H\mathbb{F}[z]^k$ is torsion free is equivalent to the right primeness of H. Since $\mathcal{M} \subset \mathcal{N}$, it follows that there exists a factorization $H = \overline{H}H_1$ with $\overline{H} \in \mathbb{F}[z]^{p\times l}$ and $H_1 \in \mathbb{F}[z]^{l\times k}$. The right primeness of H implies the right primeness of H_1. There exist therefore unimodular matrices U, V for which

$$UH_1V = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Thus

$$HV = \overline{H}U^{-1}UH_1V = \overline{H}U^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix}.$$

So $\overline{H}U^{-1} = \begin{pmatrix} HV & H' \end{pmatrix}$, or equivalently

$$\overline{H} = \begin{pmatrix} HV & H' \end{pmatrix} U = \begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} U.$$

Setting

$$W = \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} U,$$

(35) follows. □

Proposition 5 has immediate application to the analysis of inclusion of conditioned invariant subspaces.

Proposition 6: *Let $D \in \mathbb{F}[z]^{p\times p}$ be non-singular and let $\mathcal{V} = X_D \cap H\mathbb{F}^k[z]$ with H right prime. A conditioned invariant subspace $\mathcal{W}$ contains $\mathcal{V}$ if and only if*

$$\mathcal{W} = X_D \cap \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^{k+k'} \tag{36}$$

for some $H' \in \mathbb{F}^{p\times k'}[z]$.

Proof: If $\mathcal{W}$ is defined by (36), it is necessarily conditioned invariant. Moreover, as $H\mathbb{F}[z]^k \subset \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^{k+k'}$, it follows that

$$\mathcal{V} = X_D \cap H\mathbb{F}[z]^k \subset X_D \cap \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^{k+k'} = \mathcal{W}.$$

To prove the converse, assume $\mathcal{W} \supset \mathcal{V}$ is conditioned invariant. Clearly $\mathcal{O} = \langle\mathcal{W}\rangle \supset \langle\mathcal{V}\rangle = \mathcal{M} = H\mathbb{F}[z]^k$. By the right primeness of H, it follows from Proposition 3.3 that $\mathcal{O} = \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^{k+k'}$ and hence

$$\mathcal{W} = X_D \cap \mathcal{O} = X_D \cap \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^{k+k'}. \qquad \square$$

In general, for a linear transformation $T: \mathcal{X} \longrightarrow \mathcal{Y}$, $TM \subset N$ implies $T^*N^\perp \subset M^\perp$. For the (C_D, A_D) pair associated with D via the shift realization, a subspace $\mathcal{V} \subset X_D$ is conditioned invariant if and only if $\mathcal{V} = X_D \cap \mathcal{M}$ for some submodule of $\mathbb{F}[z]^m$. A conditioned invariant subspace $\mathcal{V} \subset X_D$ has several different representations

$$\begin{aligned}\mathcal{V} &= X_D \cap \mathcal{M} = X_D \cap FX_E = FX_E\\ &= X_D \cap F\mathbb{F}[z]^m = X_D \cap H\mathbb{F}[z]^k,\end{aligned} \tag{37}$$

where $H(z)$ is a basis matrix for, $\langle\mathcal{V}\rangle$, the submodule of $\mathbb{F}[z]^m$ generated by $\mathcal{V}$. The generating matrix H is essentially unique, i.e. up to a right unimodular factor. Here we assume that $D_1 = FE$ is such that $D_1^{-1}D$ is biproper. $\mathcal{V} = EX_F$ is just the representation of $\mathcal{V}$ as an invariant subspace of A_{D_1}. As a result, the equality $I(X_D \cap H\mathbb{F}[z]^k) = X_{D_1} \cap H\mathbb{F}[z]^k$ implies $I^*(X_{D_1} \cap H\mathbb{F}[z]^k)^\perp = (X_D \cap H\mathbb{F}[z]^k)^\perp$. Now

$$(X_D \cap H\mathbb{F}[z]^k)^\perp = \{h \in X^{\tilde{D}} \mid [H\mathbb{F}[z]^k, h] = 0\}.$$

Since $X^{\tilde{D}} = \operatorname{Ker}\tilde{D}(\sigma)$, we are led to

$$\begin{aligned}(X_D \cap H\mathbb{F}[z]^k)^\perp &= \operatorname{Ker}\tilde{D}(\sigma) \cap \operatorname{Ker}\tilde{H}(\sigma)\\ &= \operatorname{Ker}\begin{pmatrix}\tilde{D}(\sigma)\\ \tilde{H}(\sigma)\end{pmatrix}.\end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned}(X_{D_1} \cap H\mathbb{F}[z]^k)^{\perp} &= \operatorname{Ker} \tilde{D}_1(\sigma) \cap \operatorname{Ker} \tilde{H}(\sigma)\\ &= \operatorname{Ker} \begin{pmatrix} \tilde{D}_1(\sigma) \\ \tilde{H}(\sigma) \end{pmatrix}.\end{aligned} \tag{39}$$

Of course, the annihilators are computed in $X^{\tilde{D}}$ and $X^{\tilde{D}_1}$ respectively. Now the invariant subspace $\mathcal{V}$ has also the representation $\mathcal{V} = X_{D_1} \cap EX_F = EX_F$ for the above mentioned factorization. It follows that $\mathcal{V}^{\perp}$, in $X^{\tilde{D}_1}$, has a representation $\mathcal{V}^{\perp} = X^{\tilde{E}} \subset X^{\tilde{D}_1}$. Note that the factorization $D_1 = EF$ implies $\tilde{D}_1 = \tilde{F}\tilde{E}$, i.e. $\tilde{E}$ is a right factor of $\tilde{D}_1$. So we have

$$\begin{aligned}(X_{D_1} \cap E\mathbb{F}[z]^m)^{\perp} &= \operatorname{Ker} \begin{pmatrix} \tilde{D}_1(\sigma) \\ \tilde{E}(\sigma) \end{pmatrix}\\ &= \operatorname{Ker} \begin{pmatrix} I & -\tilde{F}(\sigma) \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{D}_1(\sigma) \\ \tilde{E}(\sigma) \end{pmatrix}\\ &= \operatorname{Ker} \begin{pmatrix} 0 \\ \tilde{E}(\sigma) \end{pmatrix} = X^{\tilde{E}}.\end{aligned} \tag{40}$$

From equations (39) and (40), we conclude

$$\operatorname{Ker} \tilde{E}(\sigma) = \operatorname{Ker} \begin{pmatrix} \tilde{D}_1(\sigma) \\ \tilde{H}(\sigma) \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} \tilde{F}(\sigma)\tilde{E}(\sigma) \\ \tilde{H}(\sigma) \end{pmatrix}.$$

This implies a factorization

$$\begin{pmatrix} \tilde{F}(z)\tilde{E}(z) \\ \tilde{H}(z) \end{pmatrix} = \begin{pmatrix} \tilde{G}(z) \\ \tilde{J}(z) \end{pmatrix} \tilde{E}(z).$$

Necessarily we have $\tilde{G} = \tilde{F}$ and $\tilde{H} = \tilde{J}\tilde{E}$. In particular, the factorization $\tilde{H} = \tilde{J}\tilde{E}$ implies

$$\operatorname{Ker} \tilde{E}(\sigma) \subset \operatorname{Ker} \tilde{H}(\sigma). \tag{41}$$

Changing notation slightly, we are in a position to recover a result of Willems (1997).

Corollary 2: *If $R(z)$ is left prime and $\operatorname{Ker} Q(\sigma) \subset \operatorname{Ker} R(\sigma)$, then there exists a polynomial matrix R' such that*

$$Q = \begin{pmatrix} R \\ R' \end{pmatrix}$$

and hence also

$$\operatorname{Ker} Q(\sigma) = \operatorname{Ker} \begin{pmatrix} R(\sigma) \\ R'(\sigma) \end{pmatrix}. \tag{42}$$

We proceed now to a more geometric analysis of the outer spectral assignment problem. It is well known, see Willems (1982), that every conditioned invariant subspace is the transversal intersection of an observability subspace and a tight condition invariant subspace. For a full discussion of the dual result, see Trentelman (1985). This means that, given an observable pair (C, A) in the state space $\mathcal{X}$, a condition invariant subspace $\mathcal{V}$ has a representation

$$\mathcal{V} = \mathcal{O} \cap \mathcal{T}, \tag{43}$$

with $\mathcal{O}$ an observability subspace and $\mathcal{T}$ a tight condition invariant subspace, recalling that tightness means $\mathcal{T} + \operatorname{Ker} C = \mathcal{X}$. That the intersection is transversal means that $\mathcal{X} = \mathcal{O} + \mathcal{T}$. Clearly (43) implies $\mathcal{V} \subset \mathcal{O}$, so a natural candidate for $\mathcal{O}$ would be $\mathcal{O}_*(\mathcal{V})$, the smallest observability subspace containing $\mathcal{V}$. The justification for this is our interest in outer spectral assignability. Applying Lemma 1, we have

$$\mathcal{X}/\mathcal{V} = \mathcal{O}_*(\mathcal{V})/\mathcal{V} \oplus \mathcal{T}/\mathcal{V}. \tag{44}$$

Now $\mathcal{X}/\mathcal{V}$ is a natural state space for constructing an observer for a linear map K satisfying $\operatorname{Ker} K \supset \mathcal{V}$. The direct sum representation (44) decomposes the state space into a part $\mathcal{T}/\mathcal{V}$ that has fixed dynamics and a part $\mathcal{O}_*(\mathcal{V})/\mathcal{V}$ where the dynamics or equivalently, the module structure, can be freely preassigned. Now the dimension formula, applied to the transversal intersection (43), yields

$$\begin{aligned}\dim \mathcal{X} &= \dim(\mathcal{O} + \mathcal{T})\\ &= \dim \mathcal{O} + \dim \mathcal{T} - \dim(\mathcal{O} \cap \mathcal{T}),\end{aligned} \tag{45}$$

and hence

$$\dim \mathcal{X} - \dim \mathcal{O}_*(\mathcal{V}) = \dim \mathcal{T} - \dim \mathcal{V}. \tag{46}$$

Thus we cannot take the tight subspace $\mathcal{T}$ to be neither too large nor too small. Furthermore, there is no uniqueness in such a representation. Using techniques originating in Hinrichsen *et al.* (1981), and further developed in Fuhrmann and Helmke (2001), we can actually parametrize all tight subspaces for which the transversal intersection representation (43) holds.

In Fuhrmann and Helmke (2001), tight conditions invariant subspaces were introduced and, with respect to the (C, A) defined via the shift realization associated with the non-singular polynomial matrix $D \in \mathbb{F}[z]^{p\times p}$, several alternative characterizations of tightness were given. We add now another polynomial characterization of tight conditioned invariant subspaces. This characterization is the dual of the characterization of coasting subspaces given in Trentelman (1985).

Proposition 7: *Given an observable pair (C, A) in the state space $\mathcal{X}$, with C assumed to be of full row rank. Then a subspace $\mathcal{V}$ is a tight conditioned invariant subspace if and only if*

$$\mathcal{O}_*(\mathcal{V}) = \mathcal{X}. \tag{47}$$

Proof: Let $D(z)^{-1}L(z)$ be a left coprime factorization of $C(zI - A)^{-1}$. Since C has full row rank, all minimal row indices of $D(z)$, i.e. the observability indices of the corresponding shift realization, are positive. Any conditioned invariant subspace $\mathcal{V}$ has, by Proposition 4, an essentially unique representation of the form $\mathcal{V} = X_D \cap H\mathbb{F}[z]^k$, with H of full column rank and $D(z)^{-1}H(z)$ strictly proper. If $\mathcal{V}$ is assumed to be tight, then $\langle\mathcal{V}\rangle$, the smallest submodule of $\mathbb{F}[z]^p$ containing $\mathcal{V}$, is a full submodule. That means that $H(z)$ is necessarily a square non-singular matrix. Applying the characterization of Corollary 1, we conclude that $\mathcal{O}_*(\mathcal{V}) = X_D \cap \mathbb{F}[z]^p = X_D$.

To prove the converse, let $\mathcal{O}_*(\mathcal{V}) = X_D$, i.e. $\mathcal{O}_*(\mathcal{V}) = X_D \cap \mathbb{F}[z]^p$ or $H_1(z) = I$, so $\mathcal{V} = X_D \cap H_0\mathbb{F}[z]^p$ with H_0 non-singular. By Theorem 6 in Fuhrmann and Helmke (2001), all reduced observability indices are positive, i.e. $\mathcal{V}$ is tight. □

Given a conditioned invariant subspace $\mathcal{V} \subset X_D$ having the representation $\mathcal{V} = X_D \cap H\mathbb{F}[z]^k = X_D \cap H_1H_0\mathbb{F}[z]^k$, with H_1H_0 an external/internal factorization of H, then there exists, using the extension procedure outlined in Fuhrmann and Helmke (2001), a not necessarily unique, extension of the form $\begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}$ such that

$$X_D \cap H_1H_0\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}\mathbb{F}[z]^p \tag{48}$$

with $D^{-1}\begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}$ proper. We point out that, in the paper quoted above, it is shown that these extensions can be parametrized and are the basis for the parametrization of all kernel representations of a given conditioned invariant subspace.

Given an observability subspace with a representation $\mathcal{V} = X_D \cap H_1\mathbb{F}[z]^k$, H_1 has a, not necessaily unique, extension of the form $\begin{pmatrix} H_1 & \hat{H} \end{pmatrix}$ such that

$$X_D \cap H_1\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1 & \hat{H} \end{pmatrix}\mathbb{F}[z]^p \tag{49}$$

with $D^{-1}\begin{pmatrix} H_1 & \hat{H} \end{pmatrix}$ proper. On the other hand, H_1 is right prime and hence has unimodular extensions of the form $\begin{pmatrix} H_1 & H' \end{pmatrix}$.

The next proposition relates the two extensions.

Proposition 8: *Let $D \in \mathbb{F}[z]^{p\times p}$ be non-singular. Let $\mathcal{O} \subset X_D$ be an observability subspace having the representation $\mathcal{O} = X_D \cap H\mathbb{F}[z]^k$ with H right prime and $D^{-1}H$ strictly proper. Let $\begin{pmatrix} H & \hat{H} \end{pmatrix}$ be a non-singular extension for which $D^{-1}\begin{pmatrix} H & \hat{H} \end{pmatrix}$ is proper and $X_D \cap H\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H & \hat{H} \end{pmatrix}\mathbb{F}[z]^p$. Let $\begin{pmatrix} H & H' \end{pmatrix}$ be an arbitrary unimodular extension of H and let*

$$\begin{pmatrix} K \\ K' \end{pmatrix}$$

be its polynomial unimodular inverse, i.e. we have

$$\begin{cases} \begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} K \\ K' \end{pmatrix} &= I \\ \begin{pmatrix} K \\ K' \end{pmatrix}\begin{pmatrix} H & H' \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{cases} \tag{50}$$

Then

1. *There exist appropriately sized and uniquely determined polynomial matrices R, S, with S square and non-singular, such that*

$$\hat{H} = HR + H'S. \tag{51}$$

Specifically, we have

$$\left.\begin{aligned} R &= K\hat{H} \\ S &= K'\hat{H}. \end{aligned}\right\} \tag{52}$$

2. *There exists a non-singular polynomial matrix $S \in \mathbb{F}[z]^{(p-k)\times(p-k)}$ for which*

$$\mathcal{O} = X_D \cap H\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p. \tag{53}$$

3. *For a non-singular polynomial matrix S, a necessary and sufficient condition for the equality (53) to hold is that all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are non-positive.*

Proof:

1. We compute, using (50),

$$\begin{pmatrix} K \\ K' \end{pmatrix}\begin{pmatrix} H & \hat{H} \end{pmatrix} = \begin{pmatrix} I & R \\ 0 & S \end{pmatrix}.$$

The non-singularity of the left side implies that of S. Multiplying on the left by $\begin{pmatrix} H & H' \end{pmatrix}$, we obtain

$$\begin{pmatrix} H & \hat{H} \end{pmatrix} = \begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} I & R \\ 0 & S \end{pmatrix}$$

and hence (51) follows.

2. Clearly, for any S we have the inclusion $H\mathbb{F}[z]^k \subset \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p$, hence $X_D \cap H\mathbb{F}[z]^k \times \subset X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p$ always holds. If $\hat{H}$ is any polynomial matrix for which $X_D \cap H\mathbb{F}[z]^k \subset X_D \cap \begin{pmatrix} H & \hat{H} \end{pmatrix}\mathbb{F}[z]^p$ and R, S as in part 1, then $\hat{H} = HR + H'S$. Clearly

$$\begin{aligned}\begin{pmatrix} H & \hat{H} \end{pmatrix}\mathbb{F}[z]^p &= \begin{pmatrix} H & \hat{H} \end{pmatrix}\begin{pmatrix} I & -R \\ 0 & S \end{pmatrix}\begin{pmatrix} I & R \\ 0 & S \end{pmatrix}\mathbb{F}[z]^p \\ &= \begin{pmatrix} H & \hat{H} \end{pmatrix}\begin{pmatrix} I & -R \\ 0 & S \end{pmatrix}\mathbb{F}[z]^p \\ &= \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p.\end{aligned}$$

This shows also that S is necessarily non-singular.

3. We prove necessity by contradiction. Assume not all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are nonpositive. By Lemma 4, not all the left Wiener-Hopf factorization indices of $D^{-1}H'S$, being a right inverse of $S^{-1}K'D$, are non-negative and hence the Toeplitz operator $\mathcal{T}_{D^{-1}H'S}$ is not injective. So there exists $0 \neq g_2 \in \operatorname{Ker} \mathcal{T}_{D^{-1}H'S}$. By Lemma 3.1 in Fuhrmann and Helmke (2001) and the injectivity of $H'S$, it follows that $0 \neq f = H'Sg_2 \in X_D \cap H'S\mathbb{F}[z]^{(p-k} \subset X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p$. If we have also $f \in X_D \cap H\mathbb{F}[z]^k$, then $f = H'Sg_1 = Hg_2$. By the unimodularity of $(H \;\; H')$ and the nonsingularity of S, we conclude that $g_i = 0$, $i = 1, 2$, and hence $f = 0$ in contradiction to f being non-zero. Thus, necessarily, the equality (53) holds.

