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A matrix Euclidean algorithm and matrix continued fraction expansions

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1. Introduction

"$V_{\max}$ is directly related to the Euclidean algorithm", Kalman [14].

The aim of this paper is to try and give a rigorous foundation to the insight carried in this remark of Kalman. In the previously quoted paper of Kalman, as well as in the strongly related paper of Gragg and Lindquist [12], the Euclidean algorithm is taken as vehicle for producing a nested sequence of partial realizations as well as for obtaining a continued fraction representation of a strictly proper transfer function. As a by-product Kalman obtains a characterization of the maximal (A, B)-invariant subspace in Ker C for a minimal realization (A, B, C) that is associated with the continued fraction expansion. While both Kalman and Gragg and Lindquist have a strong feeling that these results should generalize to the matrix case, this seems to elude them, mainly I guess, due to the fact that there does not seem to exist a suitable generalization of the Euclidean algorithm to the matrix case. Results such as in Gantmacher [11] or MacDuffee [16] are not what is needed for handling this problem.

In this paper the line of reasoning is reversed. Rather then start with the Euclidean algorithm we start with a very simple idea derived from the Morse–Wonham geometric control theory, namely the knowledge that $V^*_{\operatorname{Ker} C}$ is related to maximal McMillan degree reduction by state feedback. Indeed a minimal system is feedback irreducible iff $V^*_{\operatorname{Ker} C} = \{0\}$. Thus feedback irreducible systems provide the atoms needed for the construction of a continued fraction representation, or alternatively for a version of the Euclidean algorithm. Putting things together we obtain a recursive algorithm for the computation of $V^*_{\operatorname{Ker} C}$. The dual concept of reduction by output injection is introduced and actually most results are obtained in this setting which is technically simpler. A recursive characterization of $V_*(\mathscr{B})$ is also derived. These characterizations are related through dual direct sum decompositions.

2. A matrix Euclidean algorithm

In papers by Kalman [14] and Gragg and Lindquist [12] the Euclidean algorithm and a generalization of it introduced by Magnus [17,18] have been taken as the starting point of the analysis of continued fraction representations for rational functions, or even more generally, formal power series. As a corollary Kalman obtained a characterization of $V^*_{\operatorname{Ker} C}$. Let us analyze what is involved.

Assume g is a scalar strictly proper transfer function and let $g = p/q$ with p, q coprime. Write, following Gragg and Lindquist, the Euclidean algorithm in the following form:

Suppose s_{i-1}, s_i are given polynomials with

$$\deg s_i < \deg s_{i-1},$$

then by the division rule of polynomials there exist unique polynomials a'_{i+1} and s'_{i+1} such that

$$\deg s'_{i+1} < \deg s_i$$

and

$$s_{i-1} = a'_{i+1} s_i - s'_{i+1}.$$

Let b_i be the inverse of the highest nonzero coefficient of a'_{i+1}. Multiplying through by b_i and defining

$$a_{i+1} = b_i a'_{i+1}, \qquad s_{i+1} = b_i s'_{i+1}$$

we can write the Euclidean algorithm as

$$s_{i+1} = a_{i+1} s_i - b_i s_{i-1} \tag{2.1}$$

with

$$s_{-1} = q \quad \text{and} \quad s_0 = p.$$

Here the a_{i+1} are monic polynomials and b_i are nonzero normalizing constants. Clearly, by the definition of the algorithm, $\{s_i\}$ is a sequence of polynomials of decreasing degrees. Thus for some n, $s_{n+1} = 0$. In this case s_n, being the greatest common divisor of p and q, is a nonzero constant.

Kalman calls the a_i the atoms of the pair p, q or alternatively of the transfer function g.

In (2.1) the recursion and initial condition were used to compute the a_i and b_i. Now we use the a_i and b_i to solve the recursion relation

$$x_{i+1} = a_{i+1}x_i - b_i x_{i-1} \tag{2.2}$$

with two different sets of initial conditions.

Specifically let p_i be the solution of (2.2) with the initial conditions

$$x_{-1} = -1 \quad \text{and} \quad x_0 = 0$$

and let q_i be the solution of (2.2) with the initial conditions

$$x_{-1} = 0 \quad \text{and} \quad x_0 = 1.$$

It has been shown in both previously mentioned papers that $p = p_n$ and $q = q_n$, and we have

$$g = b_0/(a_1 - g_1),$$

i.e.

