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On the Lyapunov equation, coinvariant subspaces and some problems related to spectral factorizations

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A geometric approach to stochastic realization theory, and hence to spectral factorization problems, has been developed by Lindquist and Picci (1985, 1991) and Lindquist *et al.* (1995). Most of this work was done abstractly. Fuhrmann and Gombani (1998) adopted an entirely Hardy space approach to this set of problems, studying the set of rectangular spectral factors of given size for a weakly coercive spectral function. The parametrization of spectral factors in terms of factorizations of related inner functions, as developed in Fuhrmann (1995), had to be generalized. This led to a further understanding of the partial order introduced by Lindquist and Picci in the set of stable spectral factors.

In the present paper we study the geometry of finite dimensional coinvariant subspaces of a vectorial Hardy space H^2_+ via realization theory, emphasizing the role of the Lyapunov equation in lifting the Hardy space metric to the state space domain. We follow this by deriving state space formulas for rectangular spectral factors as well as for related inner functions arising in Fuhrmann and Gombani (1998). Finally, we develop a state space approach to the analysis of the partial order of the set of rectangular spectral factors of a given spectral function and its representation in terms of inner functions.

1. Introduction

The central theme of this paper is the study of state space aspects of the geometry of coinvariant subspaces of Hardy spaces. It is the outcome of our work on the fine structure of the class of rectangular spectral factors of a given, not necessarily coercive spectral function (see Fuhrmann and Gombani 1998 for full details). In that paper, because of necessary limitations on its length, the whole analysis was carried out using functional methods, avoiding the use of state space techniques. Now it is well known, at least since the establishment of the Kalman–Yakubovich–Popov positive real lemma, that spectral factorization problems relate to the solution of Riccati equations. A complete parametrization of all minimal, square spectral factors in terms of all solutions to a pair of Riccati equations was given in Fuhrmann (1995). In our previous work, due to the overwhelming complexity, we restricted ourselves to the analysis of stable, rectangular spectral factors. From the state space point of view, the complexity arises out of the fact that in the study of zeros of rectangular rational matrix functions there are hidden zeros that, geometrically, present themselves in the output nulling inner (anti)stabilizing subspaces. The functional representation of these spaces is naturally more complex and it relates to a more intricate factorization theory of inner functions. Our aim in this paper is to fill in some gaps, pertaining to the state space approach, that were left in Fuhrmann and Gombani (1998). Thus to a certain extent, this can be considered as a complementary paper and we suggest the consulting of that paper for more details. We did try however to make this paper as self contained as possible.

Hardy spaces were brought into the realm of operator theory by a classical paper (Beurling 1949), that literally opened up a whole new research area. What Beurling did was to study the shift operator in the Hardy space H^2 of the disc. He characterized its cyclic vectors, identifying them with outer functions, and gave the beautiful characterization, in terms of inner functions, of all the shift invariant subspaces. These results were generalized to the multivariable case by Halmos (1961) and Lax (1959) who was also working in continuous time, replacing the shift by the translation semigroup. Following this came the operator theoretic applications, especially in the analysis of the structure of classes of non-selfadjoint operators and the construction of functional models for them.

It turned out that, for functional models, the important object was not the invariant subspace itself but rather its orthogonal complement to which we refer as a coinvariant subspace. So a coinvariant subspace has the form $\{QH^2\}^{\perp}$, for some inner function Q. Since the shift is universal, its restriction or compression to a coinvariant subspace is completely determined by the space and hence by the inner function Q. This makes the connection between arithmetic, the study of operators, and the geometry of subspaces most immediate. Rationality comes in very naturally. In fact $\{QH^2\}^{\perp}$ is a finite dimensional vector space if and only if the inner function Q is rational and having a non-trivial determinant.

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It did not take long for these ideas, already in use in scattering theory and the study of electrical networks, to infiltrate the area of control theory. This was recognized by several people who made the early contribution to the use of Hardy spaces as a vehicle for developing a non-rational realization theory (see Fuhrmann 1974, 1975, 1981 a,b, Helton 1974, Baras and Brockett 1975, Dewilde 1976).

While infinite dimensional realization theory was using operators, in the finite dimensional case everything could be done using matrices. Thus writing down finite dimensional, state space realizations for inner functions became an object of study. Thus came the connection to the Lyapunov equation or to the related homogeneous Riccati equation. An early paper giving such realizations was Genin *et al.* (1983). A slight generalization of this is given in Fuhrmann and Ober (1993). In the Hardy space there is a natural metric arising from the naturally defined inner product. Thus orthogonality and orthogonal projections are naturally defined. It is clear from the state space representation that an inner function is determined by a controllable and stable pair (A, B), or rather by an equivalence class of such pairs under state space isomorphism. There is a state space based representation of coinvariant subspaces that is an offspring of a state space representation of a rational model space first derived by Hautus and Heymann (1978) (see also Fuhrmann 1994 for more details). A natural question arises as to how the Hardy space inner product induced in a coinvariant subspace is represented in terms of a state space realization. The solution, as seems well known, is that the positive definite solution of the corresponding Lyapunov equation is the Gram matrix of a natural basis of the corresponding coinvariant subspace. We will take this as a starting point to a more detailed study of the geometry of coinvariant subspaces. In particular we will arithmetize, in state space terms, the lattice operations, i.e. intersections and sums, in the set of finite dimensional coinvariant subspaces. In particular we will derive formulas for the orthogonal projections in terms of solutions to Lyapunov equations. This is taken up in § 2 which also contains a review of the relevant results about the shift realization, in particular in connection to the realization of inner functions. We believe that the material of this section may be of general interest.

In § 3 we derive state space formulas relevant to the approach to spectral factorization problems taken in Fuhrmann and Gombani (1998). The factorization theory in terms of rectangular spectral factors is considerably more complex than the standard theory, mainly through the presence of external spectral factors and the external–internal factorizations. The external part relates to the presence of non-trivial output nulling reachability subspaces in a minimal realization. These subspaces are trivial in the standard square case. We derive state space formulas not only for the set of all stable spectral factors but also for all related inner functions arising from the analysis in Fuhrmann and Gombani (1998). The formulas may look overwhelmingly complex to some but one can find consolation in the fact that the mathematical methods used are strong enough to deal with this complexity.

Finally, in § 4, we study, from the state space point of view, the partial ordering of the set of all minimal, stable spectral factors of a given spectral function.

Given a non-full rank spectral function Φ, we denote by W_-, W_+ the $p \times m_0$ minimum and maximum phase spectral factors respectively. The extended factors are the $p \times m$ factors that are obtained by adding an appropriate number of zero columns. Given any minimal stable spectral factor W of Φ, there exist, essentially unique, inner functions Q', Q'', of minimal McMillan degree, for which

$$\left.\begin{aligned} W &= W_-^e Q' \\ W_+^e &= W Q'' \end{aligned}\right\} \qquad (1)$$

The inner functions Q', Q'' are uniquely determined by the normalization $Q'(\infty) = Q''(\infty) = I$. Contrary to the standard case where $Q'Q'' = Q_+ = W_-^{-1} W_+$, we have in this case

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

where R is another inner function which can vary from factor to factor. There is however a constraint on the invariant inner factors of R. Since factorizations of inner functions relate to the solvability of Riccati equations, it is expected that the partial order introduced in the set of all spectral factors (see Anderson 1973) can be studied in state space terms, which in our exposition are focused on solutions of appropriate Lyapunov equations.

It is a pleasure for us to reiterate our indebtedness to the work of A. Lindquist and G. Picci which motivated and inspired our work in this area.

2. Coinvariant subspaces and the Lyapunov equation

Our setting will be that of vectorial Hardy spaces defined in the right half plane, in particular H_+^2 and H_+^∞. We will avoid in our notation making precise the range of values of the functions as this will always be clear from the context. We shall work mostly with row spaces, a choice dictated by reasons of compatibility with Fuhrmann and Gombani (1998). Since there is the product of an H_+^∞ function by an H_+^2 is in H_+^2, the Hardy space H_+^2 is a module over H_+^∞. We recall that a subspace $\mathcal{M} \subset H_+^2$ is called an **invariant subspace** if, for every $f \in \mathcal{M}$ and every scalar function $\phi \in H_+^\infty$, we have $\phi f \in \mathcal{M}$, i.e. if it is an H_+^∞ submodule. The principal result we use is the celebrated Beurling–Halmos–Lax representation theorem stating that $\mathcal{M}$ is an invariant subspace if and only if it has a representation of the form $\mathcal{M} = H^2_+ Q$ for some rigid function Q. Rigid functions are matrix valued functions, analytic in the right half plane whose boundary values on the imaginary axis are partial isometries with a fixed initial space. The case of interest for us is that of **inner functions**, which is distinguished by the requirement that the boundary values on the imaginary axis are unitary. We define a **coinvariant subspace** to be the orthogonal complement of an invariant subspace. We shall use the notation

$$H_r(Q) = \{H^2_+ Q\}^\perp \tag{2}$$

when dealing with *row* spaces and

$$H_c(Q) = \{Q H^2_+\}^\perp \tag{3}$$

for the standard column spaces. Rational inner functions have a simple characterization. $H_r(Q)$ is finite dimensional if and only if Q is inner and rational. For an algebraic approach to Beurling's theorem we refer to Fuhrmann (1994).

Our work is motivated by continuous time problems, so a rational inner function has always a unitary value at ∞. Since a finite dimensional coinvariant subspace determines the inner function Q only up to a right constant unitary matrix, we can use this freedom to normalize our inner functions by the requirement $Q(\infty) = I$.

As a proper, rational function a rational inner function Q has minimal realizations. The following result is standard. For generalizations, see Fuhrmann and Ober (1993).

Proposition 1: *An $m \times m$ rational matrix function Q is a normalized inner function of McMillan degree n if and only if it has a minimal realization of the form*

$$Q = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline -B^* P^{-1} & I \end{array}\right) \tag{4}$$

with A an $n \times n$ matrix and (A, B) a stable, controllable pair and with P the unique, positive definite solution of the Lyapunov equation

$$AP + PA^* + BB^* = 0 \tag{5}$$

Given a normalized inner function Q, it of course determines a unique coinvariant subspace $H_r(Q)$. We expect therefore to be able to describe elements of $H_r(Q)$ in terms of a given realization of Q. Actually we can do more. The subspace $H_r(Q)$ inherits the inner product, or metric, defined in H^2_+. This metric can be lifted to $\mathbb{C}^n_r$ and the tool to do it is the solution to the Lyapunov equation. Indeed, the next proposition gives a state space representation of finite dimensional, backward invariant subspaces in state space terms. This is an analogue, in fact it is a direct consequence, of a result of Hautus and Heymann (1978) and Wimmer (1979). The connection is obvious, since we have (see Fuhrmann 1994, 1995) that a rational model space, corresponding to a stable, non-singular polynomial matrix E, is at the same time a coinvariant subspace of H^2_+. In fact we have

$$X^E_r = H_r(K) = \{H^2_+ K\}^\perp \tag{6}$$

where $K = \bar{E} E^{-1}$ and $\bar{E}$ is the antistable solution of the spectral factorization problem $\bar{E}^* \bar{E} = E^* E$.

Let (ξ, η) be the standard inner product in $\mathbb{C}^n$. An $n \times n$ positive definite matrix P induces a new inner product in $\mathbb{C}^n$, defined by

$$\langle \xi, \eta \rangle = (P^{-1} \xi, \eta) \tag{7}$$

We will, when needed for extra clarity, denote by $(\mathbb{C}^n, P^{-1})$ the space $\mathbb{C}^n$ with the P^{-1}-induced inner product. Also, we denote by A^* the standard adjoint of A, whereas by $A^{\langle * \rangle}$ the adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$. A simple computation shows that the two adjoints are related by

$$A^{\langle * \rangle} = P A^* P^{-1} \tag{8}$$

We shall see, in Proposition 2, that the P^{-1}-induced metric in $\mathbb{C}^n$ is the lifting of the H^2_+-induced metric in the coinvariant subspace $H_r(Q)$.

We present next, for the special case of inner functions, some aspects of the realization theory based on shift operators and translation semigroups. This theory has been developed over the years by one of the authors (see in particular Fuhrmann 1976, 1977). We pay special attention to the duality theory for these realizations. This theory is of importance, as it has a direct application to the derivation of several results in the geometric control theory placed in the Hardy space context (see Fuhrmann 1994, 1998). Much of the duality theory is based on a simple unitary map defined in $L^2_r(i\mathbb{R})$. In this connection, see also Fuhrmann (1994) for other applications.

To begin, we recall the shift realization associated with a rational function G having the representation

$$G = U T^{-1} V + W \tag{9}$$

where T, U, V, W are appropriately sized polynomial matrices with T non-singular. Each representation of the form (9), is a basis for a state space realization defined in the state space X^c_T, by

$$\left.\begin{aligned} \mathbf{A}_f &= S_T^c f = \pi_T^c(sf) \\ \mathbf{B}\xi &= V\pi_T\xi \\ \mathbf{C}f &= (UT^{-1}f)_{-1} \\ \mathbf{D} &= G(\infty) \end{aligned}\right\} \tag{10}$$

Here π_T^c is the projection operator defined on polynomial vectors by $\pi_T^c f = T\pi_- T^{-1} f$. For these maps we have

$$G = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right)$$

This realization is reachable if and only if T and U are right coprime and observable if and only if T and V are left coprime.

Instead of working in the frequency domain, we could just as well start in the time domain with a $p \times m$ matrix function $\Gamma(t)$ for which its Laplace transform G is in H_+^2. This means of course that the entries of g are exponential polynomials with all exponents having negative real part, i.e. G is stable. A time domain minimal realization of Γ is obtained by taking the state space $\mathcal{M}$ to be the smallest left translation invariant subspace of $L^2(0,\infty)$ that contains all columns of Γ. The realization is defined by

$$\left.\begin{aligned} (\mathrm{e}^{\mathcal{A}t}f)(x) &= f(x+t), \qquad x \geq 0,\ t \geq 0 \\ \mathcal{B}\xi &= \Gamma(t)\xi \\ \mathcal{C}f &= f(0) \end{aligned}\right\} \tag{11}$$

Note that the infinitesimal generator $\mathcal{A}$ of the left translation semigroup is given by $\mathcal{A}f = f'$. Restricted to our finite dimensional space $\mathcal{M}$, this operator is actually bounded. Thus the realization can be written as

$$\left.\begin{aligned} \mathcal{A}f &= f' \\ \mathcal{B}\xi &= \Gamma(t)\xi \\ \mathcal{C}f &= f(0) \end{aligned}\right\} \tag{12}$$

Note, that by Laplace transform theory, we have, with $\mathcal{L}f = F$, that

$$\mathcal{L}f' = sF - f(0)$$

Also we have

$$f(0) = \lim_{s\to\infty} sF(s)$$

It is easy to check that the Laplace transform, at least for the case that $G = T^{-1}U$, of the translation realization yields the corresponding shift realization.

At this point we would like to make a general remark on the reasons for using row spaces in the rest of the paper. It is well known that given a left coprime factorization $G = T^{-1}U$ of a proper rational transfer function and a minimal realization

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

then the associated rational model space is given by $X^D = \{C(sI - A)^{-1}\xi \mid \xi \in F^n\}$ and its elements are column vectors of rational functions. This approach leads to restricted shift realizations described above. One of the appealing features of these models is that, given a transfer function W, we can define, in a Hilbert space of analytic functions, operators $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ such that $W(s) = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$, i.e. $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ is a realization of W. This definition does not require rationality of W. If, however, W is rational, the rational model X^D is a finite dimensional subspace. Therefore, if we choose a basis in X^D, the operators $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ will have matrix representations A,B,C,D and

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is a realization of W in the usual sense. As we said, however, the basis associated to this realization is $C(sI - A)^{-1}$. Now in the analysis of the set of minimal, stable spectral factors of a given spectral function Φ, undertaken in Fuhrmann and Gombani (1998), having fixed W_-, W_+ the minimum and maximum phasestable spectral factor and a minimal realization

$$W_- = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

every minimal, stable spectral factor W has a minimal realization of the form

$$W = \left(\begin{array}{c|c} A & B_W \\ \hline C & D_W \end{array}\right)$$

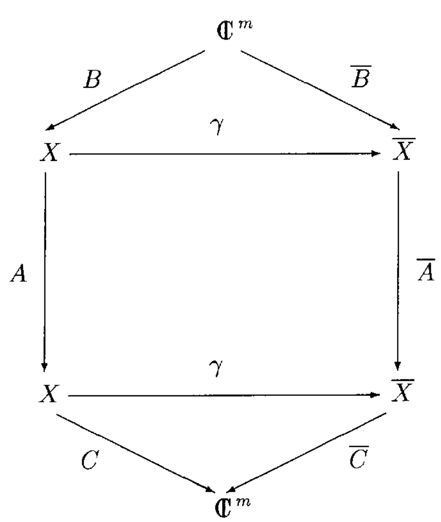
Thus all these factors have the same column model space. Therefore, if we want to study spectral factors with *column inputs* geometrically via their associated model state spaces, we are forced to the use of the (A, B_W) pairs, that is basis of the form $(sI - A)^{-1}B_W$. As a result, we naturally have to consider *row* Hardy spaces. The main problem with this approach is that, as will be seen in Proposition 2, the matrix representation of $\mathbf{A}$ in the basis $(sI - A)^{-1}B_W$ is equivalent to A^*; since the operator $\mathbf{A}$, in general, is not selfadjoint, there is no hope to get $(sI - A)^{-1}B_W$ to be a basis for the usual restricted shift realization; it will therefore turn out that we should not consider the usual functional model associated to G to work with that basis, but a slight variation of it which makes use of row vectors. This will be seen in detail in Proposition 2.