   Conversely, assume all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are non-positive. By Lemma 4, all left Wiener-Hopf factorization indices of $D^{-1}H'S$ are non-negative. Without loss of generality, assume $D^{-1}H$ is in Kronecker-Hermite canonical form, with negative column indices. We reduce $D^{-1}\begin{pmatrix} H & H'S \end{pmatrix}$ to Kronecker-Hermite canonical form by applying a unimodular matrix of the form

$$\begin{pmatrix} I & R \\ 0 & V \end{pmatrix}$$

   on the right, i.e. we have

$$D^{-1}\begin{pmatrix} H & H'S \end{pmatrix}\begin{pmatrix} I & R \\ 0 & V \end{pmatrix} = D^{-1}\begin{pmatrix} H & \hat{H} \end{pmatrix}$$

   with the column degrees of $D^{-1}\hat{H}$ non-negative.

   Now assume $f \in X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p$, i.e.

$$\begin{aligned} D^{-1}\begin{pmatrix} H & H'S \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= D^{-1}\begin{pmatrix} H & H'S \end{pmatrix}\begin{pmatrix} I & R \\ 0 & V \end{pmatrix}\begin{pmatrix} I & -RV^{-1} \\ 0 & V^{-1} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \\ &= D^{-1}\begin{pmatrix} H & \hat{H} \end{pmatrix}\begin{pmatrix} g_1 - RV^{-1}g_2 \\ g_2 \end{pmatrix}.\end{aligned}$$

   By the predictable degree property, see Forney (1972), the strict properness of

$$D^{-1}\begin{pmatrix} H & \hat{H} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

   implies $g_2 = 0$. In turn, this implies the inclusion $X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p \subset X_D \cap H\mathbb{F}[z]^k$ and, since the inverse inclusion holds always, the equality (53) follows. □

The assumption of right primeness in Proposition 8 can be easily removed.

Proposition 9: *Let $D \in \mathbb{F}[z]^{p\times p}$ be non-singular. Let $\mathcal{V} \subset X_D$ be a conditioned invariant subspace having the representation $\mathcal{V} = X_D \cap H\mathbb{F}[z]^k$ with H of full column rank and $D^{-1}H$ strictly proper. Let $H = H_1H_0$ be an external/internal factorization and let $\hat{H} \in \mathbb{F}[z]^{p\times(p-k)}$ be such that $\begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}$ is non-singular. Then*

$$X_D \cap H_1H_0\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}\mathbb{F}[z]^p \tag{54}$$

if and only if we have

$$X_D \cap H_1\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1 & \hat{H} \end{pmatrix}\mathbb{F}[z]^p. \tag{55}$$

Proof: To prove the if part, assume (55) holds. Clearly, we always have the inclusion $X_D \cap H_1H_0\mathbb{F}[z]^k \subset X_D \cap \begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}\mathbb{F}[z]^p$. To prove the converse inclusion, assume $f \in X_D \cap \begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}\mathbb{F}[z]^p$ and write, using (55), $f = H_1(H_0g_1) + \hat{H}g_2 = H_1g_0 \in X_D \cap H_1\mathbb{F}[z]^k$. Since $\begin{pmatrix} H_1H_0 & \hat{H} \end{pmatrix}$ is non-singular, so is $(H_1 \;\; \hat{H})$, which implies $g_0 = H_0g_1$ and $g_2 = 0$. This shows that $f \in X_D \cap H_1H_0\mathbb{F}[z]^k$.

To prove the only if part, assume (54) holds. Clearly, we always have the inclusion $X_D \cap H_1\mathbb{F}[z]^k \subset X_D \cap (H_1 \;\; \hat{H})\mathbb{F}[z]^p$. So let $f \in X_D \cap (H_1 \;\; \hat{H})\mathbb{F}[z]^p$ and write $f = H_1g_1 + \hat{H}g_2$. Since $f \in X_D$, $D^{-1}f$ is strictly proper. Using the direct sum decomposition $\mathbb{F}[z]^k = X_{H_0} \oplus H_0\mathbb{F}[z]^k$, we write $g_1 = g_1' + H_0g_1''$, with $g_1' \in X_{H_0}$.

Necessarily, there exists a strictly proper h' for which $g_1' = H_0 h'$. Substituting back, we have

$$\begin{aligned} D^{-1}f &= D^{-1}H_1 g_1' + D^{-1}H_1 H_0 g_1'' + D^{-1}\hat{H}g_2 \\ &= D^{-1}H_1 H_0 h' + D^{-1}H_1 H_0 g_1'' + D^{-1}\hat{H}g_2. \end{aligned}$$

it follows that $f - H_1 H_0 h' = f - H_1 g_1' \in X_D$. Hence $H_1 H_0 g_1'' + \hat{H}g_2 \in X_D \cap \begin{pmatrix} H_1 H_0 & \hat{H} \end{pmatrix}\mathbb{F}[z]^p = X_{D\,1} H_0\,\mathbb{F}[z]^k \subset X_D \cap H_1 \mathbb{F}[z]^k$. It follows that $f \in X_D \cap H_1\mathbb{F}[z]^k$. □

Corollary 3: *Under the assumptions of Proposition* 9, *we have*

$$X_D \cap H_1 H_0 \mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}\mathbb{F}[z]^p \tag{56}$$

if and only if we have

$$X_D \cap H_1 \mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p. \tag{57}$$

Proof: Follows from Propositions 8 and 9. □

Given an observability subspace in X, its image under the map ϕ defined in (20) is an observability subspace for the pair (C_D, A_D) and hence, by Proposition 4, has the essentially unique representation of the form $\mathcal{V} = X_D \cap H\,\mathbb{F}[z]^k$. We want to extend the polynomial matrix H to a nonsingular polynomial matrix of the form $\begin{pmatrix} H & H'S \end{pmatrix}$ in such a way that

1. In a sense we will make precise, S is big enough so that we have the equality
$$\mathcal{O} = X_D \cap H\,\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p.$$
2. In the same sense as before, S is small enough so that there exists a module structure on X_D so that we have the isomorphism $X_D / X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p \simeq X_S$.

The second condition can be interpreted in the following way. There exists a surjective map $\Pi : X_D \longrightarrow X_S$ with $\operatorname{Ker}\Pi = X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p$. This is also equivalent to the codimension formula $\operatorname{codim} X_D \cap (H \;\; H'S)\mathbb{F}[z]^p = \deg\det S$. This indicates that Π is related to the projection $\pi_{(H \;\; H'S)}$. If both conditions are satisfied, then the $\mathbb{F}[z]$-module structure defined on $X_D / X_D \cap (H \;\; H'S)\mathbb{F}[z]^p$ by pulling back the $\mathbb{F}[z]$-module structure on X_S will be called the induced shift module structure. Our principal effort will be to show that appropriate (equivalence classes) of extensions of the form $(H \;\; H'S)$ are in a bijective correspondence with (equivalence classes) of friends of $\mathcal{O}$, where $J_1, J_2 \in \mathcal{G}(\mathcal{O})$ are equivalent if they induce the same shift module structure on $X_D/\mathcal{O}$.

$$\begin{tikzcd}
 & X_D = \mathcal{T} + \mathcal{O}_*(\mathcal{V}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_*(\mathcal{V}) \arrow[dr, dash] & & \mathcal{T} \arrow[dl, dash] \\
 & \mathcal{V} &
\end{tikzcd}$$

Figure 2.

Now the coprime factorizations $D(z)^{-1}\Phi(z) = C(zI - A)^{-1}$ can be rewritten as $\Phi(z)(zI - A) = D(z)C$ which implies $\Phi(z)(zI - A + JC) = (D(z) + \Phi(z)J)C$ or $(D(z) + \Phi(z)J)^{-1}\Phi(z) = C(zI - A + JC)^{-1}$. If $J \in \mathcal{G}(\mathcal{O})$, then the subspace $\mathcal{O}$ is an invariant subspace, or submodule, of $X_{D(z)+\Phi(z)J}$, i.e. corresponds to a factorization

$$D(z) + \Phi(z)J = \begin{pmatrix} H & H'S \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}.$$

Obviously, $D^{-1}(D(z) + \Phi(z)J) = I + (D^{-1}\Phi(z))J$ is normalized biproper and so, as linear spaces, we have $X_D = X_{D(z)+\Phi(z)J}$.

The next theorem gives a complete analysis of the relation between the procedure of appropriately extending H to a non-singular polynomial matrix and the derivation of all friends of the given subspace $\mathcal{O}$. In order to ease the reading of the theorem, it is advisable to refer to figure 2.

Theorem 6: *Let $D \in \mathbb{F}[z]^{p\times p}$ be non-singular and let $\mathcal{V} \subset X_D$ be a conditioned invariant subspace, with respect to (C, A) defined by the shift realization* (15), *having the representation*

$$\mathcal{V} = X_D \cap H\,\mathbb{F}[z]^k, \tag{58}$$

with H of full column rank such that $D^{-1}H$ is strictly proper. Let $H = H_1 H_0$ be an external/internal factorization and let $\begin{pmatrix} H_1 & H' \end{pmatrix}$ be an arbitrary completion *of H_1 to a $p \times p$ unimodular matrix with*

$$\begin{pmatrix} K_1 \\ K' \end{pmatrix} = \begin{pmatrix} H_1 & H' \end{pmatrix}^{-1},$$

i.e.

$$\begin{pmatrix} K_1 \\ K' \end{pmatrix}\begin{pmatrix} H_1 & H' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{59}$$

Then

1. *The polynomial matrix K' is essentially uniquely determined, that is up to a left unimodular factor.*
2. *Let S be an arbitrary non-singular polynomial matrix. Then*
   (a) *The projection $\pi_{(H_1H_0 \ \ H'S)}$ can be rewritten as*
   $$\pi_{(H_1H_0 \ \ H'S)} = H_1\pi_{H_0}K_1 + H'\pi_S K', \tag{60}$$
   (b)
   $$\operatorname{Ker}\pi_{(H_1H_0 \ \ H'S)} = \begin{pmatrix} H_1H_0 & H'S \end{pmatrix}\mathbb{F}[z]^p, \tag{61}$$
   (c)
   $$\operatorname{Im}\pi_{(H_1H_0 \ \ H'S)} = H_1X_{H_0} \oplus H'X_S \simeq X_{H_0} \oplus X_S. \tag{62}$$
3. (a) *$\mathcal{O}_*(\mathcal{V})$, the smallest observability subspace containing $\mathcal{V}$, has the representations, with S a non-singular polynomial matrix,*
   $$\mathcal{O}_*(\mathcal{V}) = X_D \cap H_1\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p \tag{63}$$
   if and only if all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are non-positive.
   (b) *The following statements are equivalent:*
   (i) *All right Wiener-Hopf factorization indices of $S^{-1}K'D$ are non-negative.*
   (ii) *The Toeplitz operator $\mathcal{T}_{\widetilde{\tilde{D}K'\tilde{S}^{-1}}}$ is injective.*
   (iii) *The Toeplitz operator $\mathcal{T}_{\tilde{D}(\tilde{K}_1 \ \ \widetilde{K'\tilde{S}^{-1}})}$ is injective.*
   (iv) *All left Wiener-Hopf factorization indices of $D^{-1}(H_1 \ \ H'S)$ are non-positive.*
   (v) *We have the codimension formula*
   $$\operatorname{codim}\left(X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p\right) = \deg\det S. \tag{64}$$
   (vi) *We have the isomorphism*
   $$X_D/(X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p) \simeq H'X_S. \tag{65}$$
   (vii) *The mapping $\pi_{(H_1 \ \ H'S)}: X_D \longrightarrow X_{(H_1 \ \ H'S)} = H'X_S$ is surjective.*
4. *Define a subspace $\mathcal{T} \subset X_D$ by*
   $$\mathcal{T} = X_D \cap \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^p = X_D \cap \begin{pmatrix} H_1H_0 & H' \end{pmatrix}\mathbb{F}[z]^p. \tag{66}$$
   Then
   (a) *$\mathcal{T}$ is a tight conditioned invariant subspace of X_D, i.e.*
   $$\mathcal{T} + \operatorname{Ker} C = X_D. \tag{67}$$
   (b) *The following statements are equivalent:*
   (i) *All right Wiener-Hopf factorization indices of $H_0^{-1}K_1D$ are non-negative.*
   (ii) *We have the codimension formula*
   $$\begin{aligned}\operatorname{codim}\mathcal{T} &= \operatorname{codim}\left(X_D \cap \begin{pmatrix} H_1H_0 & H' \end{pmatrix}\mathbb{F}[z]^p\right) \\ &= \deg\det H_0.\end{aligned} \tag{68}$$
   (iii) *All left Wiener-Hopf factorization indices of D^{-1} $(H_1H_0 \ \ H')$ are non-positive.*
   (iv) *The mapping $\pi_{(H_1H_0 \ \ H')}: X_D \longrightarrow X_{(H_1H_0 \ \ H')} = H_1X_{H_0}$ is surjective.*
   (v) *We have the isomorphism*
   $$X_D/(X_D \cap \begin{pmatrix} H_1H_0 & H' \end{pmatrix}\mathbb{F}[z]^p) \simeq H_1X_{H_0}. \tag{69}$$
5. *Assume all the right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero. With the subspace $\mathcal{V}$ defined by* (58) *and $\mathcal{T}$ defined by* (66), *we have*
   (a) *All left Wiener-Hopf factorization indices of $D^{-1}\begin{pmatrix} H_1H_0 & H'S \end{pmatrix}$ are non-positive.*
   (b) *There exists a, not necessarily unique, non-singular polynomial matrix*
   $$L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$
   such that
   $$D_1 = \begin{pmatrix} H_1H_0 & H'S \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \tag{70}$$
   and $D_1^{-1}D$ is normalized biproper.
   (c) *The conditioned invariant subspaces $\mathcal{O}_*(\mathcal{V})$ and $\mathcal{T}$ are compatible.*
   (d) *$\mathcal{V}$ is the transversal intersection of $\mathcal{O}_*(\mathcal{V})$ and $\mathcal{T}$, i.e. we have*
   $$\mathcal{V} = \mathcal{O}_*(\mathcal{V}) \cap \mathcal{T} \tag{71}$$
   and
   $$X_D = \mathcal{O}_*(\mathcal{V}) + \mathcal{T}. \tag{72}$$

(e) *We have the codimension formula*

$$\operatorname{codim} \mathcal{V} = \deg \det S + \deg \det H_0. \tag{73}$$

(f) *We have the direct sum representation*

$$X_D/\mathcal{V} = \mathcal{O}_*(\mathcal{V})/\mathcal{V} \oplus \mathcal{T}/\mathcal{V}. \tag{74}$$

6. *We have the isomorphisms*

$$X_D/\mathcal{O}_*(\mathcal{V}) \simeq \mathcal{T}/\mathcal{V} \simeq X_S, \tag{75}$$

$$X_D/\mathcal{T} \simeq \mathcal{O}_*(\mathcal{V})/\mathcal{V} \simeq X_{H_0}, \tag{76}$$

and

$$X_D/\mathcal{V} = \mathcal{O}_*(\mathcal{V})/\mathcal{V} \oplus \mathcal{T}/\mathcal{V} \simeq H'X_S \oplus H_1 X_{H_0}. \tag{77}$$

7. *Assume all the right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero. Let $\Phi(z)$ be a basis matrix for the polynomial model X_D. Then there exists $J \in \mathcal{G}(\mathcal{V})$ of the form $J = \Phi(z)\hat{J}$, with $\hat{J} \in \mathbb{F}^{p\times n}$, such that*
   (a) *There exists a factorization*

$$D + \Phi\hat{J} = \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \tag{78}$$

   with $D^{-1}(D + \Phi\hat{J})$ strictly proper.
   (b) *The polynomial models X_D and $X_{D+\Phi\hat{J}}$ contain the same elements.*
   (c) *The $S_{D+\Phi\hat{J}}$-module structure on X_D is output injection equivalent to the S_D-module structure.*
   (d) *Figure 3 is commutative*:
   Thus the S-induced shift module structure on X_D is equivalent (*equal??*) *to the $S_{D+\Phi J}$-module structure.*

8. *There exists a bijective correspondence between the set of all equivalence classes of non-singular extensions of the form $\begin{pmatrix} H_1 & H'S \end{pmatrix}$ with $S \in \mathbb{F}[z]^{(p-k)\times(p-k)}$ non-singular such that all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero, where two extensions $\begin{pmatrix} H_1 & H'S_i \end{pmatrix}$, $i = 1, 2$, are considered equivalent if S_1, S_2 differ at most by a right unimodular factor, and the equivalence class of output injection maps $J \in \mathcal{G}(\mathcal{O})$, where two output injection maps are equivalent if they induce the same module structure on the quotient space $X_D/\mathcal{O}_*(\mathcal{V})$.*
   The correspondence is as follows:
   (a) *Given a non-singular $S \in \mathbb{F}[z]^{(p-k)\times(p-k)}$ for which all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero, then there exist, non-unique,*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
X_D \arrow[r, "H_1\pi_{H_0}K_1 + H'\pi_S K'"] \arrow[d, "A_D - JC_D"'] & H_1 X_{H_0} \oplus H'X_S \arrow[d, "H_1 S_{H_0} K_1 + H'S_S K'"] \\
X_D \arrow[r, "H_1\pi_{H_0}K_1 + H'\pi_S K'"] & H_1 X_{H_0} \oplus H'X_S
\end{tikzcd}$$

Figure 3.

   factorizations of the form

$$D + \Phi\hat{J} = \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}. \tag{79}$$

   All these factorizations induce the same, uniquely determined, module structure on $X_D/X_D \cap (H_1 \;\; H'S)\mathbb{F}[z]^p$, i.e. the map $J = \Phi(z)\hat{J}$ induces a unique map $\overline{J}$: $\mathbb{F}^p \longrightarrow X_D/(X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p)$.
   (b) *Given $J \in \mathcal{G}(\mathcal{V})$, write $J = \Phi(z)\hat{J}$ with $\hat{J} \in \mathbb{F}^{n\times p}$ and let the corresponding factorization be*

$$D + \Phi\hat{J} = \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \tag{80}$$

   then S is uniquely determined up to a right unimodular factor and all the right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero.