$$a_1 g - g_1 g = b_0 \quad \text{or} \quad g_1 = (a_1 g - b_0)/g.$$

Let us consider the extreme situation, namely that where the Euclidean algorithm terminates in the first step. This means that $s_1 = 0$, i.e. that

$$a_1 s_0 - b_0 s_{-1} = 0$$

or equivalently that

$$a_1 p - b_0 q = 0.$$

This in turn implies that $g = p/q = b_0/a_1$, i.e. the transfer function g has no finite zeros. Being zeroless its McMillan degree $\delta(g) = \deg a_1$ is feedback invariant, as well as output injection invariant. In the more general case if

$$g = b_0/(a_1 - g_1)$$

and writing $g_1 = r/s$ we have

$$g = b_0 s/(a_1 s - r)$$

with $\deg(r) < \deg(s)$. This indicates how we can obtain the first atom of g. Write $g = p/q$ and let $p = b_0 s$ with s monic. Let a be any monic polynomial such that

$$\deg(a) + \deg(s) = \deg(q).$$

Thus, by a result of Hautus and Heymann [13] the transfer function $b_0 s/as$ is obtainable from p/q by state feedback. Hence $q = as - r_1$ for some polynomial r_1 of degree less than that of q. If we reduce r_1 modulo s we can write $q = a_1 s - r$ and this representation isunique. Thus we have

$$g = b_0 s/(a_1 s - r) = b_0/(a_1 - (r/s)). \tag{2.3}$$

The moral of this is that given s (which in this case is just p normalized) there is a unique way of adding a polynomial r of degree less than s to q such that the resulting transfer function has smallest possible McMillan degree, i.e. we obtain b_0/a_1 and this is not further reducible. The implications are quite clear. Feedback reduction is well defined in the multivariable setting. We can use this to obtain a multivariable version of the Euclidean algorithm. It is somewhat more convenient to begin not with feedback reduction but rather with reduction by output injection. A similar simplification has been observed in Fuhrmann [8] where it turned out that the analysis of the output injection group in terms of polynomial models is significantly simpler that that of the feedback group.

Thus let G be a $p \times m$ strictly proper transfer function and assume $G = T^{-1}U$ is a left coprime factorization. Set

$$T_0 = T, \qquad U_0 = U.$$

Suppose we obtain in the i-th step T_i, U_i left coprime such that T_i is nonsingular and $T_i^{-1}U_i$ strictly proper. We describe the next step.

We state now the main technical lemma needed for our version of the Euclidean algorithm.

Lemma 2.1. *Let G_i be a strictly proper $p \times m$ transfer function and let*

$$G_i = T_i^{-1}U_i$$

be a left matrix fraction representation of G_i with T_i row proper. Then there exist a nonsingular row proper polynomial matrix T_{i+1}, a nonsingular polynomial matrix A_{i+1} with proper inverse and poly-

nomial matrices B_i and U_{i+1} such that

$$T_i = T_{i+1}A_{i+1} - U_{i+1}, \tag{2.4}$$

$$U_i = T_{i+1}B_i, \tag{2.5}$$

and the following conditions are satisfied:

(i) *$T_{i+1}^{-1}U_{i+1}$ is strictly proper.*
(ii) *$A_{i+1}^{-1}B_i$ is output injection irreducible.*
(iii) *A_i is row proper.*

Remark. Note that, as in the scalar case exemplified by equation (2.3), the idea is to obtain maximal McMillan degree reduction by adding lower order terms, i.e. U_{i+1}, to the denominator in a left matrix fraction representation. Here low order terms are interpreted in the sense that $T_i^{-1}U_{i+1}$ is strictly proper. While this can be achieved in many ways we obtain uniqueness if we add the additional requirement that T_{i+1}^{-1} is strictly proper.

Finally we point out that equations (2.4) and (2.5) taken together are the generalization to the multivariable case of (2.3).

Proof. Let $G = A^{-1}B$ be an output injection irreducible transfer function that is output injection equivalent to G_i. This means, by Theorem 3.21 in [8], that there exist polynomial matrices U and T_{i+1} such that $T_i^{-1}U$ is strictly proper and

$$T_i = T_{i+1}A - U \tag{2.6}$$

and

$$U_i = T_{i+1}E. \tag{2.7}$$

Naturally such a decomposition is not unique. However T_{i+1} is unique modulo a right unimodular factor. To see this note that if $A_1^{-1}B_1$ is output injection equivalent to $A^{-1}B$ then for some polynomial matrix Q, for which $A_1^{-1}Q$ is strictly proper, and a unimodular polynomial matrix W, we have

$$A = W(A_1 + Q)$$

for some polynomial matrix W.

Therefore we have for T_i, U_i the alternative representation

$$\begin{aligned} T_i &= T_{i+1}W(A_1 + Q) - U \\ &= (T_{i+1}W)A_1 + (T_{i+1}WQ - U) \\ &= (T_{i+1}W)A_1 - U_1 \end{aligned}$$

and

$$U_i = (T_{i+1}W)B_1.$$

Obviously $T_i^{-1}U_1$ is strictly proper, which follows from the fact that $T_i^{-1}U$ is. The simple calculation is omitted. This shows that T_{i+1} is determined uniquely up to a right unimodular factor.

Fixing T_{i+1} we reduce U modulo T_{i+1}, i.e. we write

$$U_{i+1} = T_{i+1}\pi_- T_{i+1}^{-1}U, \tag{2.8}$$

where π_- is the projection map that associates with a rational function its strictly proper part. For later use we define $\pi_+ = I - \pi_-$.