Therefore, in what follows, matrices A, B, C will act as usual on column vectors $\xi \in \mathbb{C}^n, \eta \in \mathbb{C}^m$, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ will act on column coinvariant spaces, whereas we use overbars, i.e. $\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}$ to denote operators acting on row coinvariant spaces. To this end we will need to use a slightly modified functional model which will be described next.

In the next proposition we collect all the information on the various realizations of an inner function K. In particular, we relate the Hilbert space geometry of the coinvariant subspace $H_r(K)$ to the geometry of $\mathbb{C}^n$, endowed with the metric defined by (7), with P the solution of the Lyapunov equation (5). The use of row spaces introduces a new complexity. Since we use column input and output spaces, it follows that in the shift realizations, acting in row spaces, the input and output operators are both antilinear maps. While the Markov parameters $\bar{C}\bar{A}^{i-1}\bar{B}$, $i \geq 1$ are all linear maps the transfer function, defined by $\bar{C}(sI - \bar{A})^{-1}\bar{B}$, is no longer a rational analytic function but rather an antianalytic one as it is a function of $\bar{s}$. To rectify this problematic point we can either live with the fact that $\bar{C}(sI - \bar{A})^{-1}\bar{B}$ is antianalytic or define a triple $(\bar{A}, \bar{B}, \bar{C})$ to be an **antirealization** of $G(s)$ if $G(s) = \bar{C}(\bar{s}I - \bar{A})^{-1}\bar{B}$.

However, before we pass on to the full analysis of the various models, we prove the next lemma which relates realizations and antirealizations and is in a sense a state space anti-isomorphism theorem.

Lemma 1: *Let (A, B, C) be a minimal realization of $G(s)$ in the state space X. Let $\gamma : X \to \bar{X}$ be an antilinear isomorphism, i.e. a bijective antilinear map and let $(\bar{A}, \bar{B}, \bar{C})$ be defined via the commutativity of the diagram*



Then $(\bar{A}, \bar{B}, \bar{C})$ is an antirealization of $G(s) = C(sI - A)^{-1}B$.

Proof: We compute, using the antilinearity of γ

$$G(s) = C(sI - A)^{-1}B = \bar{C}\gamma(sI - A)^{-1}B = \bar{C}(\bar{s}I - \bar{A})^{-1}\gamma B = \bar{C}(\bar{s}I - \bar{A})^{-1}\bar{B} \qquad \square$$

Proposition 2: *Let K be a $m \times m$ rational, normalized inner function of degree n and let*

$$K = \left(\begin{array}{c|c} A & B \\ \hline C & I \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right) \tag{13}$$

be a minimal realization, where P is the solution of the Lyapunov equation

$$AP + PA^* + BB^* = 0 \tag{14}$$

Then

(1) *A representation of $H_r(K)$ is given by*

$$H_r(K) = \{\xi^* P^{-1}(sI - A)^{-1}B \mid \xi \in \mathbb{C}^n\} \tag{15}$$

(2) *Let $e_i, i = 1, \ldots, n$ be the standard unit vectors of $\mathbb{C}^n$.*

$$\mathcal{B} = \{v_i = e_i^* P^{-1}(sI - A)^{-1}B \mid i = 1, \ldots, n\} \tag{16}$$

is a basis of $H_r(K)$. Moreover, for the solution P of (5), *we have*

$$[P^{-1}]_{ij} = (v_i, v_j)_{H_+^2} \tag{17}$$

i.e. P^{-1} is the Gram matrix of the basis $\mathcal{B}$.

(3) *The map $J_K : (\mathbb{C}^n, P^{-1}) \to H_r(K)$ defined by*

$$J_K\xi = \xi^* P^{-1}(sI - A)^{-1}B \tag{18}$$

is a unitary map.

(4) *Let $\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K$ be defined, for $f \in H_c(K)$ and $\eta \in \mathbb{C}^p$, by*

$$\left.\begin{aligned} \mathbf{A}_K f &= sf(s) - \lim_{s\to\infty} sf(s) \\ \mathbf{B}_K\eta &= (K(s) - I)\eta \\ \mathbf{C}_K f &= \left[\lim_{s\to\infty} sf(s)\right] \end{aligned}\right\} \tag{19}$$

Then

(a)

$$K = \left(\begin{array}{c|c} \mathbf{A}_K & \mathbf{B}_K \\ \hline \mathbf{C}_K & I \end{array}\right) \tag{20}$$

is a minimal realization of K.

(b) *The realization in* (19) *is balanced and its controllability and observability gramians are both equal to I.*

(c) *The Hilbert space adjoints of the maps in* (23) *are given by*

$$\left.\begin{aligned} \mathbf{A}_K^* f &= -sf(s) + K(s)\Big[\lim_{s\to\infty} sf(s)\Big] \\ \mathbf{B}_K^* f &= -\Big[\lim_{s\to\infty} sf(s)\Big] \\ \mathbf{C}_K^* \eta &= (I - K(s))\eta \end{aligned}\right\} \tag{21}$$

and a minimal realization of K^* *is given by*

$$K^* = \left(\begin{array}{c|c} -\mathbf{A}_K^* & \mathbf{C}_K^* \\ \hline -\mathbf{B}_K^* & I \end{array}\right) \tag{22}$$

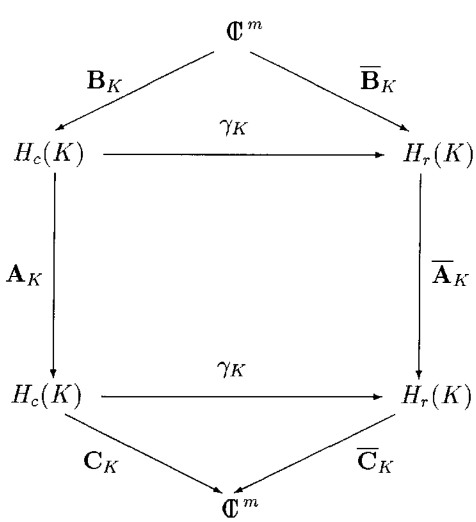
(5) *Let* $\bar{\mathbf{A}}_K, \bar{\mathbf{B}}_K, \bar{\mathbf{C}}_K$ *be defined, for* $f \in H_r(K)$ *and* $\eta \in \mathbb{C}^m$, *by*

$$\left.\begin{aligned} \bar{\mathbf{A}}_K f &= -sf(s) + \Big[\lim_{s\to\infty} sf(s)\Big] \cdot K(s) \\ \bar{\mathbf{B}}_K \eta &= \eta^*(I - K(s)) \\ \bar{\mathbf{C}}_K f &= -\Big[\lim_{s\to\infty} sf(s)\Big]^* \end{aligned}\right\} \tag{23}$$

We define the map $\gamma_K : H_c(K) \to H_r(K)$ *by*

$$\gamma_K g(s) = g^*(s)K(s) = g(-\bar{s})^* K(s) \tag{24}$$

then γ_K *is an antiunitary map, the following diagram is commutative*



and

$$\left(\begin{array}{c|c} \bar{\mathbf{A}}_K & \bar{\mathbf{B}}_K \\ \hline \bar{\mathbf{C}}_K & I \end{array}\right) \tag{25}$$

is a minimal antirealization of K.

(6) *Defining* $\tilde{K}(s) = K(\bar{s})^*$, *then* $\tilde{K}$ *is inner and we have* $K^*(s) = \tilde{K}(-s)$ *as well as the minimal antirealization*

$$K^*(s) = \left(\begin{array}{c|c} -\bar{\mathbf{A}}_{\tilde{K}} & \bar{\mathbf{B}}_{\tilde{K}} \\ \hline -\bar{\mathbf{C}}_{\tilde{K}} & I \end{array}\right) \tag{26}$$

(7) *The Hilbert space adjoints of the maps in* (23) *are given by*

$$\left.\begin{aligned} \bar{\mathbf{A}}_K^* f &= sf(s) - \Big[\lim_{s\to\infty} sf(s)\Big] \\ \bar{\mathbf{B}}_K^* f &= \Big[\lim_{s\to\infty} sf(s)\Big]^* \\ \bar{\mathbf{C}}_K^* \eta &= \eta^*(K(s) - I) \end{aligned}\right\} \tag{27}$$

and hence we have the realization

$$K^* = \left(\begin{array}{c|c} -\bar{\mathbf{A}}_K^* & \bar{\mathbf{C}}_K^* \\ \hline -\bar{\mathbf{B}}_K^* & I \end{array}\right) \tag{28}$$

(8) *Define in* $L_r^2(i\mathbb{R})$ *a map* J *by*

$$(Jf)(it) = f(-it) \tag{29}$$

Then

(a) J *is a unitary map satisfying* $J = J^* = J^{-1}$.

(b) *For an inner function* $K \in H_+^\infty$, *we set* $\tilde{K}(s) = K(\bar{s})^*$ *and* $K^*(s) = K(-\bar{s})^*$. *Then* $\tilde{K}$ *is inner in* H_+^∞ *and* K^* *is inner in* H_-^∞. *Moreover, we have*

$$K(s)K^*(s) = K(s)\tilde{K}(-s) = I \tag{30}$$

(c) *The restriction of* J *to* $H_r(\tilde{K})$ *is a unitary map onto* $H_r(K^*)$.

(d) *Define the map* $\tau_K : L_r^2(i\mathbb{R}) \to L_r^2(i\mathbb{R})$ *by*

$$(\tau_K f)(s) = (Jf)(s)\tilde{K}(s) = f(-s)\tilde{K}(s) \tag{31}$$

Then τ_K *is a unitary map and we have*

$$\left.\begin{aligned} \tau_K H_+^2 K &= H_-^2 \\ \tau_K H_r(K) &= H_r(\tilde{K}) \\ \tau_K H_-^2 &= H_+^2 \tilde{K} \end{aligned}\right\} \tag{32}$$

In particular $\tau_K : H_r(K) \to H_r(\tilde{K})$ *is a unitary map.*

(9) *For a normalized inner function* $K \in H_+^\infty$, *the function* $K^*(s) = K(-\bar{s})^*$ *is normalized inner in* H_-^∞. *Defining, in the state space* $H_r(K^*) = H_-^2 \ominus H_-^2 K^*$

$$\left.\begin{aligned} \bar{\mathbf{A}}_{K^*} h &= -sh(s) + \Big[\lim_{s\to\infty} sh(s)\Big] K^*(s) \\ \bar{\mathbf{B}}_{K^*} \eta &= \eta^*(I - K^*(s)) \\ \bar{\mathbf{C}}_{K^*} h &= -\Big[\lim_{s\to\infty} sh(s)\Big]^* \end{aligned}\right\} \tag{33}$$

then

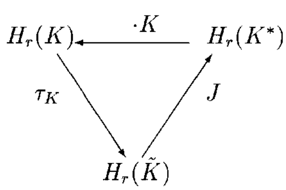
$$K^* = \left(\begin{array}{c|c} \bar{\mathbf{A}}_{K^*} & \bar{\mathbf{B}}_{K^*} \\ \hline \bar{\mathbf{C}}_{K^*} & I \end{array}\right) \tag{34}$$

is a minimal antirealization.

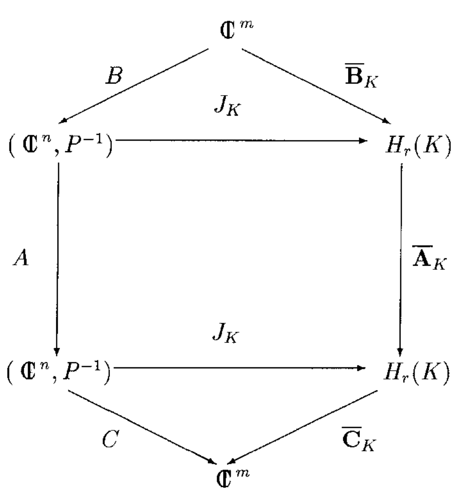
(10) *The antirealizations*

$$K^* = \left(\begin{array}{c|c} \bar{\mathbf{A}}_{K^*} & \bar{\mathbf{B}}_{K^*} \\ \hline \bar{\mathbf{C}}_{K^*} & I \end{array}\right) = \left(\begin{array}{c|c} -\bar{\mathbf{A}}_K^* & \bar{\mathbf{C}}_K^* \\ \hline -\bar{\mathbf{B}}_K^* & I \end{array}\right)$$
$$= \left(\begin{array}{c|c} -\bar{\mathbf{A}}_{\tilde{K}} & \bar{\mathbf{B}}_{\tilde{K}} \\ \hline -\bar{\mathbf{C}}_{\tilde{K}} & I \end{array}\right) \tag{35}$$

given in the state spaces $H_r(K^*)$, $H_r(K)$ and $H_r(\tilde{K})$ respectively are isomorphic and the isomorphisms are given by the unitary maps as in the following diagram.



(11) *Let A, B, C be as in* (13). *Then the following diagram is commutative.*



i.e.

$$\left.\begin{aligned} J_K A &= \bar{\mathbf{A}}_K J_K \\ J_K B &= \bar{\mathbf{B}}_K \\ C &= \bar{\mathbf{C}}_K J_K \end{aligned}\right\} \tag{36}$$

(12) *For the adjoints of the maps A, B, C with respect to the P^{-1}-inner product defined in* (7) *we have*

$$\left.\begin{aligned} A^{\langle * \rangle} &= PA^*P^{-1} \\ B^{\langle * \rangle} &= B^*P^{-1} \\ C^{\langle * \rangle} &= PC^* \end{aligned}\right\} \tag{37}$$

(13) We have

$$\left.\begin{aligned} J_K A^{\langle * \rangle} &= \bar{\mathbf{A}}_K^* \\ B^{\langle * \rangle} &= \bar{\mathbf{B}}_K^* J_K \\ J_K C^{\langle * \rangle} &= \bar{\mathbf{C}}_K^* \end{aligned}\right\} \tag{38}$$

Proof:

(1) Let $P^{-1}(sI - A)^{-1}B = H(s)E(s)^{-1}$, where the polynomial matrices H and E are right coprime. By the above-mentioned result of Heymann and Hautus and of Wimmer, the rows of H are a basis of the polynomial model X_E^r and hence the rows of HE^{-1}, and therefore also of $P^{-1}(sI - A)^{-1}B$, are a basis of the rational model space X_r^E. But, by (6), we have $X_r^E = H_r(K)$, where $K = \bar{E}E^{-1}$ is the inner function determined by E.

(2) The solution of the Lyapunov equation (14) is known to be given by

$$P = \int_0^\infty \mathrm{e}^{At} BB^* \mathrm{e}^{A^*t}\, \mathrm{d}t$$

and hence

$$[P^{-1}]_{ij} = e_i^* P^{-1} P P^{-1} e_j = \int_0^\infty e_i^* P^{-1} \mathrm{e}^{At} BB^* \mathrm{e}^{A^*t} P^{-1} e_j \, \mathrm{d}t$$
$$= (\hat{v}_i, \hat{v}_j)_{L^2(0,\infty)}$$

where $\hat{v}_i = e_i^* B \mathrm{e}^{At} \in L^2(0,\infty)$. Since the Fourier–Plancherel transform maps, unitarily, $L^2(0,\infty)$ onto H_+^2, and maps $\hat{v}_i = e_i^* \mathrm{e}^{At} B$ to $v_i = e_i^*(sI - A)^{-1}B$, we get (17).