Proof:

1. Follows from the fact that K' is a maximal left annihilator of H_1, or of H for that matter.
2. (a) We compute, for $f \in \mathbb{F}[z]^p$,

$$\begin{aligned}
\pi_{\begin{pmatrix} H_1H_0 & H'S \end{pmatrix}} f &= \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}\pi_-\begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}^{-1} f \\
&= \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}\pi_-\begin{pmatrix} H_0 & 0 \\ 0 & S \end{pmatrix}^{-1}\begin{pmatrix} K_1 \\ K' \end{pmatrix} f \\
&= \begin{pmatrix} H_1 & H' \end{pmatrix}\begin{pmatrix} H_0 & 0 \\ 0 & S \end{pmatrix}\pi_-\begin{pmatrix} H_0^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \\
&\quad\times\begin{pmatrix} K_1 \\ K' \end{pmatrix} f \\
&= \begin{pmatrix} H_1 & H' \end{pmatrix}\begin{pmatrix} H_0\pi_- H_0^{-1}K_1 f \\ S\pi_- S^{-1}K'f \end{pmatrix} \\
&= \begin{pmatrix} H_1 & H' \end{pmatrix}\begin{pmatrix} \pi_{H_0}K_1 f \\ \pi_S K'f \end{pmatrix} \\
&= H_1\pi_{H_0}K_1 f + H'\pi_S K'f.
\end{aligned}$$

This proves (60). Equality (61) is trivial.

We note that since $(H_1 \quad H')$ is unimodular, we have

$$\begin{aligned}\pi_{(H_1 H_0 \quad H'S)}\mathbb{F}[z]^p &= X_{(H_1 H_0 \quad H'S)}\\ &= \begin{pmatrix} H_1 & H' \end{pmatrix} X_{\begin{pmatrix} H_0 & 0 \\ 0 & S \end{pmatrix}}\\ &= H_1 X_{H_0} + H' X_S.\end{aligned}$$

The isomorphism $H'X_S \simeq X_S$ follows trivially from the fact that the multiplication map by K': $H'X_S \longrightarrow X_S$ is bijective.

(b) The proof is analogous.

(c) The proof is analogous.

3. (a) Follows from Proposition 8.

(b) (i) $\Leftrightarrow$ (ii)

This is standard in the analysis of Toeplitz operators.

(iii) $\Leftrightarrow$ (iv)

This is standard in the analysis of Toeplitz operators.

(i) $\Leftrightarrow$ (iv)

Assume all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are non-negative. To show that all left Wiener-Hopf factorization indices of $D^{-1}(H_1 \quad H'S)$ are non-positive is equivalent to showing that all right Wiener-Hopf factorization indices of

$$\begin{pmatrix} K_1 \\ S^{-1}K' \end{pmatrix} D$$

are non-negative. In turn this is equivalent to showing that the Toeplitz operator

$$\hat{\mathcal{T}}_{\begin{pmatrix} K_1 \\ S^{-1}K' \end{pmatrix} D}$$

is surjective. Note that both $\hat{\mathcal{T}}_{K_1 D}$ and $\hat{\mathcal{T}}_{S^{-1}K'D}$ are surjective. Thus

$$\hat{\mathcal{T}}_{\begin{pmatrix} K_1 \\ S^{-1}K' \end{pmatrix} D}$$

is surjective if and only if its adjoint $(\mathcal{T}_{\tilde{D}\tilde{K}_1} \quad \mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}})$ is injective which is the case if and only if $\operatorname{Im}\mathcal{T}_{\tilde{D}\tilde{K}_1} \cap \mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}} = \{0\}$. To see this, assume $f \in \operatorname{Im}\mathcal{T}_{\tilde{D}\tilde{K}_1} \cap \mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}}$. Therefore there exist polynomial vectors f_1, f_2, and strictly proper rational functions h_1, h_2, such that

$$\begin{aligned}f &= \pi_+\tilde{D}\tilde{K}_1 f_1 = \tilde{D}\tilde{K}_1 f_1 - h_1\\ &= \pi_+\tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 = \tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 - h_2.\end{aligned}$$

With $h = h_1 - h_2$, we can write

$$\tilde{D}\tilde{K}_1 f_1 - \tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 = h, \tag{81}$$

or

$$\tilde{K}_1 f_1 - \tilde{K}'\tilde{S}^{-1}f_2 = \tilde{D}^{-1}h. \tag{82}$$

Multiplying this on the left by $\tilde{H}_1$, and using (59), we get $f_1 = \tilde{H}_1\tilde{D}^{-1}h$ and consequently, multiplying the last equality by $\tilde{H}_0$, that

$$\tilde{H}_0 f_1 = \tilde{H}_0\tilde{H}_1\tilde{D}^{-1}h = \tilde{H}\tilde{D}^{-1}h.$$

The right hand side is strictly proper, as $D^{-1}H$ is, whereas the left hand side is polynomial, so both are necessarily zero. By the non-singularity of H_0, it follows that $f_1 = 0$. Next, we multiply (82) by $\tilde{H}'$ to get $-\tilde{S}^{-1}f_2 = \tilde{H}'\tilde{D}^{-1}h$ or $\tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 = -h$. The last equality implies $\mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}}f_2 = \pi_+\tilde{D}\tilde{K}'\tilde{S}^{-1} \times f_2 = 0$. By our assumption, all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are non-negative and hence also all left Wiener-Hopf factorization indices of $\tilde{D}\tilde{K}'\tilde{S}^{-1}$ are non-negative and $\mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}}$ is injective. This shows $f_2 = 0$ and we are done.

Conversely, assume all left Wiener-Hopf factorization indices of $D^{-1}\begin{pmatrix} H_1 & H'S \end{pmatrix}$ are non-positive. Thus all right Wiener-Hopf factorization indices of

$$\begin{pmatrix} K_1 \\ S^{-1}K' \end{pmatrix} D$$

are non-negative, i.e. the Toeplitz operator

$$\hat{\mathcal{T}}_{\begin{pmatrix} K_1 \\ S^{-1}K' \end{pmatrix} D}$$

is surjective. Necessarily, also $\hat{\mathcal{T}}_{S^{-1}K'D}$ is surjective which is equivalent to all right Wiener-Hopf factorization indices of $S^{-1}K'D$ being non-negative.

(vii) $\Rightarrow$ (vi)

Assume the map $\pi_{(H_1 \quad H'S)}$: $X_D \longrightarrow X_{(H_1 \quad H'S)} = H'X_S$ is surjective. Noting that $\operatorname{Ker}\pi_{(H_1 \quad H'S)}|X_D = X_D \cap (H_1 \quad H'S)\mathbb{F}[z]^p$, the isomorphism (65) follows.

(vi) $\Rightarrow$ (v)
Assume that we have the isomorphism (65). This implies

$$\operatorname{codim}\left(X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p\right) = \dim H'X_S = \deg\det S,$$

i.e. the codimension formula (64) holds.

(v) $\Rightarrow$ (vii)
Assume the codimension formula (64) holds. Using the fact that, for every linear transformation A defined on X, we have $\dim X = \dim \operatorname{Ker} A + \dim \operatorname{Im} A$, and applying this to the map $\pi_{(H_1 \quad H'S)}|X_D$, we have

$$\begin{aligned}\dim X_D &= \dim(X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p) \\ &\quad + \dim \pi_{(H_1 \quad H'S)}(X_D) \\ &= \dim X_D - \deg\det S + \dim \pi_{(H_1 \quad H'S)}(X_D)\end{aligned}$$

and this implies $\dim \pi_{(H_1 \quad H'S)}(X_D) = \deg\det S$. Since, by (60), we have $\pi_{(H_1 \quad H'S)}(X_D) \subset H'X_S$, we must have equality, i.e. the map $\pi_{(H_1 \quad H'S)}\colon X_D \longrightarrow H'X_S$ is surjective.

(iv) $\Leftrightarrow$ (vii)
All left Wiener-Hopf factorization indices of $D^{-1}\begin{pmatrix} H_1 & H'S \end{pmatrix}$ being non-positive is equivalent to the Toeplitz operator $\mathcal{T}_{D^{-1}(H_1 \quad H'S)} : \mathbb{F}[z]^p \longrightarrow \mathbb{F}[z]^p$ being surjective. We apply Theorem 3.3 in Fuhrmann and Helmke (2001) to conclude that this is equivalent to the projection $\pi_{(H_1 \quad H'S)} : X_D \longrightarrow X_{(H_1 \quad H'S)} = H'X_S$ being surjective. However, by (62), we have $\operatorname{Im} \pi_{(H_1 \quad H'S)} = H'X_S$ and we are done.

4. (a) Since $\begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^p$ is a submodule of $\mathbb{F}[z]^p$, it follows from Theorem 2 that $\mathcal{T} = X_D \cap \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^p$ is a conditioned invariant subspace. To show that $\mathcal{T}$ is tight, we need to show that given any $f \in X_D$, it has a decomposition of the form $f = f_1 + f_2$ with $f_1 \in \mathcal{T}$ and $f_2 \in \operatorname{Ker} C$. So let us assume $f \in X_D$. Now

$$\begin{aligned}\mathcal{T} &= X_D \cap \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^p \\ &= X_D \cap \begin{pmatrix} H_1 & H' \end{pmatrix}\begin{pmatrix} H_0 & 0 \\ 0 & I \end{pmatrix}\mathbb{F}[z]^p.\end{aligned}$$

Since $\begin{pmatrix} H_1 & H' \end{pmatrix}$ is unimodular, there exist appropriately sized polynomial vectors g_1, g_2 such that

$$f = \begin{pmatrix} H_1 & H' \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{83}$$

Using the direct sum representation $\mathbb{F}[z]^k = X_{H_0} \oplus H_0\mathbb{F}[z]^k$, we write $g_1 = g_1' + H_0 g_2'$ with $g_1' \in X_{H_0}$. This implies the existence of a strictly proper function h for which $g_1' = H_0 h$. Substituting back into (83), we have

$$\begin{aligned}f &= \begin{pmatrix} H_1 & H' \end{pmatrix}\begin{pmatrix} H_0 g_2' + g_1' \\ g_2 \end{pmatrix} \\ &= \begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} g_2' \\ g_2 \end{pmatrix} + H_1 g_1' = \begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} g_2' \\ g_2 \end{pmatrix} + H_1 H_0 h.\end{aligned}$$

Clearly, $H_1 H_0 h = Hh$ and as $D^{-1}H$ is strictly proper, $D^{-1}Hh = (D^{-1}H)h$ as a product of two strictly proper functions is in $\operatorname{Ker} C$. Moreover, $H_1 g_1' \in X_D$ as $D^{-1}H_1 g_1' = (D^{-1}H)h$ is strictly proper. Now the equality

$$f = \begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} g_2' \\ g_2 \end{pmatrix} + H_1 g_1'$$

and the assumption that $f \in X_D$ imply that also

$$\begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} g_2' \\ g_2 \end{pmatrix} \in X_D.$$

Thus

$$\begin{pmatrix} H & H' \end{pmatrix}\begin{pmatrix} g_2' \\ g_2 \end{pmatrix} \in X_D \cap \begin{pmatrix} H & H' \end{pmatrix}\mathbb{F}[z]^p$$

and hence $\mathcal{T}$ is tight.

(b) This is the counterpart of statement (3b) of this theorem with the roles of H_1, H', as well as those of H_0, S, reversed.

5. (a) We extend the method of proof used in part 3 to show that all left Wiener-Hopf factorization indices of $D^{-1}\begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}$ are non-positive. This is equivalent to all right Wiener-Hopf indices of

$$D^{-1}\begin{pmatrix} H_0^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix}\begin{pmatrix} K_1 \\ K' \end{pmatrix}D$$

being non-negative, and hence to the Toeplitz operator

$$\hat{\mathcal{T}}_{\begin{pmatrix} H_0^{-1}K_1 \\ S^{-1}K' \end{pmatrix}D}$$

being surjective. In turn, this is equivalent to the injectivity of $(\mathcal{T}_{\tilde{D}\tilde{K}_1\tilde{H}_0^{-1}} \quad \mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}})$ which is the case if and only if

$$\operatorname{Im}\mathcal{T}_{\tilde{D}\tilde{K}_1\tilde{H}_0^{-1}} \cap \operatorname{Im}\mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}} = \{0\}.$$

To prove the last identity, assume $f \in \operatorname{Im}\mathcal{T}_{\tilde{D}\tilde{K}_1\tilde{H}_0^{-1}} \cap \operatorname{Im}\mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}}$. Then there exist polynomial vectors f_1, f_2 and strictly proper rational functions h_1, h_2 such that

$$\begin{aligned} f &= \pi_+\tilde{D}\tilde{K}_1\tilde{H}_0^{-1}f_1 = \tilde{D}\tilde{K}_1\tilde{H}_0^{-1} - h_1 \\ &= \pi_+\tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 = \tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 - h_2. \end{aligned}$$

With $h = h_1 - h_2$, we can write

$$\tilde{D}\tilde{K}_1\tilde{H}_0^{-1}f_1 - \tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 = h,$$

and hence

$$\tilde{K}_1\tilde{H}_0^{-1}f_1 - \tilde{K}'\tilde{S}^{-1}f_2 = \tilde{D}^{-1}h. \tag{84}$$

Multiplying on the left by $\tilde{H}_1$ and using (59), we have $\tilde{H}_0^{-1}f_1 = \tilde{H}_1\tilde{D}^{-1}h$, or $f_1 = \tilde{H}_0\tilde{H}_1\tilde{D}^{-1}h = \tilde{H}\tilde{D}^{-1}h$. Since $D^{-1}H$ is strictly proper, the right hand side is strictly proper, while the left side is polynomial, so necessarily both vanish. Using $f_1 = 0$, it follows from (84) that $\tilde{D}\tilde{K}'\tilde{S}^{-1}f_2 = -h$ and hence that $f_2 \in \operatorname{Ker}\mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}}$. But, by assumption, all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero and so are all left Wiener-Hopf factorization indices of $\tilde{D}\tilde{K}'\tilde{S}^{-1}$ which implies the injectivity of $\mathcal{T}_{\tilde{D}\tilde{K}'\tilde{S}^{-1}}$. This means that $f_2 = 0$ and we are done.

(b) Since, by part (a), all left Wiener-Hopf factorization indices of $D^{-1}(H_1H_0 \quad H'S)$ are non-positive, the existence follows from Theorem 3.7 in Fuhrmann (1981).

(c) From the factorization (70), we obtain the two factorizations

$$\begin{aligned} D_1 &= \begin{pmatrix} H_1 & H'S \end{pmatrix}\left(\begin{pmatrix} H_0 & 0 \\ 0 & I \end{pmatrix}L\right) \\ &= \begin{pmatrix} H_1H_0 & H' \end{pmatrix}\left(\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}L\right). \end{aligned} \tag{85}$$

This shows that with respect to the X_{D_1} module structure, both $X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p$ and $X_D \cap (H_1H_0 \quad H')\mathbb{F}[z]^p$ are invariant subspaces, or with respect to the X_D module structure, they are compatible conditioned invariant subspaces.

(d) Note that the greatest common left divisor of $(H_1 \quad H'S)$ and $(H_1H_0 \quad H')$ is $(H_1 \quad H')$ which is unimodular. Applying Proposition 1, we conclude that

$$\mathcal{O}_*(\mathcal{V}) + \mathcal{T} = X_D \cap \begin{pmatrix} H_1 & H' \end{pmatrix}\mathbb{F}[z]^p = X_D.$$

Similarly, as the least common right multiple of $(H_1 \quad H'S)$ and $(H_1H_0 \quad H')$ is $(H_1H_0 \quad H'S)$, it follows that

$$\mathcal{O}_*(\mathcal{V}) \cap \mathcal{T} = X_D \cap \begin{pmatrix} H_1H_0 & H'S \end{pmatrix}\mathbb{F}[z]^p = \mathcal{V}.$$

(e) Follows from Proposition 6 in Fuhrmann and Helmke (2001) and the fact that all left Wiener-Hopf indices of $D^{-1}(H_1H_0 \quad H'S)$ are non-positive.

(f) Follows from applying Lemma 1 and using the fact that $\mathcal{V}$ is the transversal intersection of $\mathcal{O}_*(\mathcal{V})$ and $\mathcal{T}$.

6. The isomorphisms follow from the inclusions summarized in figure 2. Note that $\mathcal{V} = \mathcal{O}_*(\mathcal{V}) \cap \mathcal{T}$ implies that $(\mathcal{O}_*(\mathcal{V}) + \mathcal{T})/\mathcal{V} = \mathcal{O}_*(\mathcal{V})/\mathcal{V} \oplus \mathcal{T}/\mathcal{V}$. By part 5c, the conditioned invariant subspaces $\mathcal{O}_*(\mathcal{V})$ and $\mathcal{T}$ are compatible, so the isomorphism (77) is not only a linear subspace isomorphism but an $\mathbb{F}[z]$-module isomorphism for the module structure induced by any friend in $\mathcal{G}(\mathcal{O}_*(\mathcal{V}) \cap \mathcal{G}(\mathcal{T}))$.

7. (a) By Part 3b, the assumption that all right Wiener-Hopf indices of $S^{-1}K'D$ are non-negative is equivalent to all left Wiener-Hopf indices of $D^{-1}\begin{pmatrix} H_1 & H'S \end{pmatrix}$ being non-positive. In turn, by Theorem 3.7 in Fuhrmann (1981), this is equivalent to the existence of a non-singular polynomial matrix

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

for which

$$D^{-1}\begin{pmatrix} H_1 & H'S \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

is normalized biproper. Writing

$$D_1 = \begin{pmatrix} H_1 & H'S \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

the condition that $D^{-1}D_1$ is normalized biproper is equivalent to the existence of a representation $D_1 = D + \Phi\hat{J}$ for some constant matrix $\hat{J}$.

(b) Since $D^{-1}(D+\Phi\hat{J})$ is normalized biproper, this implies, see Lemma 5.5 in Fuhrmann and Willems (1980), that the polynomial models X_D and $X_{D+\Phi\hat{J}}$ contain the same elements. However, the respective module structures are different.