Then for some polynomial matrix A_{i+1}

$$T_i = T_{i+1}A_{i+1} - U_{i+1} \tag{2.9}$$

and $T_{i+1}^{-1}U_{i+1}$ is strictly proper by construction. Thus $A_{i+1}^{-1}B_i$ is output injection irreducible since

$$\deg(\det A_{i+1}) = \deg(\det A).$$

A representation of the form (2.9) is clearly unique.

Since T_{i+1} is only determined up to a right unimodular factor we can use this freedom to ensure that A_{i+1} is row proper.

We will call the $\{A_{i+1}, B_i\}$ the *left atoms* of the transfer function G. Notice that even if we start with a rectangular transfer function G then after the first step all the transfer functions $T_{i+1}^{-1}U_{i+1}$ are square, though not necessarily nonsingular.

We are ready to state the following matrix version of the Euclidean algorithm.

Theorem 2.2. *Let G be a $p \times m$ strictly proper transfer function. Let $G = T^{-1}U$ be a left matrix fraction representation which we do not assume to be left coprime, with T row proper. Define recursively, using the previous lemma, a sequence of polynomial matrices $\{A_{i+1}, B_i\}$, the A_{i+1} being nonsingular and properly invertible. Then*

$$\delta(T_{i+1}^{-1}U_{i+1}) < \delta(T_i^{-1}U_i). \tag{2.10}$$

Let n be the first integer for which $\delta(G_n) = 0$, i.e. for which $U_n = 0$. Then T_n is the greatest common left divisor of T and U.

Proof. Since

$$\begin{aligned}\delta(T_i^{-1}U_i) &= \deg \det T_i \\ &= \deg \det(T_{i+1}A_{i+1} - U_{i+1}) \\ &= \deg \det(T_{i+1}A_{i+1}) \\ &> \deg \det T_{i+1} = \delta(T_{i+1}^{-1}U_{i+1}),\end{aligned}$$

the decrease of the McMillan degree is proved and guarantees the termination of the process in a finite number of steps, say n. Thus $U_n = 0$ and

$$T_{n-1} = T_nA_n, \qquad U_{n-1} = T_nB_{n-1}.$$

Thus T_n is a common left divisor of T_{n-1} and U_{n-1}. In fact it is a g.c.l.d. by the output injection irreducibility of $A_n^{-1}B_{n-1}$. But

$$T_{n-2} = T_{n-1}A_{n-1} - U_{n-1},$$
$$U_{n-2} = T_{n-2}B_{n-2},$$

and so T_n is a common left divisor of T_{n-2} and U_{n-2}, and we proceed by induction.

Of course the transfer function G can be reconstructed from the atom sequence $\{A_{i+1}, B_i\}$. this is the content of Theorem 2.9.

Assume the algorithm terminates in the n-th step, i.e. $T_n^{-1}U_n$ is output injection irreducible.

Define a sequence of transfer functions Γ_i by

$$\Gamma_0 = G \tag{2.11}$$

and

$$\Gamma_i = (A_{i+1} - \Gamma_{i+1})^{-1}B_i \tag{2.12}$$

where

$$\Gamma_i = T_i^{-1}U_i. \tag{2.13}$$

Lemma 2.3. *The sequence of transfer functions $\{\Gamma_i\}$ so constructed satisfies*

$$\delta(\Gamma_{i+1}) < \delta(\Gamma_i). \tag{2.14}$$

Proof. If $\Gamma_i = A_{i+1}^{-1}B_i$ is irreducible by output injection then $\Gamma_{i+1} = 0$. Otherwise

$$\Gamma_i = T_i^{-1}U_i = (T_{i+1}A_{i+1} - U_{i+1})^{-1}T_{i+1}B_i$$

and

$$\begin{aligned}\delta(\Gamma_i) &= \deg \det T_i = \deg \det(T_{i+1}A_{i+1}) \\ &> \deg \det(T_{i+1}) = \delta(T_{i+1}^{-1}U_{i+1}) = \delta(\Gamma_{i+1}).\end{aligned}$$

We use now the $\{A_{i+1}, B_i\}$ to define recursively two sequences of polynomial matrices $\{R_i, W_i\}$ by

$$(R_i \ \ W_i) = (I \ \ 0)\begin{pmatrix} A_i & B_{i-1} \\ -I & 0 \end{pmatrix}\cdots\begin{pmatrix} A_1 & B_0 \\ -I & 0 \end{pmatrix}. \tag{2.15}$$

Obviously

$$\begin{aligned}(R_{i+1} \ \ W_{i+1}) &= (A_{i+1} \ \ B_i)\begin{pmatrix} R_i & W_i \\ -R_{i-1} & -W_{i-1} \end{pmatrix} \\ &= (A_{i+1}R_i - B_iR_{i-1} \ \ \ A_{i+1}W_i - B_iW_{i-1}),\end{aligned}$$

i.e. we solve the recursions

$$R_{i+1} = A_{i+1}R_i - B_iR_{i-1}$$

with initial conditions $R_{-1} = 0$, $R_0 = I$,

$$W_{i+1} = A_{i+1}W_i - B_iW_{i-1}$$

with initial conditions $W_{-1} = -I$, $W_0 = 0$.