(3) For the translation realization (11), the map $\xi^* P^{-1} \mathrm{e}^{At} B \mapsto \xi^* P^{-1}(sI - A)^{-1}B$ is the restriction of the Fourier–Plancherel map to the state space. Since it is unitary, we can compute

$$
\begin{aligned}
(J_K\xi, J_K\xi)_{H_r(K)} &= \left(\xi^* P^{-1}(sI-A)^{-1}B, \xi^* P^{-1} \times (sI-A)^{-1}B\right)_{H_+^2} \\
&= \left(\xi^* P^{-1} e^{At}B, \xi^* P^{-1} e^{At}B\right)_{L^2(0,\infty)} \\
&= \int_0^\infty \left(\xi^* P^{-1} e^{At}B, \xi^* P^{-1} e^{At}B\right) dt \\
&= \int_0^\infty \xi^* P^{-1} e^{At}BB^* e^{A^*t}P^{-1}\xi \, dt \\
&= \xi^* P^{-1} P P^{-1}\xi = \xi^* P^{-1}\xi = \langle \xi, \xi \rangle
\end{aligned}
$$

(4) (a) This is a special case of the shift realization.
(b) Since K is inner, we have

$$
\begin{aligned}
K^{-1} &= \left(\begin{array}{c|c} \mathbf{A}_K - \mathbf{B}_K\mathbf{C}_K & \mathbf{B}_K \\ \hline -\mathbf{C}_K & I \end{array}\right) \\
&= \left(\begin{array}{c|c} -\mathbf{A}_K^* & -\mathbf{C}_K^* \\ \hline \mathbf{B}_K^* & I \end{array}\right) = K^*
\end{aligned} \tag{39}
$$

Since the two realizations are both minimal, they are necessarily isomorphic, the isomorphism X is self adjoint and satisfies $(\mathbf{A}_K - \mathbf{B}_K\mathbf{C}_K)X = -X\mathbf{A}_K^*$ and $X\mathbf{B}_K = -\mathbf{C}_K^*$. Thus X is a solution of the Lyapunov equation

$$\mathbf{A}_K X + X\mathbf{A}_K^* + \mathbf{B}_K\mathbf{B}_K^* = 0$$

and hence is given by $X = \int_0^\infty e^{\mathbf{A}_K t}\mathbf{B}_K\mathbf{B}_K^* e^{\mathbf{A}_K^* t}\, dt$. We work now in the time domain. For $f \in \mathcal{F}^{-1}(H_c(K))$, we have

$$
\begin{aligned}
(Xf, f) &= \int_0^\infty \left(e^{\mathbf{A}_K t}\mathbf{B}_K\mathbf{B}_K^* e^{\mathbf{A}_K^* t} f, f\right) dt \\
&= \int_0^\infty \|\mathbf{B}_K^* e^{\mathbf{A}_K^* t} f\|^2 \, dt
\end{aligned}
$$

Now, in terms of the translation semigroup, we have $(e^{At}f)(\tau) = f(t+\tau) = f_t(\tau)$ and hence $f(t+\tau)B = f_t(0) = f(t)$. So $(Xf, f) = \int_0^\infty \|f(t)\|^2\, dt = \|f\|^2$, i.e. $X = I$. In particular, we have $\mathbf{A}_K + \mathbf{A}_K^* + \mathbf{B}_K\mathbf{B}_K^* = 0$. Using the identity $-\mathbf{B}_K = X\mathbf{C}_K^* = \mathbf{C}_K^*$, we get also $\mathbf{A}_K + \mathbf{A}_K^* + \mathbf{C}_K\mathbf{C}_K^* = 0$.
(c) Since $X = I$, we have $\mathbf{A}_K^* = -(\mathbf{A}_K - \mathbf{B}_K\mathbf{C}_K)$, $\mathbf{B}_K^* = -\mathbf{C}_K$ and $\mathbf{C}_K^* = -\mathbf{B}_K$. The realization follows from (39).

(5) Clearly $g \in H_c(K)$ if and only if $g^*K \in H_r(K)$. Then we can compute

$$
\begin{aligned}
\gamma_K \mathbf{B}_K \eta &= [(K(s) - I)\eta]^* K(s) \\
&= \eta^*(K(s)^* - I)K(s) \\
&= \eta^*(I - K(s)) = \bar{\mathbf{B}}_K \eta
\end{aligned}
$$

similarly, remembering that we normalize K so that $K(\infty) = I$

$$
\begin{aligned}
\gamma_K \mathbf{A}_K g &= \gamma_K\left(sg(s) - \lim_{s\to\infty} sg(s)\right) \\
&= \left(sg(s) - \lim_{s\to\infty} sg(s)\right)^* K(s) \\
&= \left(-sg^*(s) + \lim_{s\to\infty} sg^*(s)\right)K(s) \\
&= -sg^*(s)K(s) + \left(\lim_{s\to\infty} sg^*(s)\right)K(s) \\
&= -sg^*(s)K(s) + \left(\lim_{s\to\infty} sg^*(s)K^*(s)\right)K(s) \\
&= \bar{\mathbf{A}}_K \gamma_K g
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\mathbf{C}}_K \gamma_K g &= \left(\lim_{s\to\infty} sg^*(s)K(s)\right)^* \\
&= \left[-\lim_{s\to\infty} sg(s)\right] = \mathbf{C}_K g
\end{aligned}
$$

which achieves the proof.

That

$$\left(\begin{array}{c|c} \bar{\mathbf{A}}_K & \bar{\mathbf{B}}_K \\ \hline \bar{\mathbf{C}}_K & I \end{array}\right)$$

is a minimal antirealization of K follows by applying Lemma 1.

(6) We have $\tilde{K}(s) = I + C_{\tilde{K}}(sI - A_{\tilde{K}})^{-1}B_{\tilde{K}}$, so

$$
\begin{aligned}
K^*(s) = \tilde{K}(-s) &= I + C_{\tilde{K}}(-sI - A_{\tilde{K}})^{-1}B_{\tilde{K}} \\
&= I - C_{\tilde{K}}(sI + A_{\tilde{K}})^{-1}B_{\tilde{K}}
\end{aligned} \tag{40}
$$

which is equivalent to (26).

(7) It follows from part 4 by duality.

(8) (a) Clearly, $J^2 = I$, i.e. $J = J^{-1}$. For $f, g \in L_r^2(i\mathbb{R})$, we compute, using a simple change of variable,

$$
\begin{aligned}
(fJ, g) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (f(-it), g(it))\, dt \\
&= -\frac{1}{2\pi}\int_{\infty}^{-\infty} (f(is), g(-is))\, ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (f(is), g(-is))\, ds = (f, gJ)
\end{aligned} \tag{41}
$$

i.e. $J^* = J$.

(b) Since K is inner, we have $K(it)^*K(it) = I$. Now $K(it)^*$ are the boundary values of $K^*(s) = \tilde{K}(-s)$ which is inner in H^∞_-. Thus we have $\tilde{K}(-s)K(s) = I$ and also $K(-s)\tilde{K}(s) = I$.

(c) We consider the orthogonal direct sum decomposition

$$L^2_r(i\mathbb{R}) = H^2_+\tilde{K} \oplus H_r(\tilde{K}) \oplus H^2_-$$

Clearly, $JH^2_- = H^2_+$ and $J(H^2_+\tilde{K}) = H^2_-K^*$. Since we also have

$$L^2_r(i\mathbb{R}) = H^2_+ \oplus H_r(K^*) \oplus H^2_-K^*$$

we necessarily have $JH_r(\tilde{K}) = H_r(K^*)$.

(d) Since $\tilde{K}$ is inner, multiplication by $\tilde{K}$ is a unitary map in $L^2_r(i\mathbb{R})$. This shows that τ_K, as the product of two unitary maps, is also unitary.

Now we consider the orthogonal direct sum decomposition

$$L^2_r(i\mathbb{R}) = H^2_+K \oplus H_r(K) \oplus H^2_-$$

Under τ_K we obviously have $\tau_K H^2_- = H^2_+\tilde{K}$ and $\tau_K H^2_+K = H^2_-K(-s)\tilde{K}(s) = H^2_-$. Since we have also

$$L^2_r(i\mathbb{R}) = H^2_+\tilde{K} \oplus H_r(\tilde{K}) \oplus H^2_-$$

we must have $\tau_K H_r(K) = H_r(\tilde{K})$.

(9) This is the modification of part (5) to the case of inner functions in H^∞_-.

(10) We begin by showing the isomorphism of the systems defined in the state spaces $H_r(K^*)$ and $H_r(K)$. We compute, for $h \in H_r(K^*)$, $\eta \in \mathbb{C}^m$

$$\bar{\mathbf{B}}_{K^*}\eta = \eta^*(I - K^*(s))K(s) = \eta^*(K(s) - I) = \bar{\mathbf{C}}^*_K\eta$$

Next, for $h \in H_r(K^*)$

$$\begin{aligned}(\bar{\mathbf{A}}_{K^*}h)K &= \left[-sh(s) + \left(\lim_{s\to\infty} sh(s)\right)K^*(s)\right]K(s)\\ &= -\left[sh(s)K(s) - \lim_{s\to\infty} sh(s)\right]\\ &= -\left[s(h(s)K(s)) - \lim_{s\to\infty} s(h(s)K(s))\right]\\ &= -\bar{\mathbf{A}}^*_K(hK)\end{aligned}$$

and

$$\bar{\mathbf{B}}^*_K(hK) = \left[\lim_{s\to\infty} sh(s)K(s)\right]^* = \left[\lim_{s\to\infty} sh(s)\right]^* = -\bar{\mathbf{C}}_{K^*}h$$

Next, we show the isomorphism of the systems defined in the state spaces $H_r(K)$ and $H_r(\tilde{K})$. Next, for $f \in H_r(K)$ and $\eta \in \mathbb{C}^m$, we compute

$$\begin{aligned}\tau_K\bar{\mathbf{C}}^*_K\eta &= J\eta^*(K(s) - I)\tilde{K}(s) = \eta^*(K(-s) - I)\tilde{K}(s)\\ &= \eta^*(I - \tilde{K}(s)) = \bar{\mathbf{B}}_{\tilde{K}}\eta\end{aligned}$$

$$\begin{aligned}\tau_K\bar{\mathbf{A}}^*_K f &= J\left[sf(s) - \lim_{s\to\infty} sf(s)\right]\tilde{K}(s)\\ &= -\left[sf(-s) + \left(-\lim_{s\to\infty} sf(-s)\right)\right]\tilde{K}(s)\\ &= -sf(-s)\tilde{K}(s) + \left(\lim_{s\to\infty} sf(-s)\tilde{K}(s)\right)\tilde{K}(s)\\ &= -s\tau_K(f)(s) + \left(\lim_{s\to\infty} s\tau_K(f)(s)\right)\tilde{K}(s)\\ &= \bar{\mathbf{A}}_{\tilde{K}}\tau_K f\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{C}}_{\tilde{K}}\tau_K f &= -\left[\lim_{s\to\infty} sf(-s)\tilde{K}(s)\right]^* = \lim_{s\to\infty}(-sf(-s))\\ &= \lim_{s\to\infty}(sf(s)) = \bar{\mathbf{B}}^*_K f\end{aligned}$$

Finally, for $g \in H_r(\tilde{K})$ and $\eta \in \mathbb{C}^m$, we compute

$$\begin{aligned}J\bar{\mathbf{B}}_{\tilde{K}}\eta &= J\eta^*(I - \tilde{K}(s)) = \eta^*(I - \tilde{K}(-s))\\ &= \eta(I - K^*(s)) = \bar{\mathbf{B}}_{K^*}\eta\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{A}}_{K^*}(Jg) &= sg(-s) + \lim_{s\to\infty} sg(-s)K^*(s)\\ &= J\left[sg(s) - \left(\lim_{s\to\infty} sg(s)\right)\tilde{K}(s)\right]\\ &= -J\bar{\mathbf{A}}_{\tilde{K}}g\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{C}}_{K^*}Jg &= -\left[\lim_{s\to\infty} sg(-s)\right]^* = \left[\lim_{s\to\infty}(-sg(-s))\right]^*\\ &= \left[\lim_{s\to\infty}(sg(s))\right]^* = -\bar{\mathbf{C}}_{\tilde{K}}g\end{aligned}$$

(11) Let $\xi \in \mathbb{C}^n$. We compute, using the fact that $K(s) = I - B^*P^{-1}(sI - A)^{-1}B$ and remembering that, from the Lyapunov equation (14), we get $P^{-1}A + A^*P^{-1} = -P^{-1}BB^*P^{-1}$.

$$\begin{aligned}\bar{\mathbf{A}}_K(J_K\xi) &= \bar{\mathbf{A}}_K\xi^*P^{-1}(sI - A)^{-1}B\\ &= -s\xi^*P^{-1}(sI - A)^{-1}B\\ &\quad + \left[\lim_{s\to\infty} s\xi^*P^{-1}(sI - A)^{-1}B\right]\cdot K(s)\\ &= -s\xi^*P^{-1}(sI - A)^{-1}B + \xi^*P^{-1}\\ &\quad \times B[I - B^*P^{-1}(sI - A)^{-1}B]\end{aligned}$$

$$= -s\xi^* P^{-1}(sI - A)^{-1}B + \xi^* P^{-1}B$$
$$- \xi^* P^{-1}BB^* P^{-1}(sI - A)^{-1}B$$
$$= -s\xi^* P^{-1}(sI - A)^{-1}B + \xi^* P^{-1}B$$
$$+ \xi^*[P^{-1}A + A^* P^{-1}](sI - A)^{-1}B$$
$$= -\xi^* P^{-1}(sI - A)(sI - A)^{-1}B + \xi^* P^{-1}B$$
$$+ \xi^* A^* P^{-1}(sI - A)^{-1}B$$
$$= \xi^* A^* P^{-1}(sI - A)^{-1}B$$
$$= J_K(A\xi))$$

Similarly

$$J_K\bar{\mathbf{C}}_K\xi = -\Big[\lim_{s\to\infty} s\xi^* P^{-1}(sI - A)^{-1}B\Big]^* \tag{42}$$
$$= -\Big[\lim_{s\to\infty} \xi^* P^{-1}(sI - A + A)(sI - A)^{-1}B\Big]^* \tag{43}$$
$$= -[\xi^* P^{-1}B]^* = -B^* P^{-1}\xi = C\xi \tag{44}$$

and

$$J_K(B\eta) = \eta^* B^* P^{-1}(sI - A)^{-1}B = \eta^*(I - K(s)) = \bar{\mathbf{B}}_K\eta$$

(12) Let $\xi, \eta \in \mathbb{C}^n$. We compute

$$\langle A\xi, \eta\rangle = (P^{-1}A\xi, \eta) = (\xi, A^* P^{-1}\eta)$$
$$= (PP^{-1}\xi, A^* P^{-1}\eta) = (P^{-1}\xi, PA^* P^{-1}\eta)$$
$$= \langle \xi, A^{\langle * \rangle}\eta\rangle$$

that is $A^{\langle * \rangle} = PA^* P^{-1}$. Similarly, for $\xi \in \mathbb{C}^n$ and $\eta \in \mathbb{C}^m$

$$\langle \xi, B\eta\rangle = (P^{-1}\xi, B\eta) = (B^* P^{-1}\xi, \eta) = (B^{\langle * \rangle}\xi, \eta)$$

This implies that $B^{\langle * \rangle} = B^* P^{-1}$. Finally, for $\xi \in \mathbb{C}^n$ and $\eta \in \mathbb{C}^p$,

$$(C\xi, \eta) = (\xi, C^*\eta) = \langle \xi, PC^*\eta\rangle = \langle \xi, C^{\langle * \rangle}\eta\rangle$$

Thus $C^{\langle * \rangle} = PC^*$.

(13) Since J_K is a unitary map, the equalities in (36) clearly imply those in (38). It is however of interest to derive these duality relations directly using the adjoints computed in (27).

$$\bar{\mathbf{A}}_K^* J_K\xi = \bar{\mathbf{A}}_K^*\xi^* P^{-1}(sI - A)^{-1}B$$
$$= s\xi^* P^{-1}(sI - A)^{-1}B - \lim_{s\to\infty} s\xi^* P^{-1}(sI - A)^{-1}B$$
$$= \xi^* P^{-1}(sI - A + A)(sI - A)^{-1}B - \xi^* P^{-1}B$$
$$= \xi^* P^{-1}(sI - A)(sI - A)^{-1}B$$
$$+ \xi^* P^{-1}A(sI - A)^{-1}B\xi^* P^{-1}B$$
$$= \xi^* P^{-1}A(sI - A)^{-1}B$$
$$= \xi^* P^{-1}APP^{-1}(sI - A)^{-1}B$$
$$= J_K(PA^* P^{-1}\xi) = J_K(A^{\langle * \rangle}\xi)$$

Similarly, we compute

$$\bar{\mathbf{B}}_K^* J_K\xi = \bar{\mathbf{B}}_K^*\xi^* P^{-1}(sI - A)^{-1}B$$
$$= \Big[\lim_{s\to\infty} s\xi^* P^{-1}(sI - A)^{-1}B\Big]^*$$
$$= [\xi^* P^{-1}B]^* = B^{\langle * \rangle}$$

Finally,

$$\bar{\mathbf{C}}_K^*\eta = \eta^*(I - K(s)) = -\eta^* B^* P^{-1}(sI - A)^{-1}B$$
$$= -J_K(B\eta) = J_K(C^{\langle * \rangle}\xi)$$

since $C^{\langle * \rangle} = P^{-1}C^* = -B$. □

The set of coinvariant subspaces can be naturally ordered by inclusion. In view of Beurling's theorem, inclusion can be characterized via the arithmetic of inner functions. The following is well known and we omit the proof.