(c) By Theorem 3.2 in Fuhrmann (1981), it follows from the fact that $D^{-1}(D+\Phi\hat{J})$ is normalized biproper, that the $X_{D+\Phi\hat{J}}$ module structure is obtained from the X_D module structure by output injection.

(d) For $f \in X_D$, and using (60), we compute

$$\begin{aligned}
&(H_1 S_{H_0} K_1 + H'S_S K')(H_1 \pi_{H_0} K_1 + H'\pi_S K')f \\
&\quad = H_1 \pi_{H_0} z \pi_{H_0} K_1 f + H'\pi_S z \pi_S K' f \\
&\quad = H_1 \pi_{H_0} z K_1 f + H'\pi_S z K' f \\
&\quad = H_1 \pi_{H_0} K_1 (zf) + H'\pi_S K'(zf).
\end{aligned}$$

Note that $S_D f = zf - D(z)\xi_f$, with $\xi_f = (D^{-1}f)_{-1} = C_D f$. But, as $D^{-1}(D+\Phi\hat{J})$ is normalized biproper, we have also $((D+\Phi\hat{J})^{-1}f)_{-1} = (D^{-1}f)_{-1} = \xi_f$, i.e. $C_D f = C_{D+\Phi\hat{J}} f$. Thus, we can write $zf = S_{D+\Phi\hat{J}} f + (D+\Phi\hat{J})\xi_f$. Substituting back into the previous expression and noting that from the factorization (78) it follows

$$\begin{aligned}
&(H_1 S_{H_0} K_1 + H'S_S K')(H_1 \pi_{H_0} K_1 + H'\pi_S K')f \\
&\quad = (H_1 \pi_{H_0} K_1 + H'\pi_S K') S_{D+\Phi\hat{J}} f.
\end{aligned}$$

However,

$$\begin{aligned}
S_{D+\Phi\hat{J}} f &= zf - (D+\Phi\hat{J})\xi_f \\
&= zf - D\xi_f - \Phi\hat{J}\xi_f = (zf - D\xi_f) - \Phi\hat{J} C_D f \qquad (86) \\
&= (A_D - JC_D)f.
\end{aligned}$$

This proves the commutativity of figure 3.

8. (a) Assume $S \in \mathbb{F}[z]^{(p-k)\times(p-k)}$ is nonsingular and all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero. By Part 3b, all left Wiener-Hopf factorization indices of $D^{-1}\begin{pmatrix} H_1 & H'S \end{pmatrix}$ are non-positive. By Part 5, there exists a factorization

$$D_1 = \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$

with $D^{-1}D_1$ normalized biproper. This implies that $D_1(z) = D(z) + Q(z)$ with $D^{-1}Q$ strictly proper. By a result of Hautus and Heymann (1978), there exists a constant matrix $\hat{J} \in \mathbb{F}^{n\times p}$ for which $Q(z) = \Phi(z)\hat{J}$. By (86), it follows from the factorization (70) that

$$\mathcal{V} = \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix} X_{\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}}$$

is an $S_{D+\Phi\hat{J}}$-invariant subspace, hence a conditioned invariant subspace of X_D. Defining $J: \mathbb{F}^p \longrightarrow X_D$ by $J\xi = \Phi(z)\hat{J}\xi$, it follows that $J \in \mathcal{G}(\mathcal{V})$.

(b) Conversely, assume $J \in \mathcal{G}(\mathcal{V})$. Then necessarily $J\xi = \Phi(z)\hat{J}\xi$ for some, uniquely determined $\hat{J} \in \mathbb{F}^{n\times p}$. Since $\mathcal{V} = X_D \cap H_1 H_0 \mathbb{F}[z]^k$, we have, by Corollary 1, that $\mathcal{O}_*(\mathcal{V}) = X_D \cap H_1 \mathbb{F}[z]^k$ and Proposition 8 implies that $\mathcal{O}_*(\mathcal{V}) = X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p$ for some non-singular S. $\mathcal{V}$ corresponds to a factorization

$$D_1 = \begin{pmatrix} H_1 R & H'S \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

which leads to

$$D_1 = \begin{pmatrix} H_1 & H'S \end{pmatrix} \begin{pmatrix} RL_1 \\ L_2 \end{pmatrix}.$$

Therefore, we have the equality $X_D \cap H_1 \mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H_1 & H'S \end{pmatrix}\mathbb{F}[z]^p$. By Corollary 3, we have $\mathcal{V} = X_D \cap H_1 H_0 \mathbb{F}[z]^k = X_D \cap (H_1 H_0 \quad H'S)\mathbb{F}[z]^p$. Here $\begin{pmatrix} H_1 H_0 & H'S \end{pmatrix}$ is uniquely determined up to a right unimodular factor for S. □

Corollary 4: *We have*

$$\mathcal{G}(\mathcal{V}) \subset \mathcal{G}(\mathcal{O}_*(\mathcal{V})). \qquad (87)$$

Proof: Given $J \in \mathcal{G}(\mathcal{V})$ implies the factorizations

$$D + \Phi\hat{J} = \begin{pmatrix} H_1 H_0 & H'S \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

$$= \begin{pmatrix} H_1 & H'S \end{pmatrix} \begin{pmatrix} H_0 L_1 \\ L_2 \end{pmatrix}, \tag{88}$$

which shows that $J \in \mathcal{G}(\mathcal{O}_*(\mathcal{V}))$. □

The isomorphism (77) has an appealing intuitive interpretation. It shows the decomposition of the dynamics in the quotient space into the part $H_1 X_{H_0}$ with fixed spectrum and the part $H'X_S$ with assignable spectrum. The isomorphism (75) shows that the quotient space $X_D/\mathcal{O}_*(\mathcal{V})$ is a natural state space for the construction of an observer with a freely assignable spectrum. For the analysis of asymptotic observers the previous analysis can be refined, however we will not tackle this in the present paper.

The following theorem is the dual of the inner spectral assignability problem solved in Fuhrmann (2005). It is an extension of the celebrated generalized pole placement theorem of Rosenbrock (1970) to the case of quotient spaces.

Theorem 7: *Let $D \in \mathbb{F}[z]^{p\times p}[z]$ be non-singular. Let (C_D, A_D) be defined via the shift realization (15). Let $\mathcal{V} \subset X_D$ be a conditioned invariant subspace, having the representation $\mathcal{V} = X_D \cap H\mathbb{F}[z]^k$ with H of full column rank and $D^{-1}H$ strictly proper. Let $H = H_1 H_0$ be an external/internal factorization and let $S \in \mathbb{F}[z]^{(p-k)\times(p-k)}$ be non-singular for which all right Wiener-Hopf indices of $S^{-1}K'D$ are zero and let $\nu_1 \geq \cdots \geq \nu_{p-k}$ be the row indices of $K'D$. Then the invariant factors $s_1, \ldots, s_{p-k}$ of the quotient module $(A_D - C_D J)|_{X_D/X_D \cap H_1 \mathbb{F}[z]^k}$ can be freely preassigned subject to the following constraints*:

1. *The division relations* $s_{j+1} | s_j$, for $j = 1, \ldots, p-k-1$.
2. *The degree constraints*

$$\sum_{j=1}^{i} \deg s_j \begin{cases} \geq \sum_{j=1}^{i} \nu_j, & i = 1, \ldots, p-k-1 \\ = \sum_{j=1}^{i} \nu_j, & i = p-k. \end{cases} \tag{89}$$

Proof: Since all right Wiener-Hopf factorization indices of $S^{-1}K'D$ are zero, then necessarily the row indices of S are equal to $\nu_1 \geq \cdots \geq \nu_{p-k}$. However, the row indices of S are equal to the observability indices of the pair (C_S, A_S) defined via the shift realization in X_S. A straightforward application of Rosenbrock's theorem implies that the only constraints on the invariant factors of S are given by conditions 1 and 2. By Theorem 6, we have the isomorphism (65) and therefore the only constraints on the invariant factors of $\overline{(A_D - JC_D)}|_{X_D/(X_D \cap (H_1 \;\; H'S)\mathbb{F}[z]^p)}$ are the ones given. □

To illustrate the method, we resort again to the parametrization of the set of conditioned invariant subspaces given in Hinrichsen *et al.* (1981) and Fuhrmann and Helmke (2001).

Example 2: We consider an observable pair in dual Brunovsky form with observability indices $(3, 2, 1)$. This corresponds to the non-singular polynomial matrix

$$D(z) = \begin{pmatrix} z^3 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z \end{pmatrix}.$$

The corresponding state space X_D is 6-dimensional with a basis matrix

$$\begin{pmatrix} 1 & z & z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A two parameter family of 1-dimensional condition invariant subspaces, corresponding to the Kronecker-Hermite indices $(1, 0, 0)$, is given by the basis matrix

$$H(z) = \begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 \\ 0 \\ 0 \end{pmatrix},$$

or equivalently by the intersection representation

$$\mathcal{V} = X_D \cap \begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 \\ 0 \\ 0 \end{pmatrix} \mathbb{F}[z]$$

$$= \left\{ \begin{pmatrix} c(z^2 + \alpha_1 z + \alpha_0) \\ 0 \\ 0 \end{pmatrix} | c \in \mathbb{F} \right\},$$

and this implies $\dim \mathcal{V} = 1$. Clearly, as

$$\begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 \\ 0 \\ 0 \end{pmatrix}$$

is not right prime, $\mathcal{V}$ is not an observability subspace. By Corollary 1,

$$\mathcal{O}_*(\mathcal{V}) = X_D \cap \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbb{F}[z] = \left\{ \begin{pmatrix} c_0 + c_1 z + c_2 z^2 \\ 0 \\ 0 \end{pmatrix} | c_i \in \mathbb{F} \right\}, \tag{90}$$

and so $\dim \mathcal{O}_*(\mathcal{V}) = 3$. Now we fix an arbitrary choice of α_1, α_0. In order to get a module theoretic representation for $\mathcal{O}_*(\mathcal{V})$, we use the extension procedure in the above mentioned papers. As

$$H_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

a trivial extension to a unimodular polynomial matrix is given by

$$H' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies

$$K' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence

$$K'D = \begin{pmatrix} 0 & z^2 & 0 \\ 0 & 0 & z \end{pmatrix}.$$

$S^{-1}K'D$ has all its right Wiener-Hopf factorization indices zero if and only if its ordered row indices are (2, 1), i.e.

$$S(z) = \begin{pmatrix} z^2 + \xi_1 z + \xi_0 & \lambda_1 z + \lambda_0 \\ \mu_0 & z + \nu_0 \end{pmatrix},$$

which leads to

$$E_{os} = \begin{pmatrix} H_1 & H'S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^2 + \xi_1 z + \xi_0 & \lambda_1 z + \lambda_0 \\ 0 & \mu_0 & z + \nu_0 \end{pmatrix}$$

and $\mathcal{O}_*(\mathcal{V}) = X_D \cap E_{os}\mathbb{F}[z]^3$.

Next, we compute

$$E_{ts} = \begin{pmatrix} H_1 H_0 & H' \end{pmatrix} = \begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which implies $\operatorname{codim} \mathcal{T} = \deg \det E_{ts} = 2$, i.e. $\dim \mathcal{T} = 4$. Of course, as our choice of H' was convenient but rather arbitrary, this is not a unique representation. In fact, there are many other choices which we can obtain by extension. Since the first Kronecker index is 1, the Kronecker indices of the extension are necessarily given by $(1, 2, 1)$, which gives $\dim \mathcal{T} = 4$ as should come out from the dimension formula (46). The set of all such subspaces is parametrized by

$$\Gamma = D^{-1}E_{ts} = \begin{pmatrix} \frac{1}{z} + \frac{\alpha_1}{z^2} + \frac{\alpha_0}{z^3} & \frac{\epsilon_1}{z^2} + \frac{\epsilon_0}{z^3} & \frac{\eta_1}{z^2} + \frac{\eta_0}{z^3} \\ 0 & \frac{1}{z^2} & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix},$$

which leads to

$$E_{ts}(z) = \begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 & \epsilon_1 z + \epsilon_0 & \eta_1 z + \eta_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since all reduced observability indices for $\mathcal{T}$ are positive, it is a tight subspace (for all choices of $\alpha_i, \epsilon_i, \eta_i$). Next we compute $\mathcal{T} = X_D \cap E_{ts}\mathbb{F}[z]^3$. Given a polynomial vector

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \mathbb{F}[z]^3,$$

we have

$$\begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 & \epsilon_1 z + \epsilon_0 & \eta_1 z + \eta_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in X_D$$

if and only if $\deg f_1 = 0, \deg f_2 \leq 1, \deg f_3 = 0$, i.e. we have 4 free parameters at our disposal. Using the codimension formula, we get $\operatorname{codim} \mathcal{T} = \deg \det E = 2$ or $\dim \mathcal{T} = 6 - 2 = 4$.

Now

$$\begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 & \epsilon_1 z + \epsilon_0 & \eta_1 z + \eta_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \mathcal{O}_*(\mathcal{V})$$

if and only if $f_2 = f_3 = 0$ and we get

$$\mathcal{O}_*(\mathcal{V}) \cap \mathcal{T} = \left\{ c \begin{pmatrix} z^2 + \alpha_1 z + \alpha_0 \\ 0 \\ 0 \end{pmatrix} | c \in \mathbb{F} \right\} = \mathcal{V}.$$

Since $\dim \mathcal{X} = 6$ and $\dim \mathcal{V} = 1$, it follows that $\dim \mathcal{X}/\mathcal{V} = 5$. Moreover, it is easily checked directly that $\dim \mathcal{O}_*(\mathcal{V})/\mathcal{V} = 2$ and $\dim \mathcal{T}/\mathcal{V} = 3$.

4. The reversion operator

In the functional approach to system theory, duality plays a very significant role transcending the simplistic use of matrix transpositions used in the state space approach. For broader discussions of duality, see Fuhrmann (1981, 2002, 2006) and Fuhrmann and Helmke (2001). In this section we introduce and study a useful tool, namely the reversion operator.

Let $D(z)$ be a non-singular polynomial matrix in Brunovsky form, i.e.

$$D(z) = \operatorname{diag}(z^{\mu_1}, \ldots, z^{\mu_p}). \tag{91}$$

Our standing assumption is that $\mu_1 \geq \cdots \geq \mu_p > 0$. Clearly, X_D is an σ_+-invariant subspace. Using the downward shift operator σ_+, defined in (8), we define the restricted downward shift operator $\overleftarrow{S}_D : X_D \longrightarrow X_D$ by

$$\overleftarrow{A}_D = \overleftarrow{S}_D = \sigma_+ | X_D. \tag{92}$$

We define the reversion operator $\rho : X_D \longrightarrow X_D$ by

$$(\rho f)(z) = f^\sharp(z) = D(z) f(z^{-1}) z^{-1}. \tag{93}$$

Equivalently, if

$$f = \begin{pmatrix} f_1(z) \\ \cdot \\ \cdot \\ \cdot \\ f_p(z) \end{pmatrix},$$

with $f_i \in X_{z^{\mu_i}}$ then, with $f_i^\sharp(z) = z^{\mu_i - 1} f(z^{-1})$, we have

$$\rho(f) = \begin{pmatrix} f_1^\sharp(z) \\ \cdot \\ \cdot \\ \cdot \\ f_p^\sharp(z) \end{pmatrix}.$$

We define, for $f \in X_D$,

$$\overleftarrow{C}_D f = f(0). \tag{94}$$

Proposition 10: *Let $D(z)$ be a non-singular polynomial matrix in Brunovsky form, i.e.*

$$D(z) = \operatorname{diag}(z^{\mu_1}, \ldots, z^{\mu_p}). \tag{95}$$

1. *The reversion operator is an involution, i.e. satisfies $\rho^2 = I$, and in particular it is a bijective map in X_D.*
2. *For $f \in X_D$ we have*

$$C_D f = (D^{-1} f)_{-1} = f^\sharp(0), \tag{96}$$

i.e.

$$C_D f = \overleftarrow{C}_D(\rho f), \tag{97}$$

which implies

$$\rho \operatorname{Ker} C_D = \operatorname{Ker} \overleftarrow{C}_D. \tag{98}$$

3. *We have*

$$\rho S_D = \overleftarrow{S}_D \rho. \tag{99}$$

4. *ρ maps the basis matrix*

$$\begin{pmatrix} 1 & z & . & . & . & z^{\mu_1 - 1} & 0 & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 & 1 & z & . & . & . & z^{\mu_p - 1} \end{pmatrix}$$

onto the reverse basis matrix

$$\begin{pmatrix} z^{\mu_1 - 1} & . & . & . & z & 1 & 0 & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 & z^{\mu_p - 1} & . & . & . & z & 1 \end{pmatrix}.$$

We denote these bases by $\mathcal{B}$ and $\overleftarrow{\mathcal{B}}$ respectively.

5. *Define $\overleftarrow{A}_D : X_D \longrightarrow X_D$ and $\overleftarrow{C}_D : X_D \longrightarrow \mathbb{F}^p$ by*

$$\begin{aligned} \overleftarrow{A}_D &= \overleftarrow{S}_D \\ \overleftarrow{C}_D f &= f(0), \end{aligned} \tag{100}$$

then we have

$$
\begin{aligned}
\left[\overleftarrow{A}_D\right]^{\overleftarrow{\mathcal{B}}}_{\overleftarrow{\mathcal{B}}} &= [S_D]^{\mathcal{B}}_{\mathcal{B}} \\
\left[\overleftarrow{C}_D\right]^{st}_{\overleftarrow{\mathcal{B}}} &= [C_D]^{st}_{\mathcal{B}},
\end{aligned}
\tag{101}
$$

where $\mathcal{B}_{st}$ denotes the standard basis.