Lemma 2.4. *Assume $\{A_i\}$ are properly invertible and $A_{i+1}^{-1}B_i$ strictly proper. Then $R_k^{-1}W_k$ is strictly proper.*

Proof. We prove this by induction. For $k = 1$ this follows from out assumptions. Assume this holds for any $k - 1$ factors. Then

$$\begin{aligned}(R_k \ \ W_k) &= \left[(I \ \ 0)\begin{pmatrix} A_k & B_{k+1} \\ -I & 0 \end{pmatrix}\cdots\begin{pmatrix} A_2 & B_1 \\ -I & 0 \end{pmatrix}\right] \\ &\quad\cdot\begin{pmatrix} A_1 & B_0 \\ -I & 0 \end{pmatrix} \\ &= (S_{k+1} \ \ V_{k+1})\begin{pmatrix} A_1 & B_0 \\ -I & 0 \end{pmatrix}\end{aligned} \tag{2.16}$$

or

$$R_k = S_{k+1}A_1 - V_{k+1}, \tag{2.17}$$

$$W_k = S_{k+1}B_0. \tag{2.18}$$

Clearly

$$\begin{aligned}R_k^{-1} &= (S_{k+1}A_1 - V_{k+1})^{-1} \\ &= (A_1 - S_{k+1}^{-1}V_{k+1})^{-1}S_{k+1}^{-1} \\ &= (I - A_1^{-1}S_{k+1}^{-1}V_{k+1})^{-1}A_1^{-1}S_{k+1}^{-1}.\end{aligned}$$

By assumption A_1^{-1} is proper, $S_{k+1}^{-1}V_{k+1}$ is strictly proper and S_{k+1}^{-1} proper by the induction hypothesis. Since

$$\left(I - A_1^{-1}S_{k+1}^{-1}V_{k+1}\right)$$

is a bicausal isomorphism, properness of R_k^{-1} follows.

Next

$$\begin{aligned} R_k^{-1}W_k &= \left(I - A_1^{-1}S_{k+1}^{-1}V_{k+1}\right)^{-1}A_1^{-1}S_{k+1}^{-1}S_{k+1}B_0 \\ &= \left(I - A_1^{-1}S_{k+1}^{-1}V_{k+1}\right)^{-1}A_1^{-1}B_0 \end{aligned}$$

and this is clearly strictly proper.

Next we define a sequence of rational functions $\{E_i\}$ by

$$E_i = R_iG - W_i \tag{2.19}$$

with

$$E_{-1} = I \quad \text{and} \quad E_0 = G. \tag{2.20}$$

Theorem 2.5. *The E_i satisfy the recursion*

$$E_{i+1} = A_{i+1}E_i - B_iE_{i-1}. \tag{2.21}$$

Proof. We compute

$$\begin{aligned} &A_{i+1}E_i - B_iE_{i-1} \\ &\quad = A_{i+1}(R_iG - W_i) - B_i(R_{i-1}G - W_{i-1}) \\ &\quad = (A_{i+1}R_i - B_iR_{i-1})G - (A_{i+1}W_i - B_iW_{i-1}) \\ &\quad = R_{i+1}G - W_{i+1} \\ &\quad = E_{i+1}. \end{aligned}$$

Theorem 2.6. *With $\Gamma_0 = G$ and*

$$\Gamma_i = T_{i+1}^{-1}U_{i+1}$$

we have

$$E_k = \Gamma_k \cdots \Gamma_0. \tag{2.22}$$

Proof. For $k = 0$ this holds by definition. Proceed by induction. We have

$$\Gamma_i = (A_{i+1} - \Gamma_{i+1})^{-1}B_i$$

or

$$A_{i+1}\Gamma_i - B_i = \Gamma_{i+1}\Gamma_i.$$

Hence

$$\begin{aligned} \Gamma_{k+1} \cdots \Gamma_0 &= (\Gamma_{k+1}\Gamma_k)\Gamma_{k-1} \cdots \Gamma_0 \\ &= (A_{k+1}\Gamma_k - B_k)\Gamma_{k-1} \cdots \Gamma_0 \\ &= A_{k+1}\Gamma_k \cdots \Gamma_0 - B_k\Gamma_{k-1} \cdots \Gamma_0 \\ &= A_{k+1}E_k - B_kE_{k-1} = E_{k+1}. \end{aligned}$$

Corollary 2.7. *The rational matrices E_i are all strictly proper.*

Proof. Follows from the strict properness of the Γ_i.