Proposition 3: *Let Q_1, Q_2 be $p \times p$ inner functions of degree n_1 and n_2 respectively. Assume the following realizations are minimal*

$$Q_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline -B_i^* P_i^{-1} & I \end{array}\right) \tag{45}$$

where (A_i, B_i) are stable reachable pairs and P_i are the solutions of the Lyapunov equation

$$A_i P_i + P_i A_i^* + B_i B_i^* = 0 \tag{46}$$

Then

(1) *$H_r(Q_1) \supset H_r(Q_2)$ if and only if Q_2 is a right factor of Q_1, i.e. there exists a factorization $Q_1 = QQ_2$ for some inner function Q.*

(2) *$H_r(Q_1) \supset H_r(Q_2)$ if and only if there exists a minimal realization of Q_1 of the form*

$$Q_1 = \left(\begin{array}{cc|c} A_2 & 0 & B_2 \\ -BB_2^*P_2^{-1} & A & B \\ \hline -B_2^*P_2^{-1} & -B^*P^{-1} & I \end{array}\right)$$
$$= \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right) \times \left(\begin{array}{c|c} A_2 & B_2 \\ \hline -B_2^*P_2^{-1} & I \end{array}\right) \quad (47)$$

where (A, B) is a stable reachable pair and P is the solutions of the Lyapunov equation

$$AP + PA^* + BB^* = 0 \quad (48)$$

(3) *Suppose that for all $\xi \in \mathbb{C}^{n_2}$ the following inclusion holds*

$$\xi^*(sI - A_2)^{-1}B_2 \subset H_r(Q_1)$$

with (A_2, B_2) stable, reachable pair, then the inner matrix

$$Q_2 = \left(\begin{array}{c|c} A_2 & B_2 \\ \hline -B_2^*P_2^{-1} & I \end{array}\right)$$

where P_2 satisfies

$$A_2P_2 + P_2A_2^* + B_2B_2^* = 0 \quad (49)$$

is a right inner divisor of Q_1.

The sum of two finite dimensional coinvariant subspaces, having the representations $H_r(S_i)$, $i = 1, 2$, is a coinvariant subspace. Thus there exists a unique, up to a constant left unitary factor, inner function $S_1 \vee_L S_2$ for which

$$H_r(S_1 \vee_L S_2) = H_r(S_1) + H_r(S_2) \quad (50)$$

The inner function $S_1 \vee_L S_2$ is the least common left inner multiple of S_1 and S_2. In fact the inclusion $H_r(S_1 \vee_L S_2) \supset H_r(S_i)$ implies that S_i is a right inner factor of $S_1 \vee_L S_2$. Minimality follows from equality (77). In general

$$\dim H_r(S_1 \vee_L S_2) = \dim H_r(S_1) + \dim H_r(S_2) - \dim(H_r(S_1) \cap H_r(S_2))$$

By a similar argument there exists a unique, up to a constant left unitary factor, inner function $S_1 \wedge_R S_2$ for which

$$H_r(S_1 \wedge_R S_2) = H_r(S_1) \cap H_r(S_2) \quad (51)$$

$S_1 \wedge_R S_2$ is the greatest common right inner divisor of S_1 and S_2. Thus clearly we have

$$\dim H_r(S_1 \vee_L S_2) = \dim H_r(S_1) + \dim H_r(S_2) \quad (52)$$

if and only if $S_1 \wedge_R S_2$ is trivial, i.e. if and only if S_1 and S_2 are right coprime.

Our next result gives a state space method for the computation of the least common left inner multiple of two, right coprime, inner functions.

Proposition 4: *Let S_1, S_2 be two $m \times m$, right coprime, rational inner functions and let S be their least common left inner multiple. Let $S = \bar{S}_2S_1 = \bar{S}_1S_2$, with $\bar{S}_1, \bar{S}_2$ left coprime. Assume the minimal realizations*

$$S_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline C_i & I \end{array}\right) \quad (53)$$

with $C_i = -B_i^*P_i^{-1}$ and

$$A_iP_i + P_iA_i^* + B_iB_i^* = 0 \quad (54)$$

Define

$$A_e := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B_e := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad (55)$$

Then

(1) *We have*

$$H_r(S) = \left\{ (\xi_1 \quad \xi_2) \begin{pmatrix} sI - A_1 & 0 \\ 0 & sI - A_2 \end{pmatrix}^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \mid \xi_i \in \mathbb{C}^{n_i} \right\} \quad (56)$$

(2) *A minimal realization of S is given by*

$$S = \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline \hat{C}_1 & \hat{C}_2 & I \end{array}\right) = \left(\begin{array}{c|c} A_e & B_e \\ \hline \hat{C}_e & I \end{array}\right) \quad (57)$$

where

$$\hat{C}_e = (\hat{C}_1 \quad \hat{C}_2) = -(B_1^* \quad B_2^*) \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix}^{-1} \quad (58)$$

with

$$P_e = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix} \quad (59)$$

the solution of the Lyapunov equation

$$A_eP_e + P_eA_e^* + B_eB_e^* = 0 \quad (60)$$

(3) (a) *We have $P_{11} = P_1$, i.e. it is the unique solution of the equation*

$$A_1P_{11} + P_{11}A_1^* + B_1B_1^* = 0 \quad (61)$$

(b) *We have $P_{22} = P_2$, i.e. it is the unique solution of the equation*

$$A_2P_{22} + P_{22}A_2^* + B_2B_2^* = 0 \quad (62)$$

(c) *P_{12} is the unique solution of the equation*

$$A_1P_{12} + P_{12}A_2^* + B_1B_2^* = 0 \quad (63)$$

(4) *A minimal realization of $S = S_1 \vee_L S_2$ is given by*

$$S = \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ \hat{B}_2B_1^*P^{-1} & A_2 & \hat{B}_2 \\ \hline C_1 & \hat{C}_2 & I \end{array}\right)$$

$$= \left(\begin{array}{c|c} A_2 & \hat{B}_2 \\ \hline -\hat{B}_2^*(P_2 - P_{12}^*P_1^{-1}P_{12})^{-1} & I \end{array}\right)$$

$$\times \left(\begin{array}{c|c} A_1 & B_1 \\ \hline -B_1^*P_1^{-1} & I \end{array}\right) \quad (64)$$

where $\hat{B}_2 = B_2 - P_{21}P_1^{-1}B_1$ and

$$\hat{C}_2 = -\hat{B}_2^*(P_2 - P_{12}^*P_1^{-1}P_{12})^{-1}$$

(5) *We have*

$$A_2(P_2 - P_{12}^*P_1^{-1}P_{12}) + (P_2 - P_{12}^*P_1^{-1}P_{12})A_2 + \hat{B}_2\hat{B}_2^* = 0 \quad (65)$$

(6) *A minimal realization of $\bar{S_2}$ is given by*

$$\bar{S_2} = \left(\begin{array}{c|c} A_2 & \hat{B}_2 \\ \hline -\hat{B}_2^*(P_2 - P_{12}^*P_1^{-1}P_{12})^{-1} & I \end{array}\right) \quad (66)$$

where

$$\hat{B}_2 = B_2 - P_{12}^*P_1^{-1}B_1 \quad (67)$$

Similarly, a minimal realization of $\bar{S_1}$ is given by

$$\bar{S_1} = \left(\begin{array}{c|c} A_1 & \hat{B}_1 \\ \hline -\hat{B}_1^*(P_1 - P_{12}P_2^{-1}P_{12}^*)^{-1} & I \end{array}\right) \quad (68)$$

where

$$\hat{B}_1 = B_1 - P_{12}P_2^{-1}B_2 \quad (69)$$

Proof:

(1) By the right coprimeness of S_1, S_2 we have $H_r(S) = H_r(S_1) + H_r(S_2)$. Hence (56) follows.

(2) By the right coprimeness of S_1, S_2, the pair (A_e, B_e), defined in (55), is reachable. It completely determines the inner function S, by solving the Lyapunov equation (60) and applying Proposition 1.

(3) (a) Follows by computing the 1,1 term in the Lyapunov equation (60).
(b) Follows by computing the 2,2 term in the Lyapunov equation (60).
(c) Follows by computing the 1,2 term in the Lyapunov equation (60).
Note that equation (63) is solvable, and uniquely so, for the same reason that the Lyapunov equation (60) is, namely the fact that both A_1 and A_2^* are stable.

(4) We use the product of realizations formula

$$\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right) \times \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) = \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right) \quad (70)$$

Our aim is to apply an appropriate similarity to a realization of S that will exhibit the product structure. Our starting point is the following realization of S, namely

$$S = \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline \hat{C}_1 & \hat{C}_2 & I \end{array}\right)$$

where

$$\begin{pmatrix} \hat{C}_1 & \hat{C}_2 \end{pmatrix} = -\begin{pmatrix} B_1^* & B_2^* \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix}^{-1}$$

and P_e is the solution to the extended Lyapunov equation (60). Under a similarity transformation R, acting in the extended state space, the extended Lyapunov equation transforms into

$$(RA_eR^{-1})(RP_eR^*) + (RP_eR^*)(R^{-*}A_e^*R^*) + (RB_e)(B_e^*R^*) = 0 \quad (71)$$

We choose the similarity transformation R so that it block diagonalizes P_e. This is achieved by letting

$$R = \begin{pmatrix} I & 0 \\ -P_{12}^*P_1^{-1} & I \end{pmatrix}$$

We compute now

$$RB_e = \begin{pmatrix} I & 0 \\ -P_{12}^*P_1^{-1} & I \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$= \begin{pmatrix} B_1 \\ B_2 - P_{12}^* P_1^{-1} B_1 \end{pmatrix} = \begin{pmatrix} B_1 \\ \hat{B}_2 \end{pmatrix}$$

$$RA_eR^{-1} = \begin{pmatrix} I & 0 \\ -P_{12}^* P_1^{-1} & I \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ P_{12}^* P_1^{-1} & I \end{pmatrix}$$
$$= \begin{pmatrix} A_1 & 0 \\ -P_{12}^* P_1^{-1} A_1 + A_2 P_{12}^* P_1^{-1} & A_2 \end{pmatrix}$$

We compute next the solution to the transformed Lyapunov equation.

$$RP_eR^* = \begin{pmatrix} I & 0 \\ -P_{12}^* P_1^{-1} & I \end{pmatrix} \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix} \times \begin{pmatrix} I & -P_1^{-1} P_{12} \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} P_1 & 0 \\ 0 & P_2 - P_{12}^* P_1^{-1} P_{12} \end{pmatrix}$$

Therefore we have

$$S = \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ -P_{12}^* P_1^{-1} A_1 + A_2 P_{12}^* P_1^{-1} & A_2 & \hat{B}_2 \\ \hline -B_1^* P_1^{-1} & -\hat{B}_2^* (P_2 - P_{21} P_1^{-1} P_{21}^*)^{-1} & I \end{array} \right)$$

Computing the 2,1 term in the Lyapunov equation, we get

$$(-P_{12}^* P_1^{-1} A_1 + A_2 P_{12}^* P_1^{-1}) P_1 + \hat{B}_2 B_1^* = 0 \tag{72}$$

and so

$$(-P_{12}^* P_1^{-1} A_1 + A_2 P_{12}^* P_1^{-1}) = -\hat{B}_2 B_1^* P_1^{-1} = \hat{B}_2 C_1$$

Substituting this into the realization of S, (64) follows.

(5) Computing the 2,2 term in the Lyapunov equation (71), we get (65).

(6) By part (5),

$$\left(\begin{array}{c|c} A_2 & \hat{B}_2 \\ \hline -\hat{B}_2^* (P_2 - P_{12}^* P_1^{-1} P_{12})^{-1} & I \end{array} \right)$$

is an inner function. Necessarily it coincides with $\bar{S_2}$. The realization of $\bar{S_1}$ follows by symmetry. □

Once we have, via the solution of a Lyapunov equation, lifted the restriction of the Hardy space metric to a coinvariant subspace, we can compute with it orthogonal projections with respect to this metric in state space terms. This we proceed to explain.

Given two $m \times m$ inner functions S_1 and S_2, which we assume to be right coprime, we let S be their least common left inner multiple. Thus we have $S = \bar{S_2} S_1 = \bar{S_1} S_2$, with $\bar{S_1}, \bar{S_2}$ left coprime. The inner function S is essentially uniquely defined. Now the coinvariant subspace $H_r(S)$ has two natural, orthogonal direct sum representations, namely

$$H_r(S) = H_r(\bar{S_2} S_1) = H_r(\bar{S_2}) S_1 \oplus H_r(S_1)$$
$$= H_r(\bar{S_1} S_2) = H_r(\bar{S_1}) S_2 \oplus H_r(S_2)$$

This means that computing orthogonal projections, like $P_{H_r(\bar{S_2})S_1} | H_r(S_2)$ or $P_{H_r(S_2)} | H_r(S_1)$ can be done in the state space $H_r(\bar{S_2} S_1)$. Our first aim is to provide a state space based representation for it, via a minimal realization of S.

Next, we consider the geometry of the subspaces of $H_r(S)$ that correspond to the factorizations $S = \bar{S_2} S_1 = \bar{S_1} S_2$. This is best described in terms of intersections and orthogonal projections. The next lemma shows that we need not work in H_+^2, which is infinite dimensional, but can restrict ourselves to the finite dimensional space $H_r(S)$.

Lemma 2: *Let S_1, S_2 be two $m \times m$, right coprime, inner functions and let S be their least common left inner multiple. Let $S = \bar{S_2} S_1 = \bar{S_1} S_2$, with $\bar{S_1}, \bar{S_2}$ left coprime. Then,*

(1) *For the orthogonal projection $P_{H_r(\bar{S_2})S_1}$, defined in H_+^2, we have*

$$P_{H_r(\bar{S_2})S_1} | H_r(S_1) = (P_{H_r(\bar{S_2})S_1} | H_r(\bar{S_2} S_1)) | H_r(S_1) \tag{73}$$

(2) *We have*

$$\left. \begin{array}{l} H_r(S_1) \cap H_+^2 S_2 = H_r(S_1) \cap H_r(\bar{S_1}) S_2 \\ H_r(S_2) \cap H_+^2 S_1 = H_r(S_2) \cap H_r(\bar{S_2}) S_1 \end{array} \right\} \tag{74}$$

Proof:

(1) Follows from the general fact that, if $M \subset N$ are subspaces of a Hilbert space and P_M, P_N the respective orthogonal projections, then we have $P_M = P_M P_N$.

(2) Clearly, we have

$$H_r(S_1) \cap H(\bar{S_1}) S_2 \subset H_r(S_1) \cap H_+^2 S_2$$

So, it suffices to prove the inverse inclusion. Assume therefore that $f \in H_r(S_1) \cap H_+^2 S_2$. Thus there exist $g \in H_+^2$ and $h \in H_-^2$ for which $f = hS_1 = gS_2$. From $\bar{S_2} S_1 = \bar{S_1} S_2$ we obtain $S_2 S_1^* = \bar{S_1}^* \bar{S_2}$. Hence $h = g S_2 S_1^* = g \bar{S_1}^* \bar{S_2} \in H_-^2$. This means that also $g \bar{S_1}^* = h S_2^* \in H_-^2$, i.e. $g \in H_r(\bar{S_1})$. We conclude that $f \in H_r(S_1) \cap H_r(\bar{S_1}) S_2$. The other equality follows by symmetry. □

We are ready now to study the geometry of coinvariant subspaces via state space descriptions.

Proposition 5: *Let*

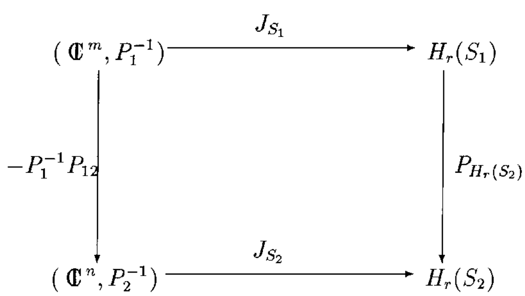
$$S_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline C_i & I \end{array}\right) \tag{75}$$

be minimal realizations of $m \times m$, right coprime, normalized inner functions and let S be their least common left inner multiple. Let $S = \bar{S}_2 S_1 = \bar{S}_1 S_2$, with $\bar{S}_1, \bar{S}_2$ normalized, left coprime inner functions. Let $J_{S_1} : (\mathbb{C}^m, P_1^{-1}) \to H_r(S_1)$ and $J_{S_2} : (\mathbb{C}^n, P_2^{-1}) \to H_r(S_2)$ be the unitary maps defined by $J_{S_i}\xi = \xi^* P_i^{-1}(sI - A_i)^{-1} B_i$. Then

(1) *Computation of $P_{H_r(S_2)} | H_r(S_1)$. We have*

$$P_{H_r(S_2)} \xi_1^* P_1^{-1}(sI - A_1)^{-1} B_1 = \xi_1^* P_1^{-1} P_{12} P_2^{-1}(sI - A_2)^{-1} B_2 \tag{76}$$

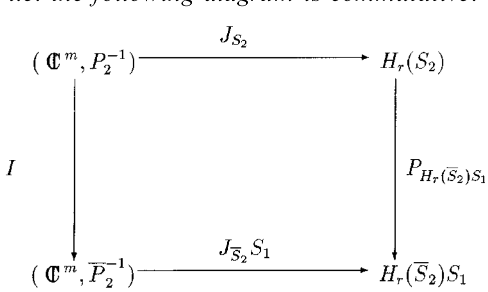
i.e. the following diagram is commutative.



(2) *Computation of $P_{H_r(\bar{S}_2)S_1} | H_r(S_2)$. With $\hat{B}_2 = B_2 - P_{12}^* P_1^{-1} B_1$ and $\bar{P}_2 = P_2 - P_{12}^* P_1^{-1} P_{12}$, we have*

$$P_{H_r(\bar{S}_2)S_1} \xi_2^* P_2^{-1}(sI - A_2)^{-1} B_2 = \xi_2^*(sI - A_2)^{-1} \hat{B}_2 S_1(s) \tag{77}$$

i.e. the following diagram is commutative.