Proof:

1. We use the fact that for D given by (95), we have $D(z)^{-1} = D(z^{-1})$. Assume $f \in X_D$, then $D^{-1}f^{\sharp} = D^{-1}D(z)z^{-1}f(z^{-1}) = z^{-1}f(z^{-1}) \in z^{-1}\mathbb{F}^p[z^{-1}]$, so $f^{\sharp} \in X_D$. Moreover, we have for $f \in X_D$,

$$
\begin{aligned}
\rho^2 f &= \rho(\rho f) = \rho(D(z)f(z^{-1})z^{-1}) \\
&= D(z)z^{-1}(D(z^{-1})f(z)z) = f(z).
\end{aligned}
$$

2. We set $\eta_f = (D^{-1}f)_{-1}$. Then we use the fact that $S_D f = zf(z) - D(z)\eta_f$ and that $D^{-1}f \in z^{-1}\mathbb{F}^p[z^{-1}]$, to compute $D^{-1}f = \sum_{i=1}^{\mu_1} \eta_i/z^i$, with $\eta_1 = \eta_f$, which implies $D(z)f(z^{-1}) = \sum_{i=1}^{\mu_1} \eta_i z^i$. In turn, we have $f^{\sharp}(z) = D(z)z^{-1}f(z^{-1}) = \sum_{i=1}^{\mu_1} \eta_i z^{i-1}$, and so $f^{\sharp}(0) = \eta_1 = \eta_f$.
3. We compute, using the equality $\eta_f = f^{\sharp}(0)$,

$$
\begin{aligned}
\rho S_D f &= \rho(zf(z) - D(z)\eta_f) \\
&= D(z)z^{-1}(z^{-1}f(z^{-1}) - D(z^{-1})\eta_f) \\
&= z^{-1}(z^{-1}D(z)f(z^{-1}) - \eta_f) = z^{-1}(f^{\sharp}(z) - f^{\sharp}(0)) \\
&= \overleftarrow{S}_D \rho f.
\end{aligned}
$$

4. Immediate.
5. Follows by a simple check. □

5. Almost observability subspaces

Much of mathematical research proceeds via analogies that lead to interesting extensions. In Fuhrmann and Willems (1980) a functional, or module theoretic, characterization of controlled invariant subspaces was obtained. This was extended to a characterization of conditioned invariant subspaces in Fuhrmann (1981). In the sequel, we will be interested, among other things, in the structure of singular as well as dead beat observers. It is well known, see Fuhrmann and Helmke (2001) or Trumpf (2002), that tracking observers correspond to conditioned invariant subspaces and asymptotic observers to the subclass of outer detectable subspaces. Thus it is quite natural to expect that the study of the classes of dead beat and singular observers would necessitate the study of some other objects arising from geometric control theory. In fact, it turns out that for these two classes of observers the corresponding subspaces are outer reconstructible and almost observability subspaces respectively. There is an interesting duality relation between these two classes of subspaces that will lead to the establishing of a duality theory between singular observers and dead-beat observers.

Dead beat observers can be viewed as an extension of the concept of asymptotic observers to the case of an arbitrary field. An infinite sequence of vectors is said to converge to zero if it is eventually zero. There is a natural analog of detectability subspaces in this context. We say a conditioned invariant subspace $\mathcal{V}$ is inner reconstructible if there exists an output injection map J such that $\mathcal{V}$ is $A + JC$-invariant and $(A + JC)|_{\mathcal{V}}$ is nilpotent. We say a subspace $\mathcal{V}$ is outer reconstructible if there exists an output injection map J such that $\mathcal{V}$ is $(A + JC)$-invariant and the induced map $(A + JC)|_{X/\mathcal{V}}$ is nilpotent.

For almost observability subspaces, we take a different route. In analogy with conditioned invariant subspaces, the almost observability subspaces can be characterized in a variety of terms. The original definition of the dual objects, namely the almost controllability subspaces, due to Willems (1980), was formulated in topological terms, followed by a purely algebraic characterization. As one of the topics we will discuss later on is that of dead beat observers, and these are important over an arbitrary field, and as a nice duality between singular observers and dead beat observers is emerging, this indicates to us that it may be advisable to define almost observability subspaces in an algebraic way and this is the direction in which we will proceed. We are well aware that the principal shortcoming of this approach is that the definition is technical rather than conceptual.

Recall, see Fuhrmann and Helmke (2001), that given the pair (C, A), a subspace $\mathcal{V}$ of the state space that has a kernel representation $\mathcal{V} = \operatorname{Ker} K$ is a conditioned invariant subspace if and only the following Sylvester equation

$$KA - FK = GC \tag{102}$$

is solvable, i.e. if and only if there exist F, G such that (102) holds. We use this as a motivation for making the following working definition of almost observability subspaces, although it is not the original definition. We will show that our definition coincides with the original one.

Definition 2: Given the pair (C, A), a subspace $\mathcal{V}$ of the state space that has a kernel representation $\mathcal{V} = \operatorname{Ker} K$ is an almost observability subspace if there exist N, L

with N nilpotent such that the following generalized Sylvester equation is satisfied

$$NKA - K = LC. \tag{103}$$

Note that we can assume without loss of generality that K is surjective.

Definition 3: Given linear spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ over the fixed field $\mathbb{F}$. Let (C, A), with $C: \mathcal{X} \longrightarrow \mathcal{Y}$ and $A: \mathcal{X} \longrightarrow \mathcal{X}$, be an observable pair with observability indices $\mu_1 \geq \cdots \geq \mu_p > 0$. Let bases $\{e_1, \ldots, e_p\}$ of $\mathcal{Y}$ and $\{g_{1\mu_1}, \ldots, g_{11}, \ldots, g_{p\mu_p}, \ldots, g_{p1}\}$ of $\mathcal{X}$ be given.

Given a pair of maps (N, L), with $N: \mathcal{Z} \longrightarrow \mathcal{Z}$ nilpotent and $L: \mathcal{Y} \longrightarrow \mathcal{Z}$, the reverse partial reachability map $\overleftarrow{R}_\mu(N, L): \mathcal{X} \longrightarrow \mathcal{Z}$ is defined by

$$\overleftarrow{R}_\mu(N, L) g_{ij} = N^{j-1} l_i, \quad i = 1, \ldots, p, j = 1, \ldots, \mu_i, \tag{104}$$

where $l_i = Le_i$.

If the spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are identified with $\mathbb{F}^n$, $\mathbb{F}^p$, $\mathbb{F}^k$ respectively, then $\overleftarrow{R}_\mu(N, L)$ is given by a matrix, naturally called the reverse partial reachability matrix.

Note that if we assume that $\overleftarrow{R}_\mu(N, L)$ is surjective on $\mathcal{Z}$, then we have the isomorphism $\mathcal{Z} \simeq \mathcal{X}/\mathcal{V}$, where $\mathcal{V} = \operatorname{Ker} \overleftarrow{R}_\mu(N, L)$. Thus $K = \overleftarrow{R}_\mu(N, L)$ is isomorphic to the canonical projection $\pi_\mathcal{V}$ of $\mathcal{X}$ onto $\mathcal{X}/\mathcal{V}$.

The following proposition sums up Propositions 5.35 and 5.38 in Trumpf (2002).

Proposition 11: *Given the pair (C_D, A_D), a subspace $\mathcal{V}$ of the state space X_D is an almost observability subspace if and only if it has the representation $\mathcal{V} = \operatorname{Ker} \overleftarrow{R}_\mu(N, L)$ for a pair (N, L) with N nilpotent. Here $\overleftarrow{R}_\mu(N, L)$ is defined using the standard bases in X_D and $\mathbb{F}^p$.*

Proof: Assume $\mathcal{V}$ is an almost observability subspace with respect to (C_D, A_D), i.e. $\mathcal{V} = \operatorname{Ker} K$ with $K: X_D \longrightarrow \mathcal{Z}$ and for a pair N, L with N nilpotent

$$NKA_D - K = LC_D \tag{105}$$

holds. Consider the standard basis of X_D given by

$$\mathcal{B}_{st} = \{g_{ij} = z^{j-1} e_i | i = 1, \ldots, p; j = 1, \ldots, \mu_i\}.$$

and the standard basis of $\mathbb{F}^p$. It is easy to check that

$$\operatorname{Ker} A_D = \operatorname{span}\{g_{i\mu_i} | i = 1, \ldots, p\}$$
$$\operatorname{Ker} C_D = \operatorname{span}\{g_{ij} | i = 1, \ldots, p; j = 1, \ldots, \mu_i - 1\}.$$

From (105) we have

$$K(z^{\mu_i - 1} e_i) = -LC_D(z^{\mu_i - 1} e_i) = -l_i.$$

For the basis elements of $\operatorname{Ker} C_D$, we compute using (105)

$$K(z^{j-1} e_i) = NKA_D(z^{j-1} e_i) = NK(z^j e_i)$$

and by induction

$$K(z^{j-1} e_i) = N^{\mu_i - j} K(z^{\mu_i - 1} e_i) = -N^{\mu_i - j} l_i.$$

This shows that $K = \overleftarrow{R}_\mu(N, -L)$.

To prove the converse, assume $\mathcal{V} = \operatorname{Ker} K$ with $K = \overleftarrow{R}_\mu(N, L)$ for a pair (N, L) with N nilpotent. With respect to the standard basis of X_D given above and the standard basis of $\mathbb{F}^p$ we have $K(z^{j-1} e_i) = N^{\mu_i - j} l_i$. Now, $z^{j-1} e_i \in \operatorname{Ker} C_D$ for $i = 1, \ldots, p$; $j = 1, \ldots, \mu_i - 1$. In this range of indices we have $A_D z^{j-1} e_i = z^j e_i$. Using this, we compute

$$\begin{aligned}(NKA_D - K) z^{j-1} e_i &= NK(z^j e_i) - K(z^{j-1} e_i) \\ &= NN^{\mu_i - j} l_i - N^{\mu_i - j + 1} l_i = 0.\end{aligned}$$

It follows that $\operatorname{Ker} C_D \subset \operatorname{Ker}(NKA_D - K)$. Hence, there exists a linear transformation $L': \mathbb{F}^p \longrightarrow \mathcal{Z}$ for which $NKA_D - K = L'C_D$ holds, i.e. $\mathcal{V}$ is an almost observability subspace with respect to (C_D, A_D). $\square$

It has been shown in Trumpf (2002, Propositions 5.7 and 5.8) that a subspace is representable by a reverse partial reachability matrix if and only if it is an almost observability subspace in the sense of Willems' original definition.

We shall now look for a functional characterization of almost observability subspaces. In view of the representation (22) of conditioned invariant subspaces, it is of interest to study subspaces of the form $\mathcal{V} = X_D \cap \mathcal{M}$, where $\mathcal{M}$ is a submodule over a ring different from $\mathbb{F}[z]$. This leads us to the following considerations. Given $D(z) = \operatorname{diag}(z^{\mu_1}, \ldots, z^{\mu_p})$, let X_D be the associated polynomial model. Let $S(z) \in \mathbb{F}[z]^{p \times k}$ be monomic, of full column rank and such that $D^{-1} S$ is proper. We define a subspace of X_D by

$$\mathcal{V} = X_D \cap S(z) z^{-1} \mathbb{F}[z^{-1}]^k. \tag{106}$$

We proceed to study this class of subspaces and especially how they transform under the reversion map in X_D.

Proposition 12: *Let $S \in \mathbb{F}[z]^{p\times k}$ have full column rank. Then*

$$\mathcal{V} = \mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k \neq \{0\} \tag{107}$$

or equivalently

$$\operatorname{Ker} S(\sigma)|z^{-1}\mathbb{F}[z^{-1}]^k \neq \{0\} \tag{108}$$

if and only if $S(z), zI$ are not right coprime, i.e. $S(z)$ has a non-trivial monomic right factor.

Proof: Let $0 \neq f \in \mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$, i.e. $f = Sh$ with $h \in z^{-1}\mathbb{F}[z^{-1}]^k$. Let $\mathcal{N}$ be the smallest $\mathbb{F}[z]$-submodule of $z^{-1}\mathbb{F}[z^{-1}]^k$ containing h, i.e. $\mathcal{N} = \pi_-\mathbb{F}[z]h$. As it is a finitely generated torsion submodule of $z^{-1}\mathbb{F}[[z^{-1}]]^k$, it is necessarily a rational model, i.e. $\mathcal{N} = X^E$ with E a non-singular polynomial matrix. If E had a non-monomial invariant factor then X^E would contain at least one element with a singularity away from zero. But $X^E \subset z^{-1}\mathbb{F}[z^{-1}]^k$ and it follows that E is monomial. Now, Sh' is polynomial for every $h' \in X^E = E^{-1}X_E$. So we have $SE^{-1}g$ is polynomial for every $g \in X_E$. On the other hand, for every $g' \in \mathbb{F}[z]^k$, we have that $SE^{-1}Eg'$ is polynomial. Since $\mathbb{F}[z]^k = X_E \oplus E\mathbb{F}[z]^k$, it follows that $SE^{-1}g$ is polynomial for every $g \in \mathbb{F}[z]^k$. This implies that necessarily $SE^{-1} = S_1$ is a polynomial matrix. Thus $S = S_1E$ and S has a non-trivial monomial right factor.

To prove the converse, assume S, zI are not right coprime. Therefore there exists a greatest common non-trivial, necessarily non-singular, right monomic factor $S_0 \in \mathbb{F}[z]^{k\times k}$. Let $X^{S_0} \subset z^{-1}\mathbb{F}[z^{-1}]^k$ be the corresponding rational model. Thus $SX^{S_0} = S_1S_0X^{S_0} = S_1X_{S_0} \subset \mathbb{F}[z]^p$ and we have $\dim SX^{S_0} = \dim S_1X_{S_0} = \deg\det S_0 > 0$. □

Proposition 13: *Let $D(z) \in \mathbb{F}[z]^{p\times p}$ be non-singular. Let $S \in \mathbb{F}[z]^{p\times k}$ have full column rank. If $D^{-1}S$ is proper, then we have*

$$\mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k = X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k. \tag{109}$$

Proof: Assume $D^{-1}S$ is proper. Clearly we have $X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k \subset \mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$. Next, let $f \in \mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$. Then $f = Sh$ with $h \in z^{-1}\mathbb{F}[z^{-1}]^k$. This implies $D^{-1}f = (D^{-1}S)h \in z^{-1}\mathbb{F}[z^{-1}]^p$, i.e. $f \in X_D$. Thus we obtain the inclusion $\mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k \subset X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$ and hence (109) follows. □

The converse of the above proposition is not true in general in the sense that equality (109) does not imply the properness of $D^{-1}S$. In fact for

$$D(z) = \begin{pmatrix} z^3 & 0 \\ 0 & z \end{pmatrix} \quad \text{and} \quad S(z) = \begin{pmatrix} 1 \\ z^2 \end{pmatrix}$$

equality (109) holds trivially but $D^{-1}S$ is not proper.

Theorem 7: *Let $D(z) \in \mathbb{F}[z]^{p\times p}$ be non-singular. Let $S \in \mathbb{F}[z]^{p\times k}$ have full column rank. Then*

1. *S has a factorization of the form $S(z) = S_1(z)S_0(z)$ with S_0 monomic non-singular and $S_1(z), zI$ right coprime. S_0 is uniquely determined up to a left unimodular factor.*
2. *If S is monomic and of full column rank, it has a factorization of the form $S(z) = S_1(z)S_0(z)$ with S_0 monomic non-singular and S_1 right prime. Moreover, we have*
$$X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k = X_D \cap S(z)z^{-1}\mathbb{F}[[z^{-1}]]^k. \tag{110}$$
3. *With the factorization of part 1, we have*
$$\dim\left(\mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k\right) = \deg\det S_0. \tag{111}$$
4. *If $D^{-1}S$ is proper, then with the factorization of Theorem 7(1), we have*
$$\dim\left(X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k\right) = \deg\det S_0. \tag{112}$$
5. *If $D^{-1}S$ is proper and S is monomic non-singular, then*
$$\dim\left(X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^p\right) = \deg\det S. \tag{113}$$

Proof:

1. Since $S(z)$ has full column rank, there exists an external/internal factorization $S(z) = H_1(z)H_0(z)$, with H_1 right prime and H_0 non-singular. Let $H_0(z) = S_{nz}(z)S_0(z)$, existing by Proposition 2, with $\det S_0(z) = z^\alpha$ and $\det S_{nz}(0) \neq 0$. Write $S_1 = H_1S_{nz}$, then $S = S_1S_0$ is the required factorization.
2. Follows from the previous part and the fact that S has no non-monomic invariant factors.

   To prove (110), note that the inclusion $z^{-1}\mathbb{F}[z^{-1}]^k \subset z^{-1}\mathbb{F}[[z^{-1}]]^k$ implies the inclusion $X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k \subset X_D \cap S(z)z^{-1}\mathbb{F}[[z^{-1}]]^k$. To prove the converse inclusion, assume $f \in X_D \cap S(z)z^{-1}\mathbb{F}[[z^{-1}]]^k$, i.e. there exists an $h \in S(z)z^{-1}\mathbb{F}[[z^{-1}]]^k$ for which $f = S_1S_0h$. Since S_1 is right prime, it has a polynomial left inverse. Thus $S_0 h$ is polynomial and hence $h \in z^{-1}\mathbb{F}[z^{-1}]^k$.

3. Assume the factorization $S = S_1 S_0$. Clearly, by Proposition 11, for $0 \neq h \in z^{-1}\mathbb{F}[z^{-1}]^k$, we have $Sh \in \mathbb{F}[z]^p$ if and only if $S_0 h \in \mathbb{F}[z]^k$, i.e. $h \in X^{S_0}$. On the other hand, for every $\in X^{S_0}$ we have $S_0 h \in X_{S_0} \subset \mathbb{F}[z]^p$ and hence $Sh \in \mathbb{F}[z]^p$. Therefore $\mathbb{F}[z]^p \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k = S_1 X_{S_0}$, and by the fact that S_1 has full column rank, the dimension formula (111) follows.
4. Follows from Proposition 12 and Theorem 7(3).
5. Follows from Theorem 7(4). □

Note that for a monomic and full column rank polynomial matrix $S(z)$, the factorization $S(z) = S_1(z)S_0(z)$ is a special case of an external/internal factorization.

Since for a unimodular matrix over the ring $\mathbb{F}[z^{-1}]$, i.e. an invertible $\Gamma \in \mathbb{F}[z^{-1}]^{k\times k}$, we have $\Gamma(z^{-1}\mathbb{F}[z^{-1}]^k) = z^{-1}\mathbb{F}[z^{-1}]^k$, we can assume without loss of generality that in the factorization $S = S_1 S_0$, $S_0 = \mathrm{diag}\,(z^{\nu_1}, \dots, z^{\nu_k})$.