Corollary 2.8. *We have $E_n = 0$ iff $\Gamma_n = 0$.*

Theorem 2.9. *Assume G is strictly proper and rational. Then if $\Gamma_n = 0$ it follows that*

$$G = R_n^{-1}W_n \tag{2.23}$$

where R_n and W_n are defined through the recursions (2.17) *and* (2.18)

We can give now a precise answer to the question of how good an approximation $R_k^{-1}W_k$ is to G.

Theorem 2.10. *Let G be a $p \times m$ strictly proper transfer function and let R_k, W_k be solutions of the recursion equations* (2.17) *and* (2.18). *Then*

$$G - R_k^{-1}W_k = R_k^{-1}E_k = R_k^{-1}\Gamma_k \cdots \Gamma_0. \tag{2.24}$$

Note that since all the Γ_i are strictly proper there is a matching of at least the first $k+1$ Markov parameters, but this of course is only a rough estimate to the more precise estimate (2.24).

3. Connections with geometric control theory

We pass now to the connection between the previously obtained matrix continued fraction representations and some problems of geometric control theory, as developed in Wonham [19].

The link between the two theories is given by the theory of polynomial models developed in a series of papers by Antoulas [1], Fuhrmann [4–8], Emre and Hautus [3], Khargonekar and Emre [15] and Fuhrmann and Willems [9,10]. The last two papers are especially relevant to the following analysis.

The power of the method of polynomial models is the fact that with any matrix fraction representation we have a closely associated realization. Thus all statements on the level of polynomial or rational matrices have an immediate interpretation in terms of state space models. That the setting up of such a complete correspondence is not a trivial matter becomes clear by a perusal of the above mentioned papers.

Recall [4] that with the left matrix fraction representation

$$G = T^{-1}U$$

of a $p \times m$ strictly proper rational function G there is associated a realization in the state space X_T given by the triple of maps (A, B, C) defined by

$$\begin{aligned} &A = S_T, \\ &Bu = Uu \quad \text{for } u \in F^m, \qquad (3.1)\\ &Cf = \left(T^{-1}f\right)_{-1} \quad \text{for } f \in X_T. \end{aligned}$$

This realization is always observable and is reachable if and only if T and U are left coprime. For the definitions of spaces X_T, X^T and maps S_T we refer to [8].

The continued fraction representation obtained previously allows us to give a finer description of this realization.

To this end let $\{A_i, B_i\}$ be the atom sequence obtained from G. Define the sequence of polynomial matrices $\{S_i, V_i\}$ by

$$(S_i \ \ V_i) = (I \ \ 0)\begin{pmatrix} A_n & B_{n-1} \\ -I & 0 \end{pmatrix} \cdots \begin{pmatrix} A_{n-i+1} & B_{n-i} \\ -I & 0 \end{pmatrix}. \qquad (3.2)$$

with

$$S_0 = I, \qquad V_0 = 0. \qquad (3.3)$$

As a special case we obtain

$$\begin{aligned} (S_n \ \ V_n) &= (I \ \ 0)\begin{pmatrix} A_n & B_{n-1} \\ -I & 0 \end{pmatrix} \cdots \\ &\cdots \begin{pmatrix} A_2 & B_1 \\ -I & 0 \end{pmatrix}\begin{pmatrix} A_1 & B_0 \\ -I & 0 \end{pmatrix} \\ &= (S_{n-1} \ \ V_{n-1})\begin{pmatrix} A_1 & B_0 \\ -I & 0 \end{pmatrix} \end{aligned}$$

or

$$S_n = S_{n-1}A_1 - V_{n-1} \qquad (3.4)$$

and in general

$$S_{n-i} = S_{n-i-1}A_{i+1} - V_{n-i-1}. \qquad (3.5)$$

These formulas lead to interesting direct sum representations for X_T. These lead, in the scalar case, directly to some canonical forms associated with the continued fraction expansion. See in this connection the papers of Kalman [14] and Gragg and Lindquist [12]. The multivariable analogs have not been clarified sofar.

Clearly $S_n = R_n$ and so if $E_n = 0$ it follows that

$$G = T^{-1}U = S_n^{-1}V_n = T_n^{-1}U_0 \qquad (3.6)$$

with S_n equal to T up to a left unimodular factor.

Theorem 3.1. *Under the previous assumptions we have*

$$\begin{aligned} X_{R_n} &= X_{S_n} \\ &= X_{A_n} \oplus S_1 X_{A_{n-1}} \oplus \cdots \oplus S_{n-1}X_{A_1}. \qquad (3.7) \end{aligned}$$

Proof. By induction. For $n = 1$ we have $T^{-1}U = A_1^{-1}B_0$ and $S_1 = A_1$ and hence

$$X_{S_1} = S_0 X_{A_1} = X_{A_1}.$$

Since

$$S_n = S_{n-1}A_1 - V_{n-1}$$

and $S_{n-1}^{-1}V_{n-1}$ is strictly proper it follows, as A_1^{-1} is proper, that $A_1^{-1}S_{n-1}^{-1}V_{n-1}$ is strictly proper. It follows from Lemma 5.5 in [10] that X_{S_n} and $X_{S_{n-1}A_1}$ are equal as sets, though they carry different module structures. But the factorization $S_{n-1}A_1$ implies a direct sum decomposition, see Theorem 2.10 in [10],

$$X_{S_n} = X_{S_{n-1}A_1} = X_{S_{n-1}} \oplus S_{n-1}X_{A_1}.$$

By induction (3.7) follows.