(3) *Computation of $H_r(\bar{S}_2)S_1 \cap H_r(S_2)$. A representation of $H_r(\bar{S}_2)S_1 \cap H_r(S_2)$ is given by*

$$\{\xi^*(sI - A_2)^{-1} B_2 \mid \xi \in \operatorname{Ker} P_{12}\} \tag{78}$$

Proof:

(1) We write

$$P_{H_r(S_2)} \xi_1^* P_1^{-1}(sI - A_1)^{-1} B_1 = \xi_2^* P_2^{-1}(sI - A_2)^{-1} B_2$$

or equivalently

$$P_{H_r(S_2)}[\xi_1^* P_1^{-1}(sI - A_1)^{-1} B_1 - \xi_2^* P_2^{-1}(sI - A_2)^{-1} B_2] = 0$$

i.e.

$$[\xi_1^* P_1^{-1}(sI - A_1)^{-1} B_1 - \xi_2^* P_2^{-1}(sI - A_2)^{-1} B_2] \perp H_r(S_2) \tag{79}$$

Now

$$\begin{aligned}
[\,\xi_1^* P_1^{-1} \quad -\xi_2^* P_2^{-1}\,] &= \left[\begin{pmatrix} \xi_1^* P_1^{-1} & -\xi_2^* P_2^{-1} \end{pmatrix} \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}\right] \\
&\quad \times \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}^{-1} \\
&= [(\xi_1^* - \xi_2^* P_2^{-1} P_{12}^*) \quad (\xi_1^* P_1^{-1} P_{12} - \xi_2^*)] \\
&\quad \times \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}^{-1}
\end{aligned}$$

i.e.

$$J_s \begin{pmatrix} \xi_1 - P_{12} P_2^{-1} \xi_2 \\ P_{12}^* P_1^{-1} \xi_1 - \xi_2 \end{pmatrix} \perp H_r(S_2)$$

Next, we compute the preimage of $H_r(S_2)$ under J_S. Let

$$\begin{aligned}
J_S \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} &= \omega_2^*(sI - A_2)^{-1} B_2 \\
&= \begin{pmatrix} \zeta_1^* & \zeta_2^* \end{pmatrix} \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} (sI - A_1)^{-1} & 0 \\ 0 & (sI - A_2)^{-1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}
\end{aligned}$$

which leads to

$$\begin{pmatrix} \zeta_1^* & \zeta_2^* \end{pmatrix} = \begin{pmatrix} \omega_2^* P_{12}^* & \omega_2^* P_2 \end{pmatrix}$$

i.e. $\zeta_1^* = \zeta_2^* P_2^{-1} P_{12}^*$. This implies

$$J_S^{-1}(\omega_2^*(sI - A_2)^{-1} B_2) = \begin{pmatrix} P_{12} P_2^{-1} \zeta_2 \\ \zeta_2 \end{pmatrix}$$

Thus the orthogonality relation (79) translates into

$$0 = \left[\left(\xi_1^* - \xi_2^* P_2^{-1} P_{12}^*\right) \quad \left(\xi_1^* P_1^{-1} P_{12} - \xi_2^*\right) \right] \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}^{-1} \times \begin{pmatrix} P_{12} P_2^{-1} \zeta_2 \\ \zeta_2 \end{pmatrix}$$

$$= \left[\left(\xi_1^* P_1^{-1} \quad -\xi_2^* P_2^{-1}\right) \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix} \right] \times \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}^{-1} \begin{pmatrix} P_{12} P_2^{-1} \zeta_2 \\ \zeta_2 \end{pmatrix}$$

$$= \left(\xi_1^* P_1^{-1} \quad -\xi_2^* P_2^{-1}\right) \begin{pmatrix} P_{12} P_2^{-1} \zeta_2 \\ \zeta_2 \end{pmatrix}$$

$$= \left(\xi_1^* P_1^{-1} P_{12} - \xi_2^*\right) P_2^{-1} \zeta_2$$

for all $\zeta_2 \in \mathbb{C}^{n_2}$. This implies $\xi_2^* = \xi_1^* P_1^{-1} P_{12}$ and (76) follows.

(2) By the right coprimeness of S_1, S_2 we have

$$H_r(S_1) + H_r(S_2) = H_r(\bar{S_2} S_1) = H_r(\bar{S_1} S_2) = H_r(\bar{S_2}) S_1 \oplus H_r(S_1) = H_r(\bar{S_1}) S_2 \oplus H_r(S_2)$$

Therefore

$$P_{H_r(\bar{S_2}) S_1} | H_r(S_2) = I - P_{H_r(S_1)} | H_r(S_2)$$

Equation (72) can be rewritten as

$$-P_{21} P_1^{-1} A_1 + A_2 P_{21} P_1^{-1} = -\hat{B}_2 B_1^* P_1^{-1}$$

which implies

$$P_{21} P_1^{-1} (sI - A_1) - (sI - A_2) P_{21} P_1^{-1} = -\hat{B}_2 B_1^* P_1^{-1}$$

In turn this leads to

$$\begin{aligned}
&(sI - A_2)^{-1} P_{21} P_1^{-1} B_1 - P_{21} P_1^{-1} (sI - A_1)^{-1} B_1 \\
&= -(sI - A_2)^{-1} \hat{B}_2 B_1^* P_1^{-1} (sI - A_1)^{-1} B_1 \\
&= (sI - A_2)^{-1} \hat{B}_2 \left(I - B_1^* P_1^{-1} (sI - A_1)^{-1} B_1\right) - (sI - A_2)^{-1} \hat{B}_2 \\
&= (sI - A_2)^{-1} \hat{B}_2 S_1(s) - (sI - A_2)^{-1} \hat{B}_2 \\
&= (sI - A_2)^{-1} \hat{B}_2 S_1(s) - (sI - A_2)^{-1} \left(B_2 - P_{21} P_1^{-1} B_1\right)
\end{aligned}$$

From this we conclude

$$(sI - A_2)^{-1} B_2 = P_{12}^* P_1^{-1} (sI - A_1)^{-1} B_1 + (sI - A_2)^{-1} \hat{B}_2 S_1(s) \qquad (80)$$

Since, for each $\xi \in \mathbb{C}^{n_2}$, we have $\xi P_{12}^* P_1^{-1} (sI - A_1)^{-1} B_1 \in H_r(S_1)$, (77) follows.

(3) A general element of $H_r(S_2)$ has a representation of the form $\eta^* (sI - A_2)^{-1} B_2$. We want to find conditions that guarantee $\eta^* (sI - A_2)^{-1} B_2 \in H_r(\bar{S_2}) S_1$. To this end we use equality (80) to get

$$\eta^* (sI - A_2)^{-1} B_2 = \eta^* P_{12}^* P_1^{-1} (sI - A_1)^{-1} B_1 + \eta^* (sI - A_2)^{-1} \hat{B}_2 S_1(s)$$

Since $\eta^* P_{12}^* P_1^{-1} (sI - A_1)^{-1} B_1 \in H_r(S_1)$ and hence is orthogonal to $\eta^* (sI - A_2)^{-1} \hat{B}_2 S_1(s)$, it follows that $\eta^* (sI - A_2)^{-1} B_2 \in H_r(\bar{S_2}) S_1$ if and only if $\eta^* P_{12}^* P_1^{-1} = 0$, which by the invertibility of P_1 is the case if and only if $\eta \in \operatorname{Ker} P_{12}$. □

3. State space formulas for spectral factors

Our aim in the following is the derivation of state space formulas for objects that arose out of the analysis of singular, i.e. rectangular and not necessarily full rank, spectral factors. To follow this part, we suggest consulting Fuhrmann and Gombani (1998). Since we are in the realm of singular transfer functions, anything related to zeros is non-trivial. We will address ourselves to two results that are needed for use in Theorem 2. These are, with a view to the characterization of a minimum-phase spectral factor, the question of characterizing the existence of a stable left inverse of a transfer function. Following that we study inner functions in terms of non-minimal realizations.

We adopt the following notation from Schumacher (1981). Given a subspace $\mathcal{V}$ of the state space of a minimal realization

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

we denote by $\langle \mathcal{V} | A \rangle$ the largest A-invariant subspace contained in $\mathcal{V}$. In particular $\langle \operatorname{Ker} C | A \rangle = 0$ is the condition for observability. Similarly $X_-(A) \supset \langle \operatorname{Ker} C | A \rangle$, with $X_-(A)$ the stable subspace of A, is the condition for detectability. We quote now the following, dualized version, of a result of Minto (1985), Chen and Francis (1987) and Fuhrmann (1989). In this connection see also Aling and Schumacher (1984).

Proposition 6: *Assume $W \in H_+^\infty$ has minimal realization*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Then the following statements are equivalent.

(1) *W has a left inverse in* H_+^∞.

(2) *D is injective and for some H we have*

$$B + HD = 0 \tag{81}$$

and

$$X_+(A + HC) = \{0\} \tag{82}$$

(3) *D is injective and*

$$X_-(A - BD^\sharp C) \supset \langle (I - D^\sharp D)C | A - BD^\sharp C \rangle \tag{83}$$

where $D^\sharp = (D^*D)^{-1}D^*$. *If H is such that* (2) *is satisfied then a* H_+^∞ *left inverse* $W^\sharp$ *of W is given by*

$$W^\sharp = \left(\begin{array}{c|c} A + HC & H \\ \hline D^\sharp C & D^\sharp \end{array}\right)$$

The following theorem generalizes, to the non-minimal case, results obtained in Fuhrmann (1995).

Theorem 1: *Let* (C, A) *be a, not necessarily observable, stable pair.*

(1) *Let Z be any solution of the homogeneous Riccati equation*

$$AZ + ZA^* + ZC^*CZ = 0 \tag{84}$$

Then

$$Q = \left(\begin{array}{c|c} A & -ZC^* \\ \hline C & I \end{array}\right) \tag{85}$$

is an inner function.

(2) *The McMillan degree of Q,* $\delta(Q)$, *is given by*

$$\delta(Q) = \operatorname{rank} Z \tag{86}$$

where $\mathcal{W}$ *is any subspace complementary to the unobservable subspace of the pair* (C, A).

(3) *The maximal McMillan degree corresponds to the maximal nonnegative definite solution of* (84).

Proof:

(1) We compute

$$QQ^* = \left(\begin{array}{c|c} A & -ZC^* \\ \hline C & I \end{array}\right) \times \left(\begin{array}{c|c} -A^* & C^* \\ \hline CZ & I \end{array}\right)$$
$$= \left(\begin{array}{cc|c} -A^* & 0 & C^* \\ -ZC^*CZ & A & -ZC^* \\ \hline CZ & C & I \end{array}\right)$$
$$= \left(\begin{array}{cc|c} -A^* & 0 & C^* \\ 0 & A & 0 \\ \hline 0 & C & I \end{array}\right) = I$$

Here we have applied the similarity

$$\begin{pmatrix} I & 0 \\ Z & I \end{pmatrix}$$

and used the fact that Z is a solution of (84).

(2) Let $\mathcal{V} = \cap \operatorname{Ker} CA^i$ be the unobservable subspace of (C, A) and let $\mathcal{W}$ be any complementary subspace. With respect to the direct sum decomposition $\mathcal{V} \oplus \mathcal{W}$ of the state space, the pair (C, A) can be written as

$$(C, A) = \left(\begin{pmatrix} 0 & C_2 \end{pmatrix}, \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \right)$$

The Riccati equation (84) can be written as

$$\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{pmatrix} + \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{pmatrix}\begin{pmatrix} A_1^* & 0 \\ A_3^* & A_2^* \end{pmatrix}$$
$$+ \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & C_2^*C_2 \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since it would be easier to work in a basis in which Z is block diagonal, we apply the change of basis transformation

$$R = \begin{pmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{pmatrix}$$

Under this C, A, Z transform into

$$CR = \begin{pmatrix} 0 & C_2 \end{pmatrix}$$

$$R^{-1}AR = \begin{pmatrix} A_1 & A_3 - A_1Z_{11}^{-1}Z_{12} - Z_{11}^{-1}Z_{12}A_2 \\ 0 & A_2 \end{pmatrix}$$

$$R^{-1}ZR^{-*} = \begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} - Z_{12}^*Z_{11}^{-1}Z_{12} \end{pmatrix}$$

Thus, using a slight change of notation, we can assume without loss of generality that

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & C_2 \end{pmatrix}$$

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

The Riccati equation now reduces to the following three equations

$$A_1 Z_1 + Z_1 A_1^* = 0$$

$$A_3 Z_2 = 0$$

$$A_2 Z_2 + Z_2 A_2^* + Z_2 C_2^* C_2 Z_2 = 0$$

Since A_1 is stable, the first equation implies $Z_1 = 0$. Thus rank Z is bounded by the codimension of the unobservable subspace of (C, A) and this bound is attained if we choose for Z_2 the positive definite solution of the above Riccati equation. This solution exists by the stability of A_2.

If we choose Z_2 to be the positive definite solution of the Riccati equation

$$A_2 Z_2 + Z_2 A_2^* + Z_2 C_2^* C_2 Z_2 = 0 \tag{87}$$

then Z is injective on any subspace complementary to the unobservable subspace $\langle \operatorname{Ker} C | A \rangle$. We compute now, for any solution Z_2 of (87)

$$Q = \left(\begin{array}{cc|c} A_1 & A_3 & 0 \\ 0 & A_2 & -Z_2 C_2^* \\ \hline 0 & C_2 & I \end{array}\right) = \left(\begin{array}{c|c} A_2 & -Z_2 C_2^* \\ \hline C_2 & I \end{array}\right)$$

Since the last realization is observable, the McMillan degree of Q is equal to the dimension of the reachable subspace of $(A_2, -Z_2 C_2^*)$. From (87) we have the relation $A_2 Z_2 = -Z_2(A_2^* + C_2^* C_2 Z_2)$. Computing

$$\sum A_2^i(-Z_2 C_2^*)\xi_i = -\sum Z_2 (-1)^i (A_2^* + C_2^* C_2 Z_2)^i C_2^* \xi_i$$

$$= \operatorname{Im} Z_2$$

In terms of the original realization we have (86).

(3) Since rank Z = rank Z_2, it is clear that the rank of Z is maximized for Z_2, the positive definite solution of the Riccati equation (87), i.e. for the maximal, nonnegative definite solution of (84). □

We proceed now, given a minimal stable spectral factor W, to compute the minimum phase, stable spectral factor W_- and the maximum phase, stable spectral factor W_+. This is effectively the computational aspect, in state space form, of the outer/inner factorization of an H_+^∞ function. We will assume $W \in H_+^\infty$ is a $p \times m$ minimal spectral factor of $\Phi = WW^*$ with rank $\Phi(i\omega) = m_0$ on the extended imaginary axis and of McMillan degree n.

Theorem 2: *Let*

$$W = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D & 0 \end{array}\right)$$

be a $p \times m$ minimal, stable spectral factor of Φ. Then

(1) *A minimal realization of the minimum phase, stable spectral factor W_- is given by*

$$W_- = \left(\begin{array}{c|c} A & B_1 + X_- C^* D (D^* D)^{-1} \\ \hline C & D \end{array}\right) \tag{88}$$

where $X_- \geq 0$ is the stabilizing solution of the Riccati equation

$$(A - B_1(D^*D)^{-1}D^*C)X + X(A^* - C^*D(D^*D)^{-1}B_1^*) + B_2 B_2^* - XC^*D(D^*D)^{-2}D^*CX = 0 \tag{89}$$

i.e. the solution for which $A - B_1(D^*D)^{-1}D^*C - X_- C^*D(D^*D)^{-2}D^*C$ is stable. Defining

$$H_- = -[B_1 + X_- C^* D(D^*D)^{-1}](D^*D)^{-1}D^* \tag{90}$$

the previous Riccati equation can be rewritten as

$$(A + H_- C)X_- + X_-(A^* + C^* H_-^*) + B_2 B_2^* + X_- C^* D (D^*D)^{-2} D^* C X_- = 0 \tag{91}$$

(2) *If X is any solution of the Riccati equation (89), then*

$$\operatorname{Ker} B_2^* \supset \operatorname{Ker} X \tag{92}$$

Equivalently,

$$\operatorname{Im} B_2 \subset \operatorname{Im} X \tag{93}$$

There exists a linear map P for which

$$B_2 = -XP^* \tag{94}$$

(3) *The minimum McMillan degree corigid function satisfying $W_- \hat{Q}' = W$ is given by*

$$\hat{Q}' = \left(\begin{array}{c|cc} A + H_- C & -X_- C^* D (D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1} D^* C & I & 0 \end{array}\right) \times \left(\begin{array}{c|cc} A - (B_1 + X_- C^* D (D^*D)^{-1})(D^*D)^{-1} D^* C & -X_- C^* D (D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1} D^* C & I & 0 \end{array}\right) \tag{95}$$

(4) *A minimal McMillan degree extension of $\hat{Q}'$ to an inner function*

$$Q' = \begin{pmatrix} \hat{Q}' \\ \tilde{Q}' \end{pmatrix}$$

is given by

$$Q' = \left(\begin{array}{c|cc} A+H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*C & I & 0 \\ P_- & 0 & I \end{array}\right) \quad (96)$$

where P_- satisfies (94). *The McMillan degree of Q' is equal to* rank X_-.