Proposition 14: *Let $D(z) = \mathrm{diag}\,(z^{\mu_1}, \dots, z^{\mu_p})$, with $\mu_1 \geq \dots \geq \mu_p > 0$ and let $S(z)$ be a full column rank, monomic, $p \times k$ polynomial matrix, such that $D^{-1}S$ is proper. We define*

$$H(z) = D(z)S(z^{-1}). \tag{114}$$

Then

1. *H is also a full column rank, monomic, $p \times k$ polynomial matrix, such that $D^{-1}H$ is proper. Moreover, we have*

$$S(z) = D(z)H(z^{-1}). \tag{115}$$

2. *Given H as above, we define the canonical projection $\pi_H : \mathbb{F}[z]^p \longrightarrow \mathbb{F}[z]^p / H(z)\mathbb{F}[z]^k$ by*

$$\pi_H f = [f]_H \tag{116}$$

for $f \in \mathbb{F}[z]^p$. Then

$$\begin{aligned} \mathrm{Ker}\,\pi_H &= H\,\mathbb{F}[z]^k \\ \mathrm{Ker}\,(\pi_H | X_D) &= X_D \cap H\,\mathbb{F}[z]^k \\ \mathrm{Im}\,(\pi_H | X_D) &= X_D / H(z)\mathbb{F}[z]^k. \end{aligned} \tag{117}$$

3. *We have*

$$\rho(X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k) = X_D \cap H(z)\mathbb{F}[z]^k. \tag{118}$$

4. *Let $S = S_1 S_0$ and $H_1 H_0$ be factorizations as in Theorem 7. Then we have*

$$\begin{aligned} \dim(X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k) &= \deg\det S_0 \\ \dim(X_D \cap S(z)\mathbb{F}[z]^k) &= \deg\det H_0. \end{aligned} \tag{119}$$

Proof:

1. Clearly, we have $D(z) = D(z^{-1})^{-1}$. Since $D(z)^{-1}S(z)$ is proper, it follows that $H(z) = D(z^{-1})^{-1}S(z^{-1}) = D(z)S(z^{-1})$ is a polynomial matrix and moreover, $D^{-1}H$ is proper. From $H(z) = D(z)S(z^{-1})$ it follows that $H(z^{-1}) = D(z^{-1})S(z)$ and hence (115) holds.
2. Clearly, $f \in \mathrm{Ker}\,(\pi_H | X_D)$ if and only if $f \in X_D$ and $f \in H\,\mathbb{F}[z]^k$. The rest is immediate.
3. Assume $f \in X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$, then $f = Sh$ with $h \in z^{-1}\mathbb{F}[z^{-1}]^k$. We compute, with $H(z) = D(z)S(z^{-1})$,

$$\begin{aligned} \rho f &= z^{-1}D(z)f(z^{-1}) = z^{-1}D(z)S(z^{-1})h(z^{-1}) \\ &= (D(z)S(z^{-1}))(z^{-1}h(z^{-1})) = H(z)g(z) \in H\,\mathbb{F}[z]^k. \end{aligned}$$

   Since $f \in X_D$ implies $\rho f \in X_D$, it follows that $\rho f \in X_D \cap H\,\mathbb{F}[z]^k$, i.e. $\rho(X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k) \subset X_D \cap H\,\mathbb{F}[z]^k$. A similar computation yields $\rho(X_D \cap H\,\mathbb{F}[z]^k) \subset X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$. The two inclusions imply the equality (118).
4. The first equality in (119) was proved in Theorem 7. The second equality follows from that, equation (118) and the fact that ρ is an involution. □

To illustrate the previous result we work out an easy example.

Example 3: Let $d(z) = z^n$, $s(z) = z^q$ with $0 \leq q < n$. Then $h(z) = z^n z^{-q} = z^{n-q}$. We have $X_{z^n} \cap z^q z^{-1}\mathbb{F}[z^{-1}] = \{c_0 + c_1 z + \dots + c_{q-1}z^{q-1} | c_i \in \mathbb{F}\}$ and, for the reversion operator ρ,

$$\begin{aligned} &\rho(X_{z^n} \cap z^q z^{-1}\mathbb{F}[z^{-1}]) \\ &\quad = \{c_0 z^{n-1} + c_1 z^{n-2} + \dots + c_{q-1}z^{n-q} | c_i \in \mathbb{F}\} \\ &\quad = X_{z^n} \cap z^{n-q}\mathbb{F}[z]. \end{aligned}$$

Theorem 8: *Let $D(z) = \mathrm{diag}\,(z^{\mu_1}, \dots, z^{\mu_p})$, with $\mu_1 \geq \dots \geq \mu_p > 0$. Let X_D be the associated polynomial model, and let the pairs (C_D, A_D) and $(\overleftarrow{C}_D, \overleftarrow{A}_D)$*

be defined by

$$\begin{aligned} A_D f &= S_D f = \pi_D z f \\ C_D f &= (D^{-1} f)_{-1} \\ \overleftarrow{A}_D f &= \overleftarrow{S}_D f = \pi_+ z^{-1} f \\ \overleftarrow{C}_D f &= f(0) \end{aligned} \tag{120}$$

and $\rho\colon X_D \longrightarrow X_D$ *the reversion operator defined by* (93). *Let the subspaces* $\mathcal{V}$ *and* $\mathcal{W}$ *of* X_D *be related by* $\mathcal{V} = \rho\mathcal{W}$. *Since* $\rho^2 = I$ *this implies also* $\mathcal{W} = \rho\mathcal{V}$. *The following statements are equivalent*:

1. *The subspace* $\mathcal{V}$ *is an almost observability subspace with respect to* (C_D, A_D).
2. *The subspace* $\mathcal{V}$ *has a representation*

$$\mathcal{V} = X_D \cap S(z) z^{-1} \mathbb{F}[z^{-1}]^k \tag{121}$$

   with $S(z)$ *a full column rank, monomic polynomial matrix for which* $D^{-1}S$ *is proper*.
3. *The subspace* $\mathcal{V}$ *is outer reconstructible with respect to* $(\overleftarrow{C}_D, \overleftarrow{A}_D)$.
4. *The subspace* $\mathcal{W}$ *is outer reconstructible with respect to* (C_D, A_D).
5. *The subspace* $\mathcal{W}$ *has a representation*

$$\mathcal{W} = X_D \cap H \mathbb{F}[z]^k, \tag{122}$$

   with $H(z)$ *a full column rank, monomic polynomial matrix for which* $D^{-1}H$ *is proper*.
6. *The subspace* $\mathcal{W}$ *is an almost observability subspace with respect to* $(\overleftarrow{C}_D, \overleftarrow{A}_D)$.

Proof:
(1) $\Rightarrow$ (3)
Assume $\mathcal{V} = \operatorname{Ker} K$ with K surjective such that the Sylvester equation

$$NKA_D - K = LC_D \tag{123}$$

is solvable with N nilpotent. Let $\{e_1, \dots, e_p\}$ be the standard basis of $\mathbb{F}^p$ and $\{g_{ij} = z^{j-1} e_i | i = 1, \dots, p; j = 1, \dots, \mu_i\}$ the standard basis of X_D. It is easy to check that

$$\begin{aligned} \operatorname{Ker} C_D &= \{g_{ij} | i = 1, \dots, p;\, j = 1, \dots, \mu_i - 1\} \\ \operatorname{Ker} \overleftarrow{C}_D &= \{g_{ij} | i = 1, \dots, p;\, j = 2, \dots, \mu_i\}. \end{aligned}$$

For every $f \in \operatorname{Ker} C_D$, we have $A_D f = zf$. Equation (123) implies, for $g_{ij} \in \operatorname{Ker} C_D$,

$$0 = NKA_D g_{ij} - K g_{ij} = NK g_{i(j+1)} - K g_{ij}. \tag{124}$$

Next, for every basis element $g_{ij} \in \operatorname{Ker} \overleftarrow{C}_D$, we have $\overleftarrow{A}_D g_{ij} = g_{i(j-1)}$. Therefore we have for these g_{ij}

$$[K\overleftarrow{A}_D - NK] g_{ij} = K\overleftarrow{A}_D g_{ij} - NK g_{ij} = K g_{i(j-1)} - NK g_{ij}$$

and (124) implies

$$[K\overleftarrow{A}_D - NK] g_{ij} \quad = NK g_{ij} - NK g_{ij} = 0.$$

This implies the inclusion $\operatorname{Ker}(K\overleftarrow{A}_D - NK) \supset \operatorname{Ker} \overleftarrow{C}_D$ and hence there exists a map G for which

$$K\overleftarrow{A}_D - NK = G\overleftarrow{C}_D$$

holds, with N nilpotent. However, this means that $\mathcal{V} = \operatorname{Ker} K$ is an outer reconstructible subspace with respect to $(\overleftarrow{C}_D, \overleftarrow{A}_D)$, cf. Theorem 11.

(3) $\Rightarrow$ (1)
Let $\mathcal{V}$ be an outer reconstructible subspace with respect to $(\overleftarrow{C}_D, \overleftarrow{A}_D)$, then there exists a $\overleftarrow{J}$ with $(\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)\mathcal{V} \subset \mathcal{V}$ and $(\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)|X_D/\mathcal{V}$ nilpotent. We define $N_J\colon X_D/\mathcal{V} \longrightarrow X_D/\mathcal{V}$ by

$$\begin{aligned} N_J[f]_{\mathcal{V}} &= [(\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D) f]_{\mathcal{V}} \\ &= (\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)|X_D/\mathcal{V}[f]_{\mathcal{V}}. \end{aligned} \tag{125}$$

Thus N_J is necessarily nilpotent. Next, we define $K\colon X_D \longrightarrow X_D/\mathcal{V}$ by

$$Kf = [f]_{\mathcal{V}}, \tag{126}$$

i.e. K is the canonical projection of X_D onto the quotient space, and $L\colon \mathbb{F}^p \longrightarrow X_D$ by

$$L\eta = -[z^{-1} D(z)\eta]_{\mathcal{V}}. \tag{127}$$

We compute, for $f \in X_D$, noting that $A_D f = zf(z) - D(z)\eta_f$ where $\eta_f = Cf = (D^{-1}f)_{-1}$,

$$\begin{aligned} N_J K A_D f - Kf &= N_J K(zf - D(z)\eta_f) - Kf \\ &= N_J [zf - D(z)\eta_f]_{\mathcal{V}} - [f]_{\mathcal{V}} \\ &= (\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)[zf - D(z)\eta_f]_{\mathcal{V}} - [f]_{\mathcal{V}} \\ &= [(\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)(zf - D(z)\eta_f)]_{\mathcal{V}} - [f]_{\mathcal{V}} \\ &= [\overleftarrow{A}_D(zf - D(z)\eta_f)]_{\mathcal{V}} - [f]_{\mathcal{V}} \\ &= [\sigma_+(zf - D(z)\eta_f) - f]_{\mathcal{V}} = -[z^{-1} D(z)\eta_f]_{\mathcal{V}} \\ &= LC_D f. \end{aligned}$$

Here we used the fact that, for $f \in X_D$, $D^{-1}f$ is strictly proper and that $\overleftarrow{C}_D(zf - D(z)\eta_f) = (zf - D(z)\eta_f)(0) = 0$. This shows that $\mathcal{V} = \operatorname{Ker} K$ is an almost observability subspace.

(3) $\Leftrightarrow$ (4)
Applying ρ to $(A_D + JC_D)\mathcal{W} \subset \mathcal{W}$, from (99) and (97) we get $(\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)\mathcal{V} \subset \mathcal{V}$ where $\overleftarrow{J} := \rho J$. From $\rho^2 = I$ we get $\rho\overleftarrow{J} = J$, while (99) implies $\rho\overleftarrow{A}_D = A_D\rho$ and (97) yields $\overleftarrow{C}_D = C_D\rho$. Applying ρ to $(\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)\mathcal{V} \subset \mathcal{V}$ we therefore get $(A_D + JC_D)\mathcal{W} \subset \mathcal{W}$. Together we have that $\mathcal{W}$ is (C_D, A_D)-invariant if and only if $\mathcal{V}$ is $(\overleftarrow{C}_D, \overleftarrow{A}_D)$-invariant.

Now $\rho\colon X_D \longrightarrow X_D$ induces a map $\overline{\rho}\colon X_D/\mathcal{W} \longrightarrow X_D/\mathcal{V}$ by

$$\overline{\rho}[f]_{\mathcal{W}} = [\rho f]_{\mathcal{V}}. \tag{128}$$

Since $\rho\mathcal{W} = \mathcal{V}$, the induced map $\overline{\rho}$ is well defined. Moreover, $\overline{\rho}$ is invertible and intertwines the induced maps $(A_D + JC_D)|X_D/\mathcal{W}$ and $(\overleftarrow{A}_D + \overleftarrow{J}\,\overleftarrow{C}_D)|X_D/\mathcal{V}$ which implies that one is nilpotent if and only if the other is. Hence $\mathcal{W}$ is outer reconstructible with respect to (C_D, A_D) if and only if $\mathcal{V}$ is outer reconstructible with respect to $(\overleftarrow{C}_D, \overleftarrow{A}_D)$.

(2) $\Leftrightarrow$ (5)
Assume that $\mathcal{V} = X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$ with $S(z)$ a full column rank, monomic polynomial matrix for which $D^{-1}S$ is proper. Proposition 12 implies $\mathcal{W} = X_D \cap H\mathbb{F}[z]^k$ with $H(z) = D(z)S(z^{-1})$ a full column rank, monomic polynomial matrix for which $D^{-1}H$ is proper.

Conversely, assume that $\mathcal{W} = X_D \cap H\mathbb{F}[z]^k$ with $H(z)$ a full column rank, monomic polynomial matrix for which $D^{-1}H$ is proper. By temporarily exchanging the roles of S and H in Proposition 12 we get that $S(z) = D(z)H(z^{-1})$ is also a full column rank, monomic polynomial matrix for which $D^{-1}S$ is proper, and moreover $H(z) = D(z)S(z^{-1})$. But then Proposition 12 in its original form together with $\rho^2 = I$ yields $\mathcal{V} = X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$.

(4) $\Leftrightarrow$ (5)
Assume $\mathcal{W} \subset X_D$ is outer reconstructible with respect to (C_D, A_D). In particular $\mathcal{W}$ is conditioned invariant, hence has a representation $\mathcal{W} = X_D \cap H\mathbb{F}[z]^k$ for H of full column rank. Let $H = H_1H_0$ be an external/internal factorization with H_1 right prime and H_0 square non-singular. Then, by Fuhrmann and Helmke (2001), there exists an extension

$$T = \begin{pmatrix} H & H' \end{pmatrix}$$

such that $X_D \cap H\mathbb{F}[z]^k = X_D \cap T\mathbb{F}[z]^p$, $D^{-1}T$ is proper and T monomic. Since $\det H_0$ is a factor of $\det T$, for T to be monomic, necessarily H_0 and hence H has to be monomic.

Assume $\mathcal{W} \subset X_D$ has the representation $\mathcal{W} = X_D \cap H\mathbb{F}[z]^k$ with H monomic and $D^{-1}H$ proper. Necessarily, $\mathcal{W}$ is conditioned invariant. Let $H = H_1H_0$ be an external/internal factorization. By the results of § 3, it has an extension of the form

$$T = \begin{pmatrix} H_1 & H' \end{pmatrix}\begin{pmatrix} H_0 & 0 \\ 0 & I \end{pmatrix}$$

with $\mathcal{W} = X_D \cap T\mathbb{F}[z]^p$.

The properness of $D^{-1}T$ implies the existence of a, not necessarily unique, non-singular polynomial matrix R such that $D^{-1}TR$ is biproper. Letting $\overline{D} = TR$, it is easy to verify that X_D and $X_{\overline{D}}$ contain the same elements, (though the module structure is different), and that there exists an output injection map J such that $A_{\overline{D}} = A_D + JC_D$. For this $A_{\overline{D}}$-module structure on $X_{\overline{D}}$, $\mathcal{W} = TX_R$ is actually an invariant subspace and moreover $X_{\overline{D}}/\mathcal{W}$ and X_T are isomorphic as modules. In particular, since S_T is nilpotent, so is the induced map $(A_D + JC_D)|X_{\overline{D}}/\mathcal{W}$. This shows that $\mathcal{W}$ is an outer reconstructible subspace with respect to (C_D, A_D).

(1) $\Leftrightarrow$ (6)
Apply ρ to the one Sylvester to get the other and vice versa. □

The previous results are very closely related to the characterization of conditioned invariant subspaces given in Theorem 5.1 of Fuhrmann and Helmke (2001).

As a corollary, we can state.

Theorem 9: *Given an observable pair (C, A), a subspace $\mathcal{V}$ of the state space is an observability subspace if and only if $\mathcal{V}$ is simultaneously a conditioned invariant subspace and an almost observability subspace.*

Proof: Without loss of generality, we assume that (C, A) is in dual Brunovsky form, i.e. $(C, A) = (C_D, A_D)$ with $D(z) = \operatorname{diag}(z^{\mu_1}, \ldots, z^{\mu_p})$. Assume $\mathcal{V} \subset X_D$ is an observability subspace. In particular it is conditioned invariant. By Proposition 4, we have $X_D \cap H\mathbb{F}[z]^k$ for some right prime $H \in \mathbb{F}[z]^{p\times k}$ that satisfies $D^{-1}H$ is strictly proper. Let $\lambda_1, \ldots, \lambda_k$ be the reduced observability indices. Then, by Theorem 7.1 in Fuhrmann and Helmke (2001), denoting by $h_1, \ldots, h_k$ the columns of H, a basis for

$\mathcal{V}$ is given by $\{h_1, \ldots, z^{\lambda_1 - 1}h_1, \ldots, h_k, \ldots, z^{\lambda_k - 1}h_k\}$. Define

$$S(z) = H(z)\begin{pmatrix} z^{\lambda_1} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & z^{\lambda_k} \end{pmatrix}$$

and let $s_1, \ldots, s_k$ the columns of S. Then it is easily checked that $\mathcal{V} = X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$ and so, by Theorem 8, it is an almost observability subspace.