This direct sum decomposition is related to geometric concepts.

Theorem 3.2. *Let (A, B, C) be the realization in X_{S_n} associated with $G = S_n^{-1}V_n$. Then the minimal (C, A)-invariant subspace containing* Im B *is $S_{n-1}X_{A_1}$, i.e.*

$$V_*(\mathscr{B}) = S_{n-1}X_{A_1}. \qquad (3.8)$$

Proof. That $S_{n-1}X_{A_1}$ is a (C, A)-invariant subspace follows from the characterization of these subspaces given by Theorem 3.3 of [8]. Also from the recursion relation (3.2) it follows that $V_n = S_{n-1}B_0$, i.e.

$$G = (S_{n-1}A_1 - V_{n-1})^{-1}S_{n-1}B_0.$$

so

$$B\xi = S_{n-1}B_0\xi \in S_{n-1}X_{A_1}$$

as $B_0\xi \in X_{A_1}$. Thus $S_{n-1}X_{A_1} \supset \mathscr{B}$. That this is the minimal subspace follows from Theorem 3.8 of [8].

We pass now to the analysis of the dual results, namely those related to feedback reduction. In analogy with Lemma 2.1 we can state, without proof, the following.

Lemma 3.3. *Let G_i be a $p \times m$ strictly proper rational matrix and let*

$$G_i = N_iD_i^{-1} \tag{3.9}$$

be a right matrix fraction representation with D_i column proper. Then there exist a nonsingular column proper matrix D_{i+1}, a nonsingular properly invertible polynomial matrix A_{i+1} and polynomial matrices N_{i+1} and B_i such that

$$D_i = A_{i+1}D_{i+1} - N_{i+1}, \tag{3.10}$$

$$N_i = B_iD_{i+1} \tag{3.11}$$

and the following conditions hold:

(i) *$G_{i+1} = N_{i+1}D_{i+1}^{-1}$ is strictly proper.*
(ii) *$B_iA_{i+1}^{-1}$ is feedback irreducible.*
(iii) *A_{i+1} is column proper.*

Starting with $G = ND^{-1}$ we can write

$$D = A_1D_1 - N_1, \qquad N = B_0D_1. \tag{3.12}$$

By transposition we obtain

$$\tilde{D} = \tilde{D}_1\tilde{A}_1 - \tilde{N}_1, \qquad \tilde{N} = \tilde{D}_1\tilde{B}_0, \tag{3.13}$$

with $(\tilde{D}_1\tilde{A}_1)^{-1}\tilde{N}_1$ strictly proper. This implies, as we saw before, the direct sum decomposition

$$X_{\tilde{D}} = X_{\tilde{D}_1} + \tilde{D}_1X_{\tilde{A}_1}. \tag{3.14}$$

We proceed to obtain the dual direct sum decomposition of X_D. Note that the annihilator of a (C, A)-invariant subspace is an (A, B)-invariant subspace. In particular the annihilator of $\tilde{D}X_{\tilde{A}_1}$ which is the minimal (C, A)-invariant subspace containing Im B is the maximal (A, E)-invariant subspace contained in Ker C.

Now every (A, B)-invariant subspace of X_D is of the form $\pi_+D\pi^DL$ for some submodule L of $z^{-1}F^m[[z^{-1}]]$, see [10]. Since

$$\dim \tilde{D}X_{\tilde{A}_1} = \deg(\det A_1)$$

the dimension of $V^*_{\mathrm{Ker}C}$ has to be deg(det D_1). This leads us to conjecture that

$$X_D \supset V^*_{\mathrm{Ker}C} = \pi_+DX^{D_1} = \pi_+(A_1D_1 - N_1)X^{D_1}.$$

Actually we can prove more.

Lemma 3.4. *Let $G = ND^{-1}$ be a strictly proper $p \times m$ rational matrix. Then the following direct sum decomposition holds:*

$$X_D = \pi_+(A_1D_1 - N_1)X^{D_1} \oplus X_{A_1}. \tag{3.15}$$

Moreover this direct sum decomposition is the dual of (3.14) *under the pairing of X_D and $X_{\tilde{D}}$ defined in* [8].