(5) *A minimal realization of the maximum phase, stable spectral factor W_+ is given by*

$$W_+ = \left(\begin{array}{c|c} A & B_1 + X_+C^*D(D^*D)^{-1} \\ \hline C & D \end{array}\right) \quad (97)$$

where X_+ is the antistabilizing solution of the Riccati equation

$$(A - B_1(D^*D)^{-1}D^*C)X_+ + X_+(A^* - C^*D(D^*D)^{-1}B_1^*) + B_2B_2^* - X_+C^*D(D^*D)^{-2}D^*CX_+ = 0 \quad (98)$$

i.e. the solution for which $A - B_1(D^*D)^{-1}D^*C - X_+C^*D(D^*D)^{-2}D^*C$ is antistable. Defining

$$H_+ = -[B_1 + X_+C^*D(D^*D)^{-1}](D^*D)^{-1}D^* \quad (99)$$

the previous Riccati equation can be rewritten as

$$(A + H_+C)X_+ + X_+(A^* + C^*H_+^*) + B_2B_2^* + X_+C^*D(D^*D)^{-2}D^*CX_+ = 0 \quad (100)$$

(6) *The minimum McMillan degree rigid function satisfying $W\hat{Q}'' = W_+$ is given by*

$$\hat{Q}'' = \left(\begin{array}{c|c} -A^* - C^*H_+^* & C^*D(D^*D)^{-1} \\ \hline -B_1^* - D^*H_+^* & I \\ -B_2^* & 0 \end{array}\right) = \left(\begin{array}{c|c} -A^* - C^*H_+^* & C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*CX_+ & I \\ -B_2^* & 0 \end{array}\right) \quad (101)$$

(7) *A minimal McMillan degree extension of $\hat{Q}''$ to an inner function $Q'' = (\hat{Q}'' \ \tilde{Q}'')$ is given by*

$$Q'' = \left(\begin{array}{c|cc} -A^* - C^*H_+^* & C^*D(D^*D)^{-1} & P_+^* \\ \hline (D^*D)^{-1}D^*CX_+ & I & 0 \\ -B_2^* & 0 & I \end{array}\right) \quad (102)$$

where P_+ satisfies (94). *The McMillan degree of Q'' is equal to* rank X_+.

(8) *The $m_0 \times m_0$ function $Q_+ := W_-^{-L}W_+$ is inner and has a, not necessarily minimal, realization given by*

$$Q_+ = \left(\begin{array}{c|c} A+H_-C & -(X_- - X_+)C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*C & I \end{array}\right) \quad (103)$$

Moreover, $X_- - X_+$ is the maximal, nonnegative definite, solution of the homogeneous Riccati equation

$$(A + H_-C)Z + Z^*(A^* + C^*H_-^*) + ZC^*D(D^*D)^{-2}D^*CZ = 0 \quad (104)$$

The McMillan degree of Q_+ is equal to rank $(X_- - X_+)$.

(9) *We have*

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix} \quad (105)$$

and realizations of R are given by

$$R = \left(\begin{array}{c|c} -A^* - C^*H_+^* & P_+^* \\ \hline -(P_- - P_+)X_+ & I \end{array}\right) \quad (106)$$

and

$$R = \left(\begin{array}{c|c} A+H_-C & -X_-(P_-^* - P_+^*) \\ \hline P_- & I \end{array}\right) \quad (107)$$

Proof:

(1) Assume

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is a minimal realization. Since W is defined only up to a constant right unitary factor, we can assume, without loss of generality, that $D = (D_1 \ 0)$, with D_1 a $p \times m_0$ full column rank matrix. Indeed, if U_2 is an orthonormal basis matrix for Ker D, and $U = (U_1 \ U_2)$ any unitary completion, we obtain $DU = (DU_1 \ 0) = (D_1 \ 0)$, with D_1 as above. So, without loss of generality, we can assume

$$W = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D & 0 \end{array}\right)$$

and D of full column rank. Now, as W and W_- have the same left pole structure (see Fuhrmann and Gombani 1998), W_- has a minimal realization of the form

$$W_- = \left(\begin{array}{c|c} A & B_- \\ \hline C & D \end{array}\right) \quad (108)$$

with B_- a $n \times m_0$ matrix. Thus all we need is to compute B_-.

To this end, we use the fact that $WW^* = W_- W_-^*$ which implies

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D & 0 \end{array}\right) \times \left(\begin{array}{c|c} -A^* & C^* \\ \hline -B_1^* & D^* \\ -B_2^* & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B_- \\ \hline C & D \end{array}\right) \times \left(\begin{array}{c|c} -A^* & C^* \\ \hline -B_-^* & D^* \end{array}\right)$$

This leads to the equality

$$\left(\begin{array}{cc|c} -A^* & 0 & C^* \\ -B_1B_1^* - B_2B_2^* & A & B_1 D^* \\ \hline -DB_1^* & C & DD^* \end{array}\right) = \left(\begin{array}{cc|c} -A^* & 0 & C^* \\ -B_-B_-^* & A & B_- D^* \\ \hline -DB_-^* & C & DD^* \end{array}\right) \quad (109)$$

By minimality, there exists a unique state space isomorphism map intertwining these realizations and it is necessarily of the form

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$$

The intertwining relations translate into

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} C^* \\ B_1 D^* \end{pmatrix} = \begin{pmatrix} C^* \\ B_- D^* \end{pmatrix} \quad (110)$$

and

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} -A^* & 0 \\ -B_1B_1^* - B_2B_2^* & A \end{pmatrix} = \begin{pmatrix} -A^* & 0 \\ -B_-B_-^* & A \end{pmatrix}\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \quad (111)$$

From (110) we get

$$XC^* + B_1 D^* = B_- D^* \quad (112)$$

Now D is left invertible, so D^* is right invertible. In fact, $D(D^*D)^{-1}$ is a right inverse of D^*. Computing now B_- from (112), we get

$$B_- = B_1 + XC^* D(D^*D)^{-1} \quad (113)$$

Computing the 2,1 term of (111), we get

$$-XA^* - B_1B_1^* - B_2B_2^* = -B_-B_-^* + AX$$

or

$$\begin{aligned} AX + XA^* &= -B_1B_1^* - B_2B_2^* + B_-B_-^* \\ &= -B_1B_1^* - B_2B_2^* \\ &\quad + B_1B_1^* + B_1(D^*D)^{-1}D^*CX \\ &\quad + XC^*D(D^*D)^{-1}B_1^* \\ &\quad + XC^*D(D^*D)^{-2}D^*CX \end{aligned}$$

This can be rewritten as the Riccati equation

$$\begin{aligned} &(A - B_1(D^*D)^{-1}D^*C)X + X(A^* - C^*D(D^*D)^{-1}B_1^*) \\ &\quad + B_2B_2^* - XC^*D(D^*D)^{-2}D^*CX = 0 \end{aligned}$$

or, equivalently

$$\begin{aligned} &(A - B_1(D^*D)^{-1}D^*C - XC^*D(D^*D)^{-2}D^*C)X \\ &\quad + X(A^* - C^*D(D^*D)^{-1}B_1^* - C^*D(D^*D)^{-2}D^*CX) \\ &\quad + B_2B_2^* + XC^*D(D^*D)^{-2}D^*CX = 0 \end{aligned}$$

Once we have computed the stabilizing solution X_- from (89), i.e. the solution for which $A - B_1(D^*D)^{-1}D^*C - X_-C^*D(D^*D)^{-2}D^*C$ is stable, B_- is given by (113).

It is easy to check now, using Proposition 6, that if W_- is given by (88), then it is left invertible in H_+^∞. In fact, if H_- is defined by (90) then, using (113), we have

$$\begin{aligned} B + H_-D &= B - (B_1 + X_-C^*D(D^*D)^{-1})(D^*D)^{-1}D^*D \\ &= B - (B_1 + X_-C^*D(D^*D)^{-1}) = 0. \end{aligned}$$

Moreover

$$\begin{aligned} &X_+(A + H_-C) \\ &\quad = X_+(A - (B_1 + X_-C^*D(D^*D)^{-1})(D^*D)^{-1}D^*C) \\ &\quad = \{0\} \end{aligned}$$

as X_- is the stabilizing solution of the Riccati equation (89). So W_- is indeed the minimum phase, stable spectral factor.

(2) This part is adapted from Pavon (1994). From the Riccati equation (89) it is clear that, for any solution X of the equation, we must have $\operatorname{Ker} B_2^* \supset \operatorname{Ker} X$. By standard linear algebra we infer the existence of a linear transformation P

for which $B_2^* = -PX$. (The minus sign is introduced for cosmetic reasons.)

(3) We compute, with H_- defined by (90), and using the similarity

$$\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$$

$$\hat{Q}' = W_-^{-L} W = \left(\begin{array}{c|c} A + H_-C & H_- \\ \hline D^\sharp C & D^\sharp \end{array}\right)$$

$$\times \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ H_-C & A + H_-C & I & 0 \\ \hline D^\sharp C & D^\sharp C & I & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ 0 & A + H_-C & B_1 + H_-D & B_2 \\ \hline 0 & D^\sharp C & I & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|cc} A + H_-C & B_1 + H_-D & B_2 \\ \hline D^\sharp C & I & 0 \end{array}\right)$$

Clearly, $\hat{Q}'$ is stable as $A + H_-C$ is. It remains to check that $\hat{Q}'$ is coisometric, i.e. that $\hat{Q}'\hat{Q}'^* = I$. Note that (90) implies $B_1 + H_-D = -X_-C^*D(D^*D)^{-1}$. So

$$-(B_1 + H_-D)(B_1^* + D^*H_-^*) - B_2B_2^*$$

$$= -X_-C^*D(D^*D)^{-2}D^*CX_- - B_2B_2^*$$

Clearly $D^\sharp = (D^*D)^{-1}D^*$ implies $(D^\sharp)^* = D(D^*D)^{-1}$. Finally, in the following computation we will use the similarity

$$\begin{pmatrix} I & 0 \\ X_- & I \end{pmatrix}$$

where X_- is the stabilizing solution of the Riccati equation (89) as well as the Riccati equation itself. So we have

$$\hat{Q}'\hat{Q}'^*$$

$$= \left(\begin{array}{c|cc} A + H_-C & B_1 + H_-D & B_2 \\ \hline D^\sharp C & I & 0 \end{array}\right) \times \left(\begin{array}{c|c} -(A^* + C^*H_-^*) & C^*(D^\sharp)^* \\ \hline -B_1^* - D^*H_-^* & I \\ -B_2^* & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -(A^* + C^*H_-^*) & 0 & C^*(D^\sharp)^* \\ -(B_1 + H_-D)(B_1^* + D^*H_-^*) - B_2B_2^* & A + H_-C & B_1 + H_-D \\ \hline -(B_1^* + D^*H_-^*) & D^\sharp C & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -(A^* + C^*H_-^*) & 0 & C^*D(D^*D)^{-1} \\ -X_-C^*D(D^*D)^{-2}D^*CX_- - B_2B_2^* & A + H_-C & -X_-C^*D(D^*D)^{-1} \\ \hline -(D^*D)^{-1}D^*CX_- & (D^*D)^{-1}D^*C & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -(A^* + C^*H_-^*) & 0 & C^*D(D^*D)^{-1} \\ 0 & A + H_-C & 0 \\ \hline 0 & (D^*D)^{-1}D^*C & I \end{array}\right) = I$$

(4) Obviously any minimal degree, normalized at infinity, inner extension of $\hat{Q}'$ is of the form

$$Q' = \left(\begin{array}{c|cc} A + H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*C & I & 0 \\ P_- & 0 & I \end{array}\right)$$

The condition for Q' to be inner is

$$Q'(Q')^*$$

$$= \left(\begin{array}{c|cc} A + H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*C & I & 0 \\ P_- & 0 & I \end{array}\right)$$

$$\times \left(\begin{array}{c|cc} -A^* - C^*H_-^* & C^*D(D^*D)^{-1} & P_-^* \\ \hline (D^*D)^{-1}D^*CX_- & I & 0 \\ -B_2^* & 0 & I \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} -A^* - C^*H_-^* & 0 & C^*D(D^*D)^{-1} & P_-^* \\ -X_-C^*D(D^*D)^{-2}D^*CX_- - B_2B_2^* & A + H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*CX_- & (D^*D)^{-1}D^*C & I & 0 \\ -B_2^* & P_- & 0 & I \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} -A^* - C^*H_-^* & 0 & C^*D(D^*D)^{-1} & P_-^* \\ 0 & A + H_-C & 0 & B_2 + X_-P_-^* \\ \hline 0 & (D^*D)^{-1}D^*C & I & 0 \\ -B_2^* - P_-X_- & P_- & 0 & I \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} -A^* - C^*H_-^* & 0 & C^*D(D^*D)^{-1} & P_-^* \\ 0 & A + H_-C & 0 & 0 \\ \hline 0 & (D^*D)^{-1}D^*C & I & 0 \\ 0 & P_- & 0 & I \end{array}\right) = I$$

Here we applied the state space similarity

$$\begin{pmatrix} I & 0 \\ X_- & I \end{pmatrix}$$

where X_- is the stabilizing solution of the Riccati equation (89) as well as the Riccati equation itself. Moreover we used the existence of a P_- satisfying (94), proved in part (2).

Another way to see that Q' is inner is to observe that we can write

$$\left(\begin{array}{c|cc} A+H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*C & I & 0 \\ P_- & 0 & I \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A} & -\mathcal{Z}\mathcal{C}^* \\ \hline \mathcal{C} & \mathcal{I} \end{array}\right)$$

where $\mathcal{Z}$ solves the homogeneous Riccati equation

$$\mathcal{A}\mathcal{Z}+\mathcal{Z}\mathcal{A}^*+\mathcal{Z}\mathcal{C}^*\mathcal{C}\mathcal{Z}=0$$

To this end we show that $\mathcal{Z}=X_-$ is a solution of this equation. Indeed

$$\begin{aligned}&(A+H_-C)X_-+X_-(A^*+C^*H_-^*)+X_-[C^*D(D^*D)^{-2}D^*C+P_-^*P_-]X_- \\ &\quad=(A+H_-C)X_-+X_-(A^*+C^*H_-^*)+X_-C^*D(D^*D)^{-2}D^*CX_-+B_2B_2^*=0.\end{aligned}$$

An application of Theorem 1 shows that the McMillan degree of Q' is equal to rank X_-.

(5) The proof is analogous to the proof of part (1).

(6) We note that W_+ has a left inverse in H_-^∞. In fact with H_+ given by (99)

$$W_+^{-L}=\left(\begin{array}{c|c} A+H_+C & H_+ \\ \hline (D^*D)^{-1}D^*C & (D^*D)^{-1}D^* \end{array}\right) \tag{114}$$

is such a left inverse. It is in H_-^∞ as, by the choice of X_+, $A+H_+C$ is antistable. We define $\bar{S}=W_+^{-L}W$. A, by now standard, computation using the realizations of W_+^{-L} and W yields

$$\bar{S}=\left(\begin{array}{c|cc} A+H_+C & -X_+C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*C & I & 0 \end{array}\right)$$

Since $\hat{Q}''=\bar{S}^*$, we get (128). By a computation similar to the one done in part (3) it is easy to show that $\bar{S}\bar{S}^*=I$. We define now $\hat{Q}''=\bar{S}^*$, which shows that $\hat{Q}''$ is rigid in H_+^∞. It remains to show that $W\hat{Q}''=W_+$. From Lemma 3.1 in Fuhrmann and Gombani (1998) we know that $W_+W_+^{-L}W_-=W_-$. Moreover, with $\hat{Q}'$ of part (3), we have $W=W_-\hat{Q}'$. Hence

$$\begin{aligned}W_+&=W_+\hat{Q}''{*}\hat{Q}''=W_+(W_+^{-L}W)\hat{Q}'' \\ &=W_+W_+^{-L}W_-\hat{Q}'\hat{Q}''=W_-\hat{Q}'\hat{Q}''=W\hat{Q}''\end{aligned}$$

(7) The proof is analogous to the proof of part (4).