To prove the converse, assume $\mathcal{V}$ is simultaneously a conditioned invariant subspace and an almost observability subspace. Thus we have

$$\mathcal{V} = X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^l = X_D \cap H\,\mathbb{F}[z]^k.$$

Here H is a basis matrix for $\langle\mathcal{V}\rangle$. S being monomic, it has a factorization $S = S_1S_0$, where S_1 is right prime and, without loss of generality, that $S_0 = \mathrm{diag}\,(z^{\nu_1}, \ldots, z^{\nu_l})$. Clearly, a basis for $\mathcal{V}$ is given by $\{s_1, \ldots, z^{\nu_1 - 1}s_1, \ldots, s_l, \ldots, z^{\nu_l - 1}s_l\}$. However, as above, another basis is given by $\{h_1, \ldots, z^{\lambda_1 - 1}h_1, \ldots, h_k, \ldots, z^{\lambda_k - 1}h_k\}$. As both $\{s_1, \ldots, s_l\}$ as well as $\{h_1, \ldots, h_k\}$ are bases for $\langle\mathcal{V}\rangle$, we have $l = k$ and $S_1 = HU$ for some unimodular U. But S_1 is right prime which implies the right primeness of H. Using the characterization of Theorem 5, it follows that $\mathcal{V}$ is an observability subspace. □

Remark 1: We note that if we remove the constraint of monomicity on S, then in general $X_D \cap S(z)z^{-1}\mathbb{F}[z^{-1}]^k$ is no longer necessarily an almost observability subspace.

Example 3: Assume $d(z) = z^2$ and $\mathcal{V} = X_d \cap (z-\alpha)\mathbb{F}[z] = \mathrm{span}\,(z - \alpha)$. Clearly $\rho(\mathcal{V}) = \mathrm{span}\,(1 - \alpha z) = X_{z^2} \cap (z - \alpha z^2)z^{-1}\mathbb{F}[z^{-1}]$. With respect to the standard basis in X_{z^2}, we have

$$\rho(\mathcal{V}) = \mathrm{span}\begin{pmatrix} 1 \\ -\alpha \end{pmatrix}$$

which has the kernel representation $\rho(\mathcal{V}) = \mathrm{Ker}\begin{pmatrix} \alpha & 1 \end{pmatrix}$. This shows that $\rho(\mathcal{V})$ is an almost observability subspace if and only if $\alpha = 0$.

Given (C, A), a subspace $\mathcal{V} \subset \mathcal{X}$ is an observability subspace if and only if there exists a friend $J \in \mathcal{G}(\mathcal{V})$ for which the characteristic polynomial of $(A + JC)|(\mathcal{X}/\mathcal{V})$ can be arbitrarily assigned, subject only to the degree constraint. Now, by Theorem 9, a subspace $\mathcal{V}$ is an observability subspace if and only if it is both conditioned invariant and almost obervable. The last two types of subspaces have been characterized, in Theorem 8, in terms of solvability of Sylvester type equations. Thus it is of interest to show how to construct a friend so that the characteristic polynomial of $(A + JC)|(\mathcal{X}/\mathcal{V})$ can be arbitrarily preassigned in terms of these Sylvester equations. This is summed up in the following theorem.

Theorem 10: *Let K be a solution to the two Sylvester equations*

$$KA - FK = GC \tag{129}$$

and

$$NKA - K = -LC, \tag{130}$$

where N is nilpotent and F a $k \times k$ matrix. Then for every monic polynomial p of degree k there exist F_p and G_p such that the characteristic polynomial of F_p is equal to p and

$$KA - F_pK = G_pC\ . \tag{131}$$

Proof: We want to construct P such that (P, F) is observable and such that $F_Q := F - QP$ solves

$$KA - F_QK = G_QC$$

for every choice of Q and an appropriate G_Q. Using (129) the latter is equivalent to the following: for every Q we have to find $\tilde{G}_Q$ such that

$$QPK = \tilde{G}_QC.$$

But this is equivalent to the existence of $\tilde{G}_I$ such that $PK = \tilde{G}_IC$ which in turn is equivalent to

$$\mathrm{Ker}\,P \supset K(\mathrm{Ker}\,C).$$

On the other hand (P, F) being observable is equivalent to

$$\mathrm{Ker}\,O(P, F) = \mathrm{Ker}\begin{pmatrix} P \\ PF \\ \vdots \\ PF^{k-1} \end{pmatrix} = \{0\},$$

which suggests to choose $\mathrm{Ker}\,P$ as small as possible.

Now let P be such that $\mathrm{Ker}\,P = K(\mathrm{Ker}\,C)$ and let $x \in \mathrm{Ker}\,O(P, F)$ be arbitrary. It will be shown by induction that $x = 0$. Let $i \in \mathbf{N}$ and assume that $x = N^{i-1}F^{i-1}x$ (which is obviously true for $i = 1$). Then $x \in \mathrm{Ker}\,O(P, F)$ implies $PF^{i-1}x = 0$ which yields

$F^{i-1}x = Ky$ for a $y \in \operatorname{Ker} C$. But then (130) implies $F^{i-1}x = Ky = NKAy + LCy = NKAy$ and multiplying (129) by N yields $F^{i-1}x = NKAy = NFKy + NGCy = NFKy = NFF^{i-1}x = NF^ix$. Multiplying the last equation by N^{i-1} and using the induction hypothesis it follows that $x = N^iF^ix$. By induction it follows $x = N^kF^kx$ and since N is nilpotent this yields $x = 0$. Hence (P, F) is observable.

Now the statement follows from the pole placement theorem setting $F_p := F_Q$ and $G_p := G_Q$ for an appropriate Q. □

We note that this is a slightly weaker result than Theorem 6 where also the fine structure given by the invariant factors was taken into account.

6. On observers

In this section we will explain how the various types of invariant subspaces that have been discussed in the previous sections and their spectral properties relate to observer theory. We review the definitions of tracking observers and singular tracking observers, whose existence is equivalent to the existence of certain conditioned invariant or almost observability subspaces, respectively. We discuss in detail how the spectral properties of these subspaces relate to the observer dynamics and give existence conditions in form of solvability of Sylvester type, as well as rational and polynomial matrix equations.

6.1 *Dead-beat tracking observers*

Definition 6.1: Given the linear systems

$$\begin{cases} \sigma x = Ax + Bu \\ y = Cx \\ z = Kx \end{cases} \tag{132}$$

in the state space $\mathbb{F}^n$ and

$$\begin{cases} \sigma\xi = F\xi + Gy + Hu \\ \zeta = J\xi \end{cases} \tag{133}$$

in the state space $\mathbb{F}^q$. System (133) will be called a tracking observer for K if for every $x(0) \in \mathbb{F}^n$ there exists a $\xi(0) \in \mathbb{F}^q$ such that, for the solutions $x(t)$ and $\xi(t)$ of (132) and (133) respectively, we have $e(t) = z(t) - \zeta(t) = Kx(t) - J\xi(t) = 0$ for all $t \geq 0$. Here e is called the tracking error.

A tracking observer is called a dead-beat tracking observer if for all initial conditions of the states x and ξ and all inputs u there exists a time T such that $e(t) = 0$ for all $t \geq T$, i.e. the tracking error is eventually zero.

Tracking observers have been discussed in Fuhrmann and Helmke (2001) under the name "preobservers". There they have been defined via the following characterization.

Proposition 15: *The observable system* (133) *is a tracking observer for* K *if and only if there exists a transformation* Z *such that*

$$\begin{cases} ZA - FZ = GC \\ H = ZB \\ K = JZ \end{cases} \tag{134}$$

holds. Furthermore, the map Z *is uniquely determined and the dynamics of* $d(t) = \xi(t) - Zx(t)$ *is governed by* $\sigma d = Fd$ *with the tracking error being* $e = Jd$.

Proof: Let equations (134) be fullfilled and let $d(t) := \xi(t) - Zx(t)$. Then

$$\begin{aligned} d(t+1) &= F\xi(t) + Gy(t) + Hu(t) - Z(Ax(t) + Bu(t)) \\ &= F\xi(t) - FZx(t) + FZx(t) + GCx(t) \\ &\quad + Hu(t) - ZAx(t) - ZBu(t) \\ &= Fd(t) - (ZA - FZ - GC)x(t) + (H - ZB)u(t) \\ &= Fd(t). \end{aligned}$$

For given $x(0)$ set $\xi(0) := Zx(0)$ then $d(0) = 0$. It follows $d(t) = 0$ for all $t \geq 0$, especially $e(t) = \zeta(t) - Kx(t) = J\xi(t) - JZx(t) = Jd(t) = 0$ for all $t \geq 0$. Hence system (133) is a tracking observer for Kx.

Conversely, let system (133) be a tracking observer for Kx. Let $\mathcal{B}$ be a basis of the state space of the observed system and let $x(0) \in \mathcal{B}$ be arbitrary. Then there exists $\xi(0)$ such that $e(t) = \zeta(t) - Kx(t) = 0$ for all $t \in \mathbf{R}$. Taking for every $x(0) \in \mathcal{B}$ the corresponding $\xi(0)$, the assignment $\xi(0) =: Zx(0)$ defines a linear map Z which fullfills $JZx(0) = J\xi(0) = \zeta(0) = Kx(0)$ for all $x(0) \in \mathcal{B}$ and hence $K = JZ$. Furthermore, choosing an arbitrary $x(0) \in \mathcal{B}$, taking the corresponding $\xi(0)$ and setting $d(t) := \xi(t) - Zx(t)$ it follows $Jd(t) = J\xi(t) - JZx(t) = \zeta(t) - Kx(t) = 0$ for all $t \geq 0$ and hence

$$\begin{aligned} 0 &= Jd(t+1) \\ &= J[Fd(t) - (ZA - FZ - GC)x(t) + (H - ZB)u(t)] \end{aligned}$$

for all $t \geq 0$, especially $(t := 0)$

$$0 = JFd(0) - J(ZA - FZ - GC)x(0) + J(H - ZB)u(0).$$

Note that by definition $d(0) = \xi(0) - Zx(0) = 0$. Setting $u(0) := 0$ and using the fact that $x(0) \in \mathcal{B}$ was arbitrary, this yields $J(ZA - FZ - GC) = 0$. But then $J(H - ZB) = 0$ since $u(0)$ can be chosen at will. It follows $0 = Jd(t+1) = JFd(t)$ for all $t \geq 0$. Now let $i \in \mathrm{N}$ and assume that

$$\begin{aligned} JF^{i-1}(ZA - FZ - GC) &= 0 \\ JF^{i-1}(H - ZB) &= 0 \\ JF^{i}d(t) = Jd(t+i) &= 0 \quad \text{for all } t \geq 0. \end{aligned}$$

Then it follows

$$\begin{aligned} 0 &= Jd(t+i+1) = JF^{i}d(t+1) \\ &= JF^{i}[Fd(t) - (ZA - FZ - GC)x(t) + (H - ZB)u(t)] \end{aligned}$$

for all $t \geq 0$, especially $(t := 0)$

$$\begin{aligned} 0 = JF^{i+1}d(0) &- JF^{i}(ZA - FZ - GC)x(0) \\ &+ JF^{i}(H - ZB)u(0). \end{aligned}$$

As before it follows $JF^{i}(ZA - FZ - GC) = 0$, $JF^{i}(H - ZB) = 0$ and $0 = Jd(t+i+1) = JF^{i+1}d(t)$ for all $t \geq 0$. By induction and using the fact that (F, J) was observable this yields $ZA - FZ - GC = 0$ and $H - ZB = 0$.

Let Z_1, Z_2 be two solutions of equations (134). Then the difference $\Delta Z := Z_2 - Z_1$ fullfills $J\Delta Z = 0$ and $\Delta ZA - F\Delta Z = 0$. Now let $i \in \mathrm{N}$ and assume that $JF^{i-1}\Delta Z = 0$. Then $JF^{i}\Delta Z = JF^{i-1}F\Delta Z = JF^{i-1}\Delta ZA = 0$. By induction and again using the fact that (F, J) was observable this yields $\Delta Z = 0$. □

Remark 1: If equations (134) hold for system (133) then it is a tracking observer for K even if it is not observable. Observability has not been used for that conclusion.

Note that requiring observers to be observable systems is not a grave restriction since we are designing them ourselves. Furthermore, it follows from the (dual) Kalman decomposition that we can always make an observer observable by reducing its order. Its observable subsystem has the same input output behaviour and hence the same observer properties as the original observer. Using Proposition 15 it is easy to derive also a similar characterization for dead-beat tracking observers.

Proposition 16: *The observable system* (133) *is a dead-beat tracking observer for K if and only if it is a tracking observer for K and F is nilpotent*, *i.e. if and only if F is nilpotent and there exists a transformation Z such that equations* (134) *hold.*

Proof: Let the observable system (133) be a tracking observer for K and let F be nilpotent. According to Proposition 15 the dynamics of $d(t) = \xi(t) - Zx(t)$ is governed by $\sigma d = Fd$. Since F is nilpotent this implies that for every $x(0)$, $\xi(0)$ and u there exists a $T \geq 0$ such that $d(t) = 0$ for $t \geq T$. But then also the tracking error $e(t) = Jd(t) = 0$ for all $t \geq T$ and system (133) is a dead-beat tracking observer for K.

Conversely, let the observable system (133) be a dead-beat tracking observer for K then it is clearly a tracking observer for K. Choose $x(0) = 0$ and $u \equiv 0$ then $d = \xi$ and $e = J\xi$. Assume that F is not nilpotent then there exists a $\xi(0)$ such that $\xi(t) = F^{t}\xi(0) \neq 0$ for all $t \geq 0$. But since system (133) is a dead-beat tracking observer there exists a $T \geq 0$ such that $J\xi(t) = 0$ for all $t \geq T$. This implies $JF^{i}F^{T}\xi(0) = 0$ for all $i \geq 0$ and since (F, J) is observable it follows $F^{T}\xi(0) = 0$, a contradiction. Hence F was nilpotent. □

Remark 2: If system (133) is a tracking observer for K with F nilpotent then it is a dead-beat tracking observer for K even if it is not observable. Observability has not been used for that conclusion.

The theory of the various types of invariant subspaces discussed in the first part of this paper comes into play when one is interested in existence conditions for observers. In view of equation (102), the Sylvester equation in (134) is equivalent to $\operatorname{Ker} Z$ being conditioned invariant. Furthermore, we have an equation of the type $K = JZ$ if and only if $\operatorname{Ker} Z \subset \operatorname{Ker} K$.

In order to be able to link spectral properties of the observer matrix F to outer spectral properties of the subspace $\operatorname{Ker} Z$, we want Z to be surjective. The next result shows that this can always be achieved by reducing the order of the observer.

Proposition 17: *There exist transformations F, G, H, J and Z such that equations* (134) *hold if and only if there exist transformations $\bar{F}$, $\bar{G}$, $\bar{H}$, $\bar{J}$ and $\bar{Z}$, with $\bar{Z}$ surjective*, *such that*

$$\begin{cases} \bar{Z}A - \bar{F}\bar{Z} = \bar{G}C \\ \bar{H} = \bar{Z}B \\ K = \bar{J}\bar{Z} \end{cases} \tag{135}$$

holds. *Furthermore*, *we can choose $(\bar{J}, \bar{F})$ to be observable.*

Proof: Let F, G, H, J and Z such that equations (134) hold. If Z is not surjective we can choose a basis such that

$$Z = \begin{pmatrix} Z_1 \\ 0 \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & J_2 \end{pmatrix}$$

and Z_1 is surjective. Then equations (134) imply

$$\begin{cases} Z_1A - F_{11}Z_1 = G_1C \\ \qquad\quad H_1 = Z_1B \\ \qquad\quad K = J_1Z_1 \end{cases}$$

If (J_1, F_{11}) is not observable then there exists a basis (dual Kalman decomposition) in which

$$Z_1 = \begin{pmatrix} \bar{Z} \\ \bar{Z}_2 \end{pmatrix}, \quad F_{11} = \begin{pmatrix} \bar{F} & 0 \\ \bar{F}_{21} & \bar{F}_{22} \end{pmatrix}, \quad G_1 = \begin{pmatrix} \bar{G} \\ \bar{G}_2 \end{pmatrix},$$
$$H_1 = \begin{pmatrix} \bar{H} \\ \bar{H}_2 \end{pmatrix}, \quad J_1 = \begin{pmatrix} \bar{J} & 0 \end{pmatrix}$$

and $\bar{Z}$ is surjective since Z_1 is surjective. Now equations (135) follow. □

An immediate consequence is the following existence condition for tracking observers.

Theorem 11: *Let p be a monic polynomial of degree q. If* $\operatorname{Ker} K$ *contains a codimension q conditioned invariant subspace $\mathcal{V}$ which has a friend $L \in \mathcal{G}(\mathcal{V})$ such that p is the characteristic polynomial of the induced map $(A - LC)|_{\mathbb{F}^n/\mathcal{V}}$ then there exists a tracking observer for K of order q with the characteristic polynomial of F being p.*

Conversely, if there exists a tracking observer for K of order q with the characteristic polynomial of F being p then $\operatorname{Ker} K$ *contains a conditioned invariant subspace $\mathcal{V}$ of codimension less or equal than q which has a friend $L \in \mathcal{G}(\mathcal{V})$ such that the characteristic polynomial of the induced map $(A - LC)|_{\mathbb{F}^n/\mathcal{V}}$ divides p.*

Proof: Let $\mathcal{V} \subset \operatorname{Ker} K$ be conditioned invariant of codimension q then $\mathcal{V} = \operatorname{Ker} Z$ for a suitable surjective $Z \in \mathbb{F}^{q \times n}$. Let $L \in \mathcal{G}(\mathcal{V})$ be a friend of $\mathcal{V}$ then $(A - LC)\mathcal{V} \subset \mathcal{V}$ implies that there exists a matrix $F \in \mathbb{F}^{q \times q}$ such that $Z(A - LC) = FZ$, i.e. such that figure 4 commutes.