Proof. Assume f and g are in X_{A_1} and $X_{\tilde{D}_1}$ respectively. Thus

$$f = A_1h \quad \text{with } h \in X^{A_1}$$

and

$$g = \tilde{D}_1k \quad \text{with } k \in X^{\tilde{D}_1}.$$

We compute

$$\begin{aligned}\langle f, g\rangle &= \left[(A_1D_1 - N_1)^{-1}f, g\right]\\ &= \left[(A_1D_1 - N_1)^{-1}A_1h, \tilde{D}_1k\right]\\ &= \left[D_1(A_1D_1 - N_1)^{-1}A_1h, k\right]\\ &= \left[(I - A_1^{-1}N_1D_1^{-1})^{-1}h, k\right] = 0\end{aligned}$$

by the causality of $A_1^{-1}N_1D_1^{-1}$. Also for $h \in X^{D_1}$ and $k \in X^{\tilde{A}_1}$ we have

$$\begin{aligned}&\langle \pi_+(A_1D_1 - N_1)h, \tilde{D}_1\tilde{A}_1k\rangle\\ &= \left[(A_1D_1 - N_1)^{-1}\pi_+(A_1D_1 - N_1)h, \tilde{D}_1\tilde{A}_1k\right]\\ &= \left[A_1D_1(A_1D_1 - N_1)^{-1}(A_1D_1 - N_1)h, k\right]\\ &= \left[A_1D_1(A_1D_1 - N_1)^{-1}(A_1D_1 - N_1)h, k\right]\\ &= \left[D_1h, \tilde{A}_1k\right].\end{aligned}$$

The removal of the projection π_+ is permissible by the causality of

$$A_1 D_1 (A_1 D_1 - N_1)^{-1}.$$

This ends the proof.

We note that in $X_{\tilde{D}}$, with the realization associated with $\tilde{D}^{-1}\tilde{N}$ we have

$$V_*(\operatorname{Im} B) = \tilde{D}_1 X_{\tilde{A}_1} \tag{3.16}$$

whereas in X_D, with the realization associated with ND^{-1}, we have

$$V^*_{\operatorname{Ker} C} = \pi_+ D X^{D_1} = \pi_+ (A_1 D_1 - N_1) X^{D_1}. \tag{3.17}$$

The preceding result can be easily generalized to yield the following.

Theorem 3.5. *Given $G = ND^{-1}$ with the right atom sequence $\{A_{i+1}, B_i\}$ and the relations*

$$D_i = A_{i+1} D_{i+1} - N_{i+1} \tag{3.18}$$

and

$$N_i = B_i D_{i+1}. \tag{3.19}$$

Then the direct sum decompositions

$$X_D = \pi_+ D D_1^{-1} \pi_+ D_1 D_2^{-1} \cdots \pi_+ D_{n-1} D_n^{-1} X_{A_n} \oplus \cdots \oplus X_{A_1} \tag{3.20}$$

and

$$X_{\tilde{D}} = X_{\tilde{A}_n} \oplus \tilde{D}_{n-1} X_{\tilde{A}_{n-1}} \oplus \cdots \oplus \tilde{D}_1 X_{\tilde{A}_1} \tag{3.21}$$

are dual direct sum decompositions.

Proof. By induction. For $k = 1$ we proved the result in the previous lemma. Assume we proved the result for k. Then, since

$$D_k = A_{k+1} D_{k+1} - N_{k+1} \tag{3.22}$$

and

$$\tilde{D}_k = \tilde{D}_{k+1} \tilde{A}_{k+1} - \tilde{N}_{k+1}, \tag{3.23}$$

it follows that

$$X_{\tilde{D}_k} = X_{\tilde{D}_{k+1}} \oplus \tilde{D}_{k+1} X_{\tilde{A}_{k+1}} \tag{3.24}$$

and

$$X_{D_k} = \pi_+ D_k D_{k+1}^{-1} X_{D_{k+1}} \oplus X_{A_{k+1}}. \tag{3.25}$$

Hence

$$\begin{aligned} X_D = {} & \pi_+ D D_1^{-1} \cdots \pi_+ D_{k-1} D_k^{-1} \left(\pi_+ D_k D_{k+1}^{-1} X_{D_{k+1}} \oplus X_{A_{k+1}} \right) \\ & + \pi_+ D D_1^{-1} \cdots \pi_+ D_{k-2} D_{k-1}^{-1} \cdot X_{A_{k-1}} \oplus \cdots \oplus X_{A_1} \end{aligned} \tag{3.26}$$

and

$$X_{\tilde{D}} = X_{\tilde{D}_{k+1}} \oplus \tilde{D}_{k+1} X_{\tilde{A}_{k+1}} \oplus \tilde{D}_k X_{\tilde{A}_k} \oplus \cdots \oplus \tilde{D}_1 X_{\tilde{A}_1}. \tag{3.27}$$

Since $N_{n-1} D_{n-1}^{-1} = B_{n-1} A_n^{-1}$ the direct sum decomposition follows.