(8) We compute

$$\begin{aligned}&W_-^{-L}W_+ \\ &=\left(\begin{array}{c|c} A+H_-C & H_- \\ \hline D^\sharp C & D^\sharp \end{array}\right)\times\left(\begin{array}{c|c} A & B_1-X_+C^*D(D^*D)^{-1} \\ \hline C & D \end{array}\right) \\ &=\left(\begin{array}{cc|c} A & 0 & B_1-X_+C^*D(D^*D)^{-1} \\ H_-C & A+H_-C & H_-D \\ \hline D^\sharp C & D^\sharp C & I \end{array}\right) \\ &=\left(\begin{array}{cc|c} A & 0 & B_1+X_+C^*D(D^*D)^{-1} \\ 0 & A+H_-C & (B_1+X_+C^*D(D^*D)^{-1})-(B_1+X_-C^*D(D^*D)^{-1}) \\ \hline 0 & (D^*D)^{-1}D^*C & I \end{array}\right) \\ &=\left(\begin{array}{c|c} A+H_-C & -(X_--X_+)C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*C & I \end{array}\right)\end{aligned}$$

There is another way to compute Q_+, using the fact that $\hat{Q}'\hat{Q}''=Q_+$. We have, using the state space representations of $\hat{Q}'$ and $\hat{Q}''$

$$\begin{aligned}&\hat{Q}'\hat{Q}'' \\ &=\left(\begin{array}{c|cc} A+H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*C & I & 0 \end{array}\right) \\ &\quad\times\left(\begin{array}{c|c} -A^*-C^*H_+^* & C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*CX_+ & I \\ -B_2^* & 0 \end{array}\right) \\ &=\left(\begin{array}{cc|c} -A^*-C^*H_+^* & 0 & C^*D(D^*D)^{-1} \\ -B_2B_2^*-X_-C^*D(D^*D)^{-2}D^*CX_+ & A+H_-C & -X_-C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*CX_+ & (D^*D)^{-1}D^*C & I \end{array}\right)\end{aligned}$$

We apply now the state space similarity

$$\begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}$$

and compute

$$\begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}\begin{pmatrix} C^*D(D^*D)^{-1} \\ -X_-C^*D(D^*D)^{-1} \end{pmatrix} = \begin{pmatrix} C^*D(D^*D)^{-1} \\ (X_+-X_-)C^*D(D^*D)^{-1} \end{pmatrix}$$

Similarly,

$$((D^*D)^{-1}D^*CX_+ \;\; (D^*D)^{-1}D^*C)\left(\begin{pmatrix} I & 0 \\ -X_+ & I \end{pmatrix}\right) = \begin{pmatrix} 0 & C^*D(D^*D)^{-1} \end{pmatrix}$$

Finally, using the definitions of H_-, H_+ and the Riccati equation we get

$$\begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}\begin{pmatrix} -A^* - C^*H_+^* & 0 \\ -B_2B_2^* - X_-C^*D(D^*D)^{-2}D^*CX_+ & A + H_-C \end{pmatrix}\begin{pmatrix} I & 0 \\ -X_+ & I \end{pmatrix} = \begin{pmatrix} -A^* - C^*H_+^* & 0 \\ 0 & A + H_-C \end{pmatrix}$$

Thus

$$Q_+ = \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & C^*D(D^*D)^{-1} \\ -B_2B_2^* - X_-C^*D(D^*D)^{-2}D^*CX_+ & A + H_-C & -X_-C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*CX_+ & (D^*D)^{-1}D^*C & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & C^*D(D^*D)^{-1} \\ 0 & A + H_-C & (X_+ - X_-)C^*D(D^*D)^{-1} \\ \hline 0 & (D^*D)^{-1}D^*C & I \end{array}\right)$$

$$= \left(\begin{array}{c|c} A + H_-C & -(X_- - X_+)C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*C & I \end{array}\right)$$

That $Q_+ = \hat{Q}'\hat{Q}''$ is inner has been proved in Fuhrmann and Gombani (1998), however we give here also a computational proof (see equation below).

Applying the similarity

$$\begin{pmatrix} I & 0 \\ X_+ - X_- & I \end{pmatrix}$$

and going through some algebra, we get

$$Q_+Q_+^* = \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & -C^*D(D^*D)^{-1} \\ (X_+ - X_-)C^*D(D^*D)^{-2}D^*C(X_+ - X_-) & A + H_-C & (X_+ - X_-)C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*C(X_+ - X_-) & (D^*D)^{-1}D^*C & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & -C^*D(D^*D)^{-1} \\ 0 & A + H_-C & 0 \\ \hline 0 & (D^*D)^{-1}D^*C & I \end{array}\right) = I$$

The crux of the matter is the fact, which can be proved by direct computation, that $X_- - X_+$ is a solution of the homogeneous Riccati equation (104). Note that

$$(A + H_-C)X_+ + (X_- - X_+)C^*D(D^*D)^{-2}D^*CX_+ = (A + H_+C)X_+$$

Using this equality, we compute

$$\begin{aligned}
&(A + H_-C)(X_- - X_+) + (X_- - X_+)(A^* + C^*H_-^*) \\
&\quad + (X_- - X_+)C^*D(D^*D)^{-2}D^*C(X_- - X_+) \\
&= [(A + H_-C)X_- + X_-(A^* + C^*H_-^*) \\
&\quad + X_-C^*D(D^*D)^{-2}D^*CX_-] \\
&\quad - [(A + H_-C)X_+ + (X_- - X_+)C^*D(D^*D)^{-2}D^*CX_+] \\
&\quad - [X_+(A^* + C^*H_-^*) + X_+C^*D(D^*D)^{-2} \\
&\quad \times D^*C(X_- - X_+)] - X_+C^*D(D^*D)^{-2}D^*CX_+ \\
&= -B_2B_2^* - (A + H_+C)X_+ - X_+(A^* + C^*H_+^*) \\
&\quad - X_+C^*D(D^*D)^{-2}D^*CX_+ = 0
\end{aligned}$$

(9) We already computed $\hat{Q}'\hat{Q}'' = Q_+$. Next we compute

$$\tilde{Q}'\hat{Q}'' = \left(\begin{array}{c|cc} A + H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1}D^*C & I & 0 \end{array}\right) \times \left(\begin{array}{c|c} -A^* - C^*H_+^* & P_+^* \\ \hline (D^*D)^{-1}D^*CX_+ & 0 \\ -B_2^* & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & P_+^* \\ -B_2^*B_2 - X_-C^*D(D^*D)^{-2}D^*CX_+ & A + H_-C & B_2 \\ \hline (D^*D)^{-1}D^*CX_+ & (D^*D)^{-1}D^*C & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & P_+^* \\ 0 & A + H_-C & 0 \\ \hline 0 & (D^*D)^{-1}D^*C & 0 \end{array}\right) = 0$$

Here we used the state space similarity

$$Q_+Q_+^* = \left(\begin{array}{c|c} A + H_-C & (X_+ - X_-)C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*C & I \end{array}\right) \times \left(\begin{array}{c|c} -A^* - C^*H_+^* & -C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*C(X_+ - X_-) & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & -C^*D(D^*D)^{-1} \\ (X_+ - X_-)C^*D(D^*D)^{-2}D^*C(X_+ - X_-) & A + H_-C & (X_+ - X_-)C^*D(D^*D)^{-1} \\ \hline (D^*D)^{-1}D^*C(X_+ - X_-) & (D^*D)^{-1}D^*C & I \end{array}\right)$$

$$\begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}$$

with the following calculations

$$\begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}\begin{pmatrix} P_+^* \\ B_2 \end{pmatrix} = \begin{pmatrix} P_+^* \\ 0 \end{pmatrix}$$

$$\left((D^*D)^{-1}D^*CX_+ \;\; (D^*D)^{-1}D^*C\right)\begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix} = \left(0 \;\; (D^*D)^{-1}D^*C\right)$$

and

$$\begin{pmatrix} I & 0 \\ X_+ & I \end{pmatrix}\begin{pmatrix} -A^* - C^*H_+^* & 0 \\ -B_2^*B_2 - X_-C^*D(D^*D)^{-2}D^*CX_+ & A+H_-C \end{pmatrix}\begin{pmatrix} I & 0 \\ -X_+ & I \end{pmatrix} = \begin{pmatrix} -A^* - C^*H_+^* & 0 \\ 0 & A+H_-C \end{pmatrix}$$

The last equality follows by computing the 2,1 term, i.e.

$$\begin{aligned} &-X_+(A^* + C^*H_+^*) - B_2^*B_2 - X_-C^*D(D^*D)^{-2}D^*CX_+ - (A+H_-C)X_+ \\ &= -X_+(A^* + C^*H_+^*) - B_2^*B_2 - [(X_- - X_+)C^*D(D^*D)^{-2}D^*CX_+ \\ &\quad + (A+H_-C)X_+] - X_+C^*D(D^*D)^{-2}D^*CX_+ \\ &= -X_+(A^* + C^*H_+^*) - B_2^*B_2 - (A+H_+C)X_+ - X_+C^*D(D^*D)^{-2}D^*CX_+ = 0 \end{aligned}$$

Finally, using the same state space similarity as before, we compute

$$\begin{aligned} R &= \tilde{Q}'\tilde{Q}'' \\ &= \left(\begin{array}{c|cc} A+H_-C & -X_-C^*D(D^*D)^{-1} & B_2 \\ \hline P_- & 0 & I \end{array}\right) \\ &\times \left(\begin{array}{c|c} -A^* - C^*H_+^* & P_+^* \\ \hline (D^*D)^{-1}D^*CX_+ & 0 \\ -B_2^* & I \end{array}\right) \\ &= \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & P_+^* \\ -B_2B_2^* - X_-C^*D(D^*D)^{-2}D^*CX_+ & A+H_-C & B_2 \\ \hline -B_2^* & P_- & I \end{array}\right) \\ &= \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & P_+^* \\ 0 & A+H_-C & 0 \\ \hline -B_2^* - P_-X_+ & P_- & I \end{array}\right) \\ &= \left(\begin{array}{c|c} -A^* - C^*H_+^* & P_+^* \\ \hline -(P_- - P_+)X_+ & I \end{array}\right) \end{aligned}$$

Similarly, applying the similarity

$$\begin{pmatrix} I & 0 \\ X_- & I \end{pmatrix}$$

to

$$R = \left(\begin{array}{cc|c} -A^* - C^*H_+^* & 0 & P_+^* \\ -B_2B_2^* - X_-C^*D(D^*D)^{-2}D^*CX_+ & A+H_-C & B_2 \\ \hline -B_2^* & P_- & I \end{array}\right)$$

we get

$$R = \left(\begin{array}{c|c} A+H_-C & -X_-(P_-^* - P_+^*) \\ \hline P_- & I \end{array}\right) \qquad \square$$

The observation (93) as well as its proof are adapted from Pavon (1994). In this connection see also Lindquist and Picci (1991) and the reference to Molinari (1977). The dimensions of Q', Q'' that we computed have a geometric control significance. For this interpretation we refer to Fuhrmann and Gombani (1998).

4. Partially ordering inner functions

In the set of $p \times p$ inner functions in the right half plane there exists a natural partial order. Since we are working with row Hardy spaces, we say that $Q_1 \geq Q_2$ if $H_r(Q_1) \supset H_r(Q_2)$. In view of Beurling's theorem, this is equivalent to Q_2 being a right factor of Q_1, i.e the existence of a factorization of the form $Q_1 = QQ_2$ for some inner function Q. If we fix the inner function Q_1, the set of all its right factors is a lattice and can be parametrized in terms of the set of non-negative solutions of a corresponding homogeneous Riccati equation. For a complete analysis of this issue see Fuhrmann (1995) and Michaletzky (1998) who analyses the discrete time case. For some purposes, this partial order is satisfactory. In fact, with respect to this partial order, the set of $p \times p$ inner functions is clearly a lattice.

Independently of the Hardy space analysis, it has been known for a long time that the set of symmetric solutions of a Riccati equation is partially ordered under the usual partial order where $Q \leq P$ if and only if $(Qx, x) \leq (Px, x)$ for every vector x. In the standard coercive case, the analysis of spectral factorization problems via the positive real lemma led to the introduction of a partial order in the set of all stable, minimal spectral factors. This partial order also had a geometric representation (see Anderson 1973). Motivated by recent work on stochastic realization theory and spectral factorizations which deals with the weakening of the full rank assumptions on the spectral function and the squareness of the spectral factors, it became natural to consider more refined partial orders in the set of all rectangular, stable spectral factors, and hence in certain sets of inner functions. For this we refer to Lindquist and Picci (1985, 1991) and Fuhrmann and Gombani

(1998). We try to give a brief account of the background. We assume we are given a rational spectral function Φ, that is a $p \times p$ proper rational matrix function which is non-negative on the imaginary axis. We do not assume, as is usual, that Φ is regular on the imaginary axis. We do assume however that it is **weakly coercive**, in the sense that Φ has constant rank, $m_0 \le p$, on the extended imaginary axis, i.e. including at the point of infinity. Furthermore, we assume that Φ, which clearly satisfies $\Phi(s) = \Phi(-\bar{s})^*$, has McMillan degree $2n$. A $p \times m$ proper rational matrix function W is called a **spectral factor** of Φ if $\Phi = WW^*$. Here, as elsewhere $W^*(s) = W(-\bar{s})^*$. A spectral factor W is called stable (antistable) if $W \in H_+^\infty$ ($W \in H_-^\infty$).

It turns out that the study of the set of minimal spectral factors is facilitated if we study it in relation to four extremal spectral factors. These four spectral factors are determined by the requirement that all their poles are located in either the left or right half planes, and the same for the zeros.

It is well known that spectral factors exist. Moreover, using the Beurling–Lax–Halmos theorem, there exists a stable, minimum phase, or outer spectral factor, which we denote by W_-. Our weak coercivity assumption implies actually that W_- is left invertible over H_+^∞. In a completely analogous way, there exists an essentially unique antistable and maximum phase spectral factor $\bar{W}_+$, which has also dimension $p \times m_0$. By the same argument as before $\bar{W}_+$ has an antistable left inverse. The maximum phase, stable spectral factor W_+ can be obtained from $\bar{W}_+$ by a DSS factorization $\bar{W}_+ = W_+ K_+^*$ over H_+^∞. Similarly, the minimum phase, antistable spectral factor $\bar{W}_-$ is obtained from W_- by a DSS factorization $\bar{W}_- = W_- K_-$ over H_-^∞. Any stable, $p \times m$ spectral factor W has an essentially unique representation of the form $W = W_- \hat{Q}$, where $\hat{Q} \in H_+^\infty$ is an $m_0 \times m$ **rigid function**, i.e. satisfies $\hat{Q}(i\omega)\hat{Q}^*(i\omega) = I_{m_0}$. We suppose now that the column dimension m of the spectral factors is fixed. Nevertheless, it is convenient to have the extremal factors belong to this class. To obtain this, we define, given m, the **extended extremal spectral factors** by

$$W_-^e := \begin{pmatrix} W_- & 0 \end{pmatrix}, \qquad W_+^e := \begin{pmatrix} W_+ & 0 \end{pmatrix}$$

where both zero matrices are of size $p \times (m - m_0)$. The rigid function $\hat{Q}$ can be extended in an essentially unique way, to an inner function Q' of the same McMillan degree. This is summarized in the following proposition, quoted from Fuhrmann and Gombani (1998).

Proposition 7:

(1) *Let W_-^e and W_+^e be the extended, stable, minimum and maximum phase respectively, spectral factors. Given any minimal stable spectral factor W, there exist, essentially unique, inner functions Q', Q'', of minimal McMillan degree, for which*

$$\left.\begin{aligned} W &= W_-^e Q' \\ W_+^e &= W Q'' \end{aligned}\right\} \tag{115}$$

The inner functions Q', Q'' are uniquely determined by the normalization $Q'(\infty) = Q''(\infty) = I$. We shall refer to the factorization $W = W_-^e Q'$ as an **outer-inner factorization**.

(2) *Let $\bar{W}_-^e$ and $\bar{W}_+^e$ be the extended, antistable, minimum and maximum phase respectively, spectral factors. Given any minimal antistable spectral factor $\bar{W}$, there exist essentially unique inner functions $\bar{Q}', \bar{Q}''$ for which*

$$\left.\begin{aligned} \bar{W} &= \bar{W}_-^e \bar{Q}' \\ \bar{W}_+^e &= \bar{W} \bar{Q}'' \end{aligned}\right\} \tag{116}$$

The inner functions $\bar{Q}', \bar{Q}''$ are uniquely determined by the normalization $\bar{Q}'(\infty) = \bar{Q}''(\infty) = I$.

In the standard regular case, let W_-, W_+ be the stable, minimum and maximum phase spectral factor respectively. W_+ can be obtained from W_- by right multiplication by Q_+. The left factors Q' of Q_+ parametrize, for the spectral factor $W = W_- Q'$ the number of stable zeros that have been replaced by their antistable reflections. Since the factors W are regular, there is no difference between the treatment of zeros and poles. Order between two factors is determined essentially by how many zeros have been flipped which is a measure of closeness to the maximum phase factor W_+. The situation changes dramatically when the factors are rectangular or less than full rank. From the geometric control theory point of view, given a minimal realization of a minimal stable spectral factor W, then the maximal, output nulling, reachability subspace may be non-trivial. Thus, by multiplication on the right by inner functions, transmission zeros can decrease in number. The closeness to W_+ is now measured geometrically in terms of properties of subspaces associated with spectral factors. These subspaces are coinvariant subspaces canonically associated with the spectral factors. Of course each coinvariant subspace is associated with a unique, normalized inner function. Thus we expect to be able to describe the partial order in terms of inner functions.

This analysis was carried out in Fuhrmann and Gombani (1998) by functional techniques. In the rest of this paper we will redo some of the analysis, however using this time state space techniques.