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "A - LC"] \arrow[d, "Z"'] & \mathbb{F}^n \arrow[d, "Z"] \\ \mathbb{F}^q \arrow[r, "F"] & \mathbb{F}^q \end{tikzcd}$$

Figure 4.

This induces a quotient diagram with the induced map $\bar{Z}$ an isomorphism.

$$\begin{tikzcd} \mathbb{F}^n/V \arrow[r, "(A - LC)_{\mathbb{F}^n/V}"] \arrow[d, "\bar{Z}"'] & \mathbb{F}^n/V \arrow[d, "\bar{Z}"] \\ \mathbb{F}^q \arrow[r, "F"] & \mathbb{F}^q \end{tikzcd} \tag{136}$$

But then F is similar to $(A - LC)|_{\mathbb{F}^n/\mathcal{V}}$. Define $G := ZL$ then the first diagram yields $ZA - GC = FZ$. Define $H := ZB$. Since $\operatorname{Ker} Z = \mathcal{V} \subset \operatorname{Ker} K$ there exists a matrix J such that $K = JZ$. Remark 1 now states that system (133) is a tracking observer for K as required.

Conversely, let there exist an order q tracking observer for K. Let system (133) be its observable subsystem which then has order less or equal than q and is also a tracking observer for K. Furthermore, the characteristic polynomial of F divides the original one. It follows from Propositions 15 and 16 that there exists a surjective Z such that $ZA - \bar{F}Z = \bar{G}C$, where the size of $\bar{F}$ is less or equal than that of F and its characteristic polynomial divides that of F. Since Z is surjective there exists L such that $\bar{G} = ZL$. But then $Z(A - LC) = \bar{F}Z$ and with $\mathcal{V} := \operatorname{Ker} Z$ it follows $(A - LC)\mathcal{V} \subset \mathcal{V}$, i.e. L is a friend of $\mathcal{V}$. Furthermore, diagram (136) with F replaced by $\bar{F}$ yields that $(A - LC)|_{\mathbb{F}^n/\mathcal{V}}$ is similar to $\bar{F}$. Since the number of rows of Z is less or equal than q and Z is surjective, it follows that $\mathcal{V}$ has codimension less or equal than q. □

The apparent asymmetry in the last result is only overcome in the minimal order case.

Corollary 5: *The minimal order of a tracking observer for K is equal to the minimal codimension of a conditioned invariant subspace contained in* $\operatorname{Ker} K$. *Let this order be q_{min} and let p be a monic polynomial of degree q_{min}. There exists a tracking observer for K with the characteristic polynomial of F being p if and only if* $\operatorname{Ker} K$ *contains a codimension q_{min} conditioned invariant subspace which has a friend $L \in \mathcal{G}(\mathcal{V})$ such that the characteristic polynomial of the induced map $(A - LC)|_{\mathbb{F}^n/\mathcal{V}}$ is p.*

Note that the previous theorem together with Theorem 7 completely solves the question of possible observer dynamics for tracking observers. The invariant factors can be freely preassigned subject to the constraints given in Theorem 7.

The previous results are easily applied to derive existence conditions for dead-beat tracking observers.

Corollary 6: *If* $\operatorname{Ker} K$ *contains a codimension q outer reconstructible subspace then there exists a dead-beat* *tracking observer for K of order q. If there exists a dead-beat tracking observer for K of order q then* $\operatorname{Ker} K$ *contains an outer reconstructible subspace $\mathcal{V}$ of codimension less or equal than q. If q is the minimal order of a dead-beat tracking observer for K then* $\operatorname{codim} \mathcal{V} = q$.

Again, there is an apparent asymmetry in this result which in general can not be overcome in the non-minimal order case as the following example shows.

Example 4: Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

then the spectrum of $A - LC$ is $\{0, 1\}$, independent of L, and hence the trivial subspace is not outer reconstructible. Hence $\operatorname{Ker} K$ contains no codimension 2 outer reconstructible subspace. However,

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$J = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

fullfill equations (134), F is nilpotent and (J, F) is observable, so there exists an order 2 observable dead-beat tracking observer for K. The minimal order for a dead-beat tracking observer for K would be 1 in this case.

There are many equivalent ways of expressing the existence of conditioned invariant subspaces in polynomial terms, see Fuhrmann and Helmke (2001). The theorem below is a (slightly corrected) variant of their Theorem 10.

Theorem 12: *Define*

$$\begin{aligned} Z_K(z) &= K(zI - A)^{-1} \\ Z_C(z) &= C(zI - A)^{-1} \end{aligned} \tag{137}$$

Consider the following statements:

1. *There exists a codimension q outer reconstructible subspace $\mathcal{V} \subset \operatorname{Ker} K$.*
2. *There exist linear transformations Z, F, G, H, J, with Z surjective of rank q and F nilpotent, such that equations* (134) *hold.*
3. *There exists an order r dead-beat tracking observer for K.*
4. *There exist strictly proper, rational functions M, N with monomic denominator and the McMillan degree of $\begin{pmatrix} M & N \end{pmatrix}$ equal to s that solve*

$$\begin{pmatrix} M & N \end{pmatrix} \begin{pmatrix} zI - A \\ C \end{pmatrix} = K. \tag{138}$$

5. *There exist strictly proper, rational functions Z_1, Z_2 with monomic denominator and the McMillan degree of $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$ equal to s that solve*

$$Z_K = Z_1 Z_C + Z_2. \tag{139}$$

Then (1) *and* (2) *are equivalent and so are* (4) *and* (5). *Furthermore,* (2) *implies* (3) *with $r = q$ and* (4) *with $s \le q$. Finally,* (3) *implies* (2) *with $q \le r$ and* (4) *implies* (3) *with $r = s$.*

Proof:
(1) $\Leftrightarrow$ (2)
This is exactly the construction in Theorem 11.

(4) $\Leftrightarrow$ (5)
Set $Z_1 = N$ and $Z_2 = M$ and the result follows.

(2) $\Rightarrow$ (3) with $r = q$ and (3) $\Rightarrow$ (2) with $q \le r$.
This is the statement of Corollary 6.1 combined with (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (4) with $s \le q$

$$\begin{pmatrix} M & N \end{pmatrix} := \left(\begin{array}{c|cc} F & Z & G \\ \hline J & 0 & 0 \end{array}\right)$$

has McMillan degree $s \le q$ (since it has an order q realization) and a monomic denominator (since F is nilpotent). Furthermore,

$$\begin{aligned} &\begin{pmatrix} M & N \end{pmatrix} \begin{pmatrix} zI - A \\ C \end{pmatrix} \\ &= J(zI - F)^{-1} Z(zI - A) + J(zI - F)^{-1} GC \\ &= J(zI - F)^{-1} [Z(zI - A) + GC] \\ &= J(zI - F)^{-1} [zIZ - ZA) + (ZA - FZ)] \\ &= J(zI - F)^{-1} [(zI - F)Z] \\ &= JZ \\ &= K. \end{aligned}$$

(4) $\Rightarrow$ (3) with $r = s$
Let

$$\begin{pmatrix} M & N \end{pmatrix} = \left(\begin{array}{c|cc} F & Z & G \\ \hline J & 0 & 0 \end{array}\right)$$

be an order $r = s$ observable realization. It follows that F is nilpotent since $\begin{pmatrix} M & N \end{pmatrix}$ has a monomic denominator. Then

$$K = J(zI - F)^{-1}[GC + Z(zI - A)]$$

and hence $\operatorname{Im} K \subset \operatorname{Im} J$. There exists a transformation T such that $K = JT$. It follows

$$J(zI - F)^{-1}[(zI - F)T - GC - Z(zI - A)] = 0. \quad (140)$$

Since (J, F) is observable there exists L such that $F - LJ$ and A have disjoint spectra. Hence the Sylvester equation

$$(F - LJ)X - XA = -TA + FT + GC \quad (141)$$

has a unique solution X. Set $Y := Z + X - T$ then

$$\begin{aligned}
(zI - (F - LJ))X - Y(zI - A) &\\
= zX - (F - LJ)X - zY + YA &\\
= zX + TA - FT - GC - XA &\\
\quad - zZ - zX + zT + ZA + XA - TA &\\
= (zI - F)T - GC - Z(zI - A) &
\end{aligned}$$

and hence (140) and

$$J(zI - (F - LJ))^{-1} = (I + J(zI - F)^{-1}L)^{-1}J(zI - F)^{-1}$$

imply

$$\begin{aligned}
0 &= J(zI - (F - LJ))^{-1}[(zI - (F - LJ))X \\
&\quad - Y(zI - A)](zI - A)^{-1} \\
&= JX(zI - A)^{-1} - J(zI - (F - LJ))^{-1}Y
\end{aligned}$$

Since the spectra of $F - LJ$ and A are disjoint this yields

$$JX(zI - A)^{-1} = J(zI - (F - LJ))^{-1}Y = 0$$

But then $JX = 0$ and the observability of (J, F) and of $(J, F - LJ)$ implies $Y = 0$. Now (141) becomes

$$FX - XA + TA - FT - GC = 0$$

and with $0 = Y = Z + X - T$ and hence $X = T - Z$ this implies $ZA - FZ = GC$. Furthermore, $JZ = JT - JX = JT = K$. Set $H := ZB$. Now the statement follows from Proposition 16. □

Example indicates that the complicated relationship between observer order, subspace codimension and McMillan degree of the involved rational functions in this theorem is the best we can hope for.

6.2 *Singular tracking observers*

Definition 5: Given the linear system (132) in the state space $\mathbb{F}^n$ and the singular linear system

$$N\sigma\xi = \xi + Ly + Mu \quad (142)$$

in the state space $\mathbb{F}^q$, where N is nilpotent. System (142) will be called a singular tracking observer for K if for every $x(0) \in \mathbb{F}^n$ setting $\xi(0) := Kx(0)$ is consistent, i.e. allows a solution $\xi(t)$ of (142), and if furthermore for the solution $x(t)$ of (132) we have $e(t) = z(t) - \xi(t) = Kx(t) - \xi(t) = 0$ for all $t \geq 0$. Here e is called the tracking error.

We have the following characterization of singular tracking observers.

Proposition 18: *The system* (142) *is a tracking observer for K if and only if*

$$\begin{cases} NKA - K = LC \\ \quad\;\; M = NKB \end{cases} \quad (143)$$

holds. Furthermore, the dynamics of the tracking error is governed by $N\sigma e = e$.

Proof: Let equations (143) be fulfilled then $\xi(t) := Kx(t)$ is a solution of the observer equation (142) since

$$\begin{aligned}
N\sigma\xi &= NK\sigma x \\
&= NKAx + NKBu \\
&= Kx + LCx + Mu \\
&= \xi + Ly + Mu.
\end{aligned}$$

Since solutions are uniquely determined by the initial value this means that (142) is a singular tracking observer for K.

Conversely, let system (142) be a singular tracking observer for K. Choose $\xi(0) := Kx(0)$. Then $\xi(t) = Kx(t)$ for all $t \geq 0$, i.e. $e(t) = 0$ for all $t \geq 0$, and

$$\begin{aligned}
N\sigma e &= NK\sigma x - N\sigma\xi \\
&= NKAx + NKBu - \xi - Ly - Mu \\
&= NKAx - Kx - LXx + NKBu - Mu \\
&= (NKA - K - LC)x + (NKB - M)u.
\end{aligned}$$

Especially, for $x(0)=0$ it follows $0=N(\sigma e)(0)=(NKB-M)u(0)$ and hence $NKB=M$ since $u(0)$ is arbitrary. But then $0=N(\sigma e)(0)=(NKA-K-LC)x(0)$ implies $NKA-K=LC$ since $x(0)$ is arbitrary.

If equations (143) are fulfilled we have

$$\begin{aligned} N\sigma e &= NKAx+NKBu-\xi-Ly-Mu \\ &= (NKA-LC)x-\xi+(NKB-M)u \\ &= Kx-\xi=e. \end{aligned} \qquad \square$$

Recalling the definition of almost observability subspaces we immediately have the following characterization of the existence of singular tracking observers.

Corollary 7: *There exists a singular tracking observer for K if and only if* $\operatorname{Ker} K$ *is an almost observability subspace.*

In the following we state a polynomial characterization of almost observability subspaces which in view of the previous corollary yields an existence criterion for singular tracking observers.

Theorem 13: *Define*

$$\begin{aligned} Z_K(z) &= K(zI-A)^{-1} \\ Z_C(z) &= C(zI-A)^{-1}. \end{aligned} \tag{144}$$

The equation

$$P_1 Z_C + P_2 = Z_K \tag{145}$$

has a polynomial solution $\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ *of the form*

$$\left(\begin{array}{c|cc} N & L & L' \\ \hline I & 0 & 0 \end{array}\right)$$

if and only if $\operatorname{Ker} K$ *is an almost observability subspace.*

Proof: Assume

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix} = \left(\begin{array}{c|cc} N & L & L' \\ \hline I & 0 & 0 \end{array}\right) = (zN-I)^{-1}\begin{pmatrix} L & L' \end{pmatrix} \tag{146}$$

is polynomial, i.e. N is nilpotent, and solves (139). We have

$$\begin{aligned} 0 &= Z_K - P_1 Z_C - P_2 \\ &= K(zI-A)^{-1} - (zN-I)^{-1}LC(zI-A)^{-1} \\ &\quad - (zN-I)^{-1}L' \end{aligned}$$

i.e.

$$(zN-I)K - LC - L'(zI-A) = 0.$$

Equating coefficients, we conclude that $L'=NK$ and $-K-LC+NKA=0$, i.e. $\operatorname{Ker} K$ is an almost observability subspace.

Conversely, assume that $\operatorname{Ker} K$ is an almost observability subspace, i.e. there exist N nilpotent and L such that $NKA-K=LC$. We define

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix} := (zN-I)^{-1}\begin{pmatrix} L & NK \end{pmatrix}.$$

The nilpotency of N guarantees that P_1, P_2 are polynomial matrices. We compute

$$\begin{aligned} & P_1 Z_C + P_2 - Z_K = \\ & (zN-I)^{-1}LC(zI-A)^{-1} + (zN-I)^{-1}NK - K(zI-A)^{-1} = \\ & (zN-I)^{-1}[LC+NK(zI-A)-(zN-I)K](zI-A)^{-1} = \\ & (zN-I)^{-1}[LC-NKA+K](zI-A)^{-1} = 0, \end{aligned}$$

i.e. P_1, P_2 solve (145). $\square$

Remark 3: $\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ has a realization of the form

$$\left(\begin{array}{c|cc} N & L & L' \\ \hline I & 0 & 0 \end{array}\right)$$

with N nilpotent if and only if it has a realisation of the form

$$\left(\begin{array}{c|cc} N & L & L' \\ \hline P & 0 & 0 \end{array}\right)$$

with N nilpotent and P invertible.

6.3 *Tracking observers with arbitrary dynamics*

Going back to Theorem 11, it is easy to derive a sufficient condition for the existence of tracking observers with arbitrary dynamics, i.e. where the designer can freely choose the spectrum of F.

Theorem 14: *If* $\operatorname{Ker} K$ *contains a codimension q observability subspace then for every monic polynomial p of degree q there exists an order q tracking observer for K such that the characteristic polynomial of F is p.*

Note that the existence of fixed order tracking observers with arbitrary spectrum does not necessarily imply the existence of a suitable observability subspace, not even in the minimal order case. For every given characteristic polynomial there exists a conditioned invariant subspace with the respective outer spectrum but they could all be different. Furthermore, there is even no guarantee that all these subspaces can be found with the right codimension, cf. Example 4. This is not the case, though, if J is invertible (e.g. $J = I$ and hence effectively the observer *state* tracks Kx). Then the subspace under consideration is $\operatorname{Ker} K$ itself and it is possible to work around potential rank defects in Z. For the details see Theorem 3.38 in Trumpf (2002).

A polynomial characterization of the existence of observability subspaces and hence a sufficient existence criterion for tracking observers with arbitrary dynamics can be formulated as follows.

Theorem 15: *There exists an observability subspace $\mathcal{V} \subset \operatorname{Ker} K$ if and only if there exists a strictly proper solution*

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \left(\begin{array}{c|cc} F & G & Z \\ \hline J & 0 & 0 \end{array}\right)$$

of

$$Z_1 Z_C + Z_2 = Z_K$$

and there exists a polynomial solution

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix} = \left(\begin{array}{c|cc} N & L & L' \\ \hline I & 0 & 0 \end{array}\right)$$

of

$$P_1 Z_C + P_2 = Z_Z$$

Here $Z_Z(z) = Z(zI - A)^{-1}$. The subspace is then given by $\mathcal{V} = \operatorname{Ker} Z$.

Proof: Put together Theorem 12 (without the spectral requirements, cf. also Theorem 5.4 in Fuhrmann and Helmke (2001)) and Theorem 13, and use the fact that a subspace is an observability subspace if and only if it is conditioned invariant and at the same time an almost observability subspace. □

Corollary 8: $\operatorname{Ker} K$ *is an observability subspace if and only if there exists a strictly proper solution*

$$\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} = \left(\begin{array}{c|cc} F & G & Z \\ \hline I & 0 & 0 \end{array}\right)$$

of

$$Z_1 Z_C + Z_2 = Z_K$$

and there exists a polynomial solution

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix} = \left(\begin{array}{c|cc} N & L & L' \\ \hline I & 0 & 0 \end{array}\right)$$

of

$$P_1 Z_C + P_2 = Z_K$$

Proof: From the proof of Theorem 12 it follows $K = IZ$, and we apply the previous theorem. □

7. Summary

This paper is a contribution to the field of geometric control in general and to observer theory in particular. Its principal contributions are to the functional, or module theoretic, characterizations of the classes of observability, almost observability and reconstructibility subspaces. In our opinion the results on spectral assignability for observers of partial states are definitive, solving the problem completely. In the analysis of almost observability subspaces we took a formal approach with the conceptual foundation missing. This gap should eventually be closed. Due to the already significant size of this paper, the discussion of observers in §6 has been limited. A topic that has not been addressed at all in this paper is the relative advantages and disadvantages of developing observer theory from the point of view of state space theory in comparison to a behavioral point of view as in Valcher and Willems (1999a, 1999b). It seems to us that this is a far from finished area of research.

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