To show the duality of the two direct sum decompositions it suffices, by induction, to prove that the orthogonality relations

$$X_{\tilde{D}_{k+1}} \perp \pi_+ D D_1^{-1} \cdots \pi_+ D_{k-1} D_k^{-1} X_{A_{k+1}} \tag{3.28}$$

and

$$\tilde{D}_{k+1} X_{\tilde{A}_{k+1}} \perp \pi_+ D D_1^{-1} \cdots \pi_+ D_k D_{k+1}^{-1} X_{D_{k+1}} \tag{3.29}$$

hold.

Assume first

$$f \in \pi_+ D D_1^{-1} \cdots \pi_+ D_{k-1} D_k^{-1} X_{A_{k+1}}, \quad g \in X_{\tilde{D}_{k+1}}.$$

Thus there exist $h, k \in z^{-1} F^m[[z^{-1}]]$ such that

$$f = \pi_+ D D_1^{-1} \cdots \pi_+ D_{k-1} D_k^{-1} A_{k+1} h, \quad g = \tilde{D}_{k+1} k.$$

Hence

$$\begin{aligned} \langle f, g \rangle & = \left[D^{-1} \pi_+ D D_1^{-1} \cdots \pi_+ D_{k-1} D_k^{-1} A_{k+1} h, \tilde{D}_{k+1} k \right] \\ & = \left[D_{k+1} D^{-1} \pi_+ D D_1^{-1} \cdots \cdots \pi_+ D_{k-1} D_k^{-1} A_{k+1} h, k \right] \\ & = \left[D_{k+1} D^{-1} D D_1^{-1} \cdots \pi_+ D_{k-1} D_k^{-1} A_{k+1} h, k \right] \\ & = \left[D_{k+1} D_k^{-1} A_{k+1} h, k \right] \\ & = \left[D_{k+1} (A_{k+1} D_{k+1} - N_{k+1})^{-1} A_{k+1} h, k \right] \\ & = \left[\left(I - A_{k+1}^{-1} N_{k+1} D_{k+1}^{-1} \right) h, k \right] \\ & = 0. \end{aligned}$$

Similarly we want to compute

$$\left[D^{-1} \pi_+ D D_1^{-1} \cdots \pi_+ D_k D_{k+1}^{-1} D_{k+1} h, \tilde{D}_{k+1} \tilde{A}_{k+1} k \right]$$

To this end we note that, since

$$D_i = A_{i+1} D_{i+1} - N_{i+1},$$

it follows that

$$\begin{aligned} D_{i+1} D_i^{-1} &= D_{i+1} \left(A_{i+1} D_{i+1} - N_{i+1} \right)^{-1} \\ &= \left(I - A_{i+1}^{-1} N_{i+1} D_{i+1}^{-1} \right)^{-1} A_{i+1}^{-1} \\ &= A_{i+1} \left(I - N_{i+1} D_{i+1}^{-1} A_{i+1}^{-1} \right)^{-1} \end{aligned}$$

is proper, and so is

$$A_{i+1} D_{i+1} D_i^{-1} = \left(I - N_{i+1} D_{i+1}^{-1} A_{i+1}^{-1} \right)^{-1}.$$

Also, for $i > j$, $A_i D_i D_j^{-1}$ is proper since

$$A_i D_i D_j^{-1} = \left(A_i D_i D_{i-1}^{-1} \right) \left(D_{i-1} D_{i-2}^{-1} \right) \cdots \left(D_{j+1} D_j^{-1} \right)$$

and the product of proper matrices is proper. Using these properties it follows that

$$\begin{aligned} &\left[D^{-1} \pi_- D D_1^{-1} \cdots \pi_+ D_k D_{k+1}^{-1} D_{k+1} h,\ \tilde{D}_{k+1} \tilde{A}_{k+1} k \right] \\ &\quad = \left[A_{k+1} D_{k+1} D^{-1} \pi_- D D_1^{-1} \cdots \right. \\ &\qquad \left. \cdots \pi_+ D_k D_{k+1}^{-1} D_{k+1} h,\ k \right] \\ &\quad = 0. \end{aligned}$$

It follows, proceeding inductively, that

$$\begin{aligned} &\left[D^{-1} \pi_+ D D_1^{-1} \cdots \pi_+ D_{k+1}^{-1} D_{k+1} h,\ \tilde{D}_{k+1} \tilde{A}_{k+1} k \right] \\ &\quad = \left[D_1^{-1} \pi_+ D_1 D_2^{-1} \cdots \pi_+ D_k D_{k+1}^{-1} D_{k+1} h, \right. \\ &\qquad \left. \tilde{D}_{k+1} \tilde{A}_{k+1} k \right] \\ &\quad = \cdots = \left[D_{k+1}^{-1} D_{k+1} h,\ \tilde{D}_{k+1} \tilde{A}_{k+1} k \right] \\ &\quad = \left[D_{k+1} h,\ \tilde{A}_{k+1} k \right] = 0. \end{aligned}$$

This completes the proof of the theorem.

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