In preparation for the main result concerning the equivalence of the various partial orders in the general case, we derive some preliminary results.

Proposition 8: *Let π_1 be a constant, rank m_0, projection on $\mathbb{C}^m$. We assume*

$$\pi_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

Let Q_α and Q_β be $m \times m$ normalized inner functions. Then the following conditions are equivalent.

(1) *We have*

$$H_r(Q_\beta)\pi_1 \subset H_r(Q_\alpha)\pi_1 \tag{117}$$

and the maps $H_r(Q_i)|\pi_1$ are injective.

(2) *Given a minimal realization*

$$Q_\beta = \left(\begin{array}{c|c} A_\beta & B_\beta \\ \hline -B_\beta P_\beta^{-1} & I \end{array}\right) = \left(\begin{array}{c|cc} A_\beta & B'_\beta & B''_\beta \\ \hline -B'^*_\beta P_\beta^{-1} & I & 0 \\ -B''^*_\beta P_\beta^{-1} & 0 & I \end{array}\right) \tag{118}$$

where $B'_\beta = B_\beta \pi_1$ and $B''_\beta = B_\beta(I - \pi_1)$, the pair (A_β, B'_β) is controllable and P_β is the positive definite solution of the Lyapunov equation

$$A_\beta P_\beta + P_\beta A_\beta^* + B_\beta B_\beta^* = 0 \tag{119}$$

There exists a minimal realization of Q_α of the form

$$Q_\alpha = \left(\begin{array}{c|c} A_\alpha & B_\alpha \\ \hline -B_\alpha^* P_\alpha^{-1} & I \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} A_\beta & 0 & B'_\beta & B''_\beta \\ A_{\beta\sigma} & A_\sigma & B'_\sigma & B''_\sigma \\ \hline -B'_\beta{*}\hat{P}_\beta^{-1} & -B'_\sigma{*}P_\sigma^{-1} & I & 0 \\ -B''_\beta{*}\hat{P}_\beta^{-1} & -B''_\sigma{*}P_\sigma^{-1} & 0 & I \end{array}\right) \tag{120}$$

with the pair

$$\begin{pmatrix} A_\beta & 0 \\ A_{\beta\sigma} & A_\sigma \end{pmatrix}, \begin{pmatrix} B'_\beta \\ B'_\sigma \end{pmatrix}$$

controllable. Moreover, the realization of Q_α can be chosen so that

$$P_\alpha = \begin{pmatrix} P_\beta & 0 \\ 0 & P_\sigma \end{pmatrix}$$

Proof: (2) $\Rightarrow$ (1)

Assume Q_α has the minimal realization as in (120). By (15), we have the following representation for $H_r(Q_\alpha)$, namely

$$H_r(Q_\alpha) = \{\xi(sI - A_\alpha)^{-1}B_\alpha | \xi \in \mathbb{C}^{n_1}\}$$

Using (120), it follows that

$$H_r(Q_\alpha)\pi_1 = \left\{\xi \begin{pmatrix} sI - A_\beta & 0 \\ -A_{\beta\sigma} & sI - A_\sigma \end{pmatrix}^{-1} \begin{pmatrix} B'_\beta \\ B'_\sigma \end{pmatrix} | \xi \in \mathbb{C}^{n_1}\right\}$$

$$\supset \{\eta(sI - A_\beta)^{-1}B'_\beta | \eta \in \mathbb{C}^{n_2}\} = H_r(Q_\beta)\pi_1$$

$H_r(Q_\alpha)|\pi_1$ is injective because of our assumption that the pair A_α, B_α is controllable. Similarly, $H_r(Q_\beta)|\pi_1$ is injective because of our assumption that the pair A_β, B'_β is controllable.

(1) $\Rightarrow$ (2)

We assume the inclusion (117) as well as the injectivity of the maps $H_r(Q_i)|\pi_1$. Clearly $H_r(Q_i)\pi_1$ are both coinvariant subspaces, hence of the form $H_r(\hat{Q}_i)$ for some $m_0 \times m_0$ inner functions $\hat{Q}_i$. Since $H_r(Q_i)\pi_1 = \{\xi(sI - A_i)^{-1}B_i | \xi \in \mathbb{C}^{n_i}\}$ and $B_i = \begin{pmatrix} B'_i & B''_i \end{pmatrix}$, we have

$$H_r(\hat{Q}_i) = H_r(Q_i)\pi_1 = \{\xi(sI - A_i)^{-1}B'_i | \xi \in \mathbb{C}^{n_i}\}$$

By our assumption of the injectivity of the restricted projections, the pairs (A_i, B'_i) are controllable. Moreover, the inclusion (117) translates into $H_r(\hat{Q}_\beta) \subset H_r(\hat{Q}_\alpha)$. This however is equivalent to the existence of an inner function S for which the factorization $\hat{Q}_\alpha = S\hat{Q}_\beta$ holds. We have of course

$$\hat{Q}_\alpha = \left(\begin{array}{c|c} A_\alpha & B'_\alpha \\ \hline -B'_\alpha{*}\hat{P}_\alpha^{-1} & I \end{array}\right), \quad \hat{Q}_\beta = \left(\begin{array}{c|c} A_\beta & B'_\beta \\ \hline -B'_\beta{*}\hat{P}_\beta^{-1} & I \end{array}\right)$$

where $\hat{P}_\alpha, \hat{P}_\beta$ are the positive definite solutions of the Lyapunov equations

$$\left.\begin{array}{r} A_\alpha \hat{P}_\alpha + \hat{P}_\alpha A_\alpha^* + B'_\alpha (B'_\alpha)^* = 0 \\ A_\beta \hat{P}_\beta + \hat{P}_\beta A_\beta^* + B'_\beta (B'_\beta)^* = 0 \end{array}\right\}$$

respectively. Let

$$S = \left(\begin{array}{c|c} A_\sigma & B'_\sigma \\ \hline -B'_\sigma{*}P'_\sigma{-1} & I \end{array}\right)$$

be a minimal realization. Then

$$\hat{Q}_\alpha = \left(\begin{array}{c|c} A_\alpha & B'_\alpha \\ \hline -B'_\alpha{*}\hat{P}_\alpha^{-1} & I \end{array}\right) = \left(\begin{array}{c|c} A_\sigma & B'_\sigma \\ \hline -B'_\sigma{*}\hat{P}_\sigma^{-1} & I \end{array}\right) \times \left(\begin{array}{c|c} A_\beta & B'_\beta \\ \hline -(B'_\beta)^*\hat{P}_\beta^{-1} & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} A_\beta & 0 & B'_\beta \\ A_{\beta\sigma} & A_\sigma & B'_\sigma \\ \hline -(B'_\beta)^*\hat{P}_\beta^{-1} & -B'_\sigma{*}\hat{P}_\sigma^{-1} & I \end{array}\right)$$

where $A_{\beta\sigma} = -B'_\sigma(B'_\beta)^*\hat{P}_\beta^{-1}$. This is a minimal realization of $\hat{Q}_\alpha$, for in the product of two inner functions there can be no zero-pole cancellations. Therefore the pair

$$\left(\begin{pmatrix} A_\beta & 0 \\ A_{\beta\sigma} & A_\sigma \end{pmatrix}, \begin{pmatrix} B'_\beta \\ B'_\sigma \end{pmatrix}\right)$$

is necessarily controllable. This implies that (A_α, B_α) can be chosen as in (120). Let us use the block description

$$P_\alpha = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix}$$

for the positive definite solution of the Lyapunov equation corresponding to (A_α, B_α). It is immediate that we have $P_{11} = P_\beta$. Applying a compatible, lower block triangular state space isomorphism of the form

$$\begin{pmatrix} I & 0 \\ -P_{12}^*P_{22} & I \end{pmatrix}$$

we can reduce P_α to block diagonal form

$$\begin{pmatrix} P_\beta & 0 \\ 0 & P_\sigma \end{pmatrix}$$

without changing the lower block triangularity of A_α and also leaving B'_β unchanged. Of course, $A_{\beta\sigma}$ and B'_σ have to be redefined appropriately. □

The next lemma gives some equivalent characterizations of a set of inner functions on which we will define a partial order.

Lemma 3: *Let Q_+ be a given inner function, and let π_1 be a constant projection such that $Q_+\pi_1Q_+^* = \pi_1$. Then the following sets are equal:*

(1) *The set of inner functions Q such that*

$$H_r(Q)\mathbf{P}_{H_+^2\pi_1} \subset H_r(Q_+) \quad (121)$$

and the projection is injective.

(2) *The set of inner functions Q such that $H_r(Q)|\mathbf{P}_{H_+^2\pi_1}$ is injective and*

$$\pi_1Q(s)^*Q(s)\pi_1 \geq \pi_1Q_+(s)^*Q_+(s)\pi_1 \quad \text{for } s \in \mathbb{C}_+ \quad (122)$$

(3) *The set of inner functions Q such that $H_r(Q)|\mathbf{P}_{H_+^2\pi_1}$ is injective and*

$$\pi_1Q(s)^{-1}Q(s)^{-*}\pi_1 \leq \pi_1Q_+(s)^{-1}Q_+(s)^{-*}\pi_1 \quad \text{for } s \in \mathbb{C}_+; \quad Q_+^*(s) \text{ is analytic} \quad (123)$$

Proof: $1 \Leftrightarrow 3$

The assumption $H_r(Q)\pi_1 \subset H_r(Q_+)$ means that $H_r(Q)\pi_1Q_+^* \subset H_-^2$ or, equivalently, that $Q\pi_1Q_+^* \in H_-^\infty$, which is the same as $Q_+\pi_1Q^* \in H_+^\infty$. We claim that this last function is contractive in the right half plane. By the maximum principle, it suffices to show that it is contractive on the imaginary axis. Indeed, for $f \in H_+^2$, we have

$$\|fQ_+\pi_1Q^*\| = \|fQ_+\pi_1\| \leq \|fQ_+\|\|\pi_1\| = \|f\|$$

This implies

$$Q_+(s)\pi_1Q(s)^{-1}Q(s)^{-*}\pi_1Q_+(s)^* \leq I \quad (124)$$

Multiplying this inequality on both sides by π_1 and using the commutation relation $\pi_1Q_+ = Q_+\pi_1$, we get (123).

Conversely, (123) implies (124). This in turn means that $Q\pi_1Q_+^* \in H_-^\infty$. Let $h \in H_-^2 \ominus H_-^2Q^*$, i.e. $hQ \in H_r(Q) \subset H_+^2$. Now $hQ\pi_1Q_+^* \in H_-^2$, i.e. $H_r(Q)\pi_1 \subset H_r(Q_+)$.

$2 \Rightarrow 1$

A finite dimensional coinvariant subspace $H_r(Q)$ is spanned by eigenfunctions and generalized eigenfunctions. Eigenfunctions are of the form $\eta/(s+\omega)$, with $\omega \in \mathbb{C}_+$ and satisfying $\eta Q(\omega)^* = 0$. Similarly, generalized eigenfunctions are of the form $\eta/(s+\bar{\omega})^k$, with $\omega \in \mathbb{C}_+$ and satisfying $\eta Q^{(j)}(\omega)^* = 0$ for $j = 0, \ldots, k-1$. Now inequality (122) implies

$$Q(s)^*Q(s) \geq \pi_1Q(s)^*Q(s)\pi_1 \geq \pi_1Q_+(s)^*Q_+(s)\pi_1$$

and hence $\eta Q(\omega)^* = 0$ implies $\eta\pi_1Q_+(\omega)^* = 0$. This means that $\eta/(s+\bar{\omega}) \in H_r(Q))$ implies that $\eta\pi_1/(s+\bar{\omega}) \in H_r(Q_+)$. This holds true also for generalized eigenfunctions. Indeed $\eta Q^{(j)}(\omega)^* = 0$ for $j = 0, \ldots, k-1$ implies $\eta\pi_1Q_+^{(j)}(\omega)^* = 0$ for $j = 0, \ldots, k-1$. Therefore $\eta/(s+\bar{\omega})^k \in H_r(Q))$ implies $\eta\pi_1/(s+\bar{\omega})^k \in H_r(Q_+)$, i.e. $H_r(Q)\pi_1 \subset H_r(Q_+)$.

$3 \Rightarrow 2$

Using the commutation relation $\pi_1Q_+ = Q_+\pi_1$, it is easily seen that (123) implies

$$Q_+(s)\pi_1Q(s)^{-1}Q(s)^{-*}\pi_1Q_+(s)^* \leq \pi_1$$

Now the function $T(s) = Q_+(s)\pi_1Q(s)^{-1}$ is meromorphic in the open right half plane, and by the previous inequality is actually bounded, even contractive. Thus all singularities are removable and T is analytic.

We have therefore $Q_+(s)\pi_1 = T(s)Q(s)$. From this relation we conclude that $Q(\omega)\eta^* = 0$ implies that $Q_+(\omega)\pi_1\eta^* = 0$. An easy induction argument shows that $Q^{(j)}(\omega)\eta^* = 0$ for $j = 0,\dots,k-1$ implies that $Q_+^{(j)}(\omega)\pi_1\eta^* = 0$ for $j = 0,\dots,k-1$. As before, this means that $H_r(Q)\pi_1 \subset H_r(Q_+)$. □

We will explicitly need the following.

Lemma 4: *Suppose*

$$\|g\mathbf{P}_{H_r(Q_\alpha)}\| \geq \|g\mathbf{P}_{H_r(Q_\beta)}\| \qquad \forall g \in H_r(Q_+) \qquad (125)$$

Then $H_r(Q_\beta)\mathbf{P}_{H_r(Q_+)} \subset H_r(Q_\alpha)\mathbf{P}_{H_r(Q_+)}$.

Proof: Since (125) implies that $g\mathbf{P}_{H(Q_\alpha)} = 0$ for $g \in H_r(Q_+)$ whenever $g\mathbf{P}_{H(Q_\beta)} = 0$, if we take the kernels of the previous projections in $H(Q_+)$, we obtain $[H_r(Q_\alpha)^\perp \cap H_r(Q_+)] \subset [H_r(Q_\beta)^\perp \cap H_r(Q_+)]$, or equivalently, $[(H_+^2 Q_\alpha \cap H_r(Q_+)] \supset [H_+^2 Q_\beta \cap H_r(Q_+)]$. This in turn, taking orthogonal complements in H_+^2, is equivalent to $H_r(Q_\beta) + H_+^2 Q_+ \subset H_r(Q_\alpha) + H_+^2 Q_+$. Applying the projection $P_{H_r(Q_+)}$ to this inclusion, we get $H_r(Q_\beta)\mathbf{P}_{H_r(Q_+)} \subset H_r(Q_\alpha)\mathbf{P}_{H_r(Q_+)}$, as wanted. □

We have next a theorem about the partial ordering on inner functions induced by a projection matrix π_1 acting in $\mathbb{C}^m$. This projection can be extended to all of H_+^2 by defining, for $f \in H_+^2$, $f\pi_1 := fP_{H_+^2\pi_1}$.

Theorem 3: *Let Q_+ be given, and let π_1 be a constant projection such that $H_r(Q_+) \subset \operatorname{Im}\pi_1$. Then the following partial orderings are equivalent:*

(1) *The set of inner functions Q such that $H_r(Q)\mathbf{P}_{H^2\pi_1}$ is injective and*

$$H_r(Q)\mathbf{P}_{H_+^2\pi_1} \subset H_r(Q_+) \qquad (126)$$

with the partial order defined by

$$Q_\beta \leq Q_\alpha \quad \text{if} \quad \|g\mathbf{P}_{H_r(Q_\beta)}\| \leq \|g\mathbf{P}_{H_r(Q_\alpha)}\| \qquad \forall g \in H_r(Q_+) \qquad (127)$$

(2) *The set of inner functions Q such that $H_r(Q)|\mathbf{P}_{H_+^2\pi_1}$ is injective and*

$$\pi_1 Q(s)^* Q(s)\pi_1 \geq \pi_1 Q_+(s)^* Q_+(s)\pi_1 \qquad \forall s \in \mathbb{C}_+ \quad (128)$$

with the partial order defined by

$$Q_\beta \leq Q_\alpha \quad \text{if} \quad \pi_1 Q_\beta(s)^* Q_\beta(s)\pi_1 \geq \pi_1 Q_\alpha(s)^* Q_\alpha(s)\pi_1 \qquad \forall s \in \mathbb{C}_+ \quad (129)$$

(3) *The set of inner functions Q such that $H_r(Q)|\pi_1$ is injective and*

$$\pi_1 Q(s)^{-1} Q(s)^{-*}\pi_1 \leq \pi_1 Q_+(s)^{-1} Q_+(s)^{-*}\pi_1 \qquad \forall s \in \mathbb{C}_+ \text{ for which } Q_+(s)^{-1} \text{ is analytic} \quad (130)$$

with the partial ordering defined by

$$Q_\beta \leq Q_\alpha \ \text{ if } \ \pi_1 Q_\beta(s)^{-1} Q_\beta(s)^{-*}\pi_1 \leq \pi_1 Q_\alpha(s)^{-1} Q_\alpha(s)^{-*}\pi_1 \qquad \forall s \in \mathbb{C}_+ \text{ for which } Q_+(s)^{-1} \text{ is analytic} \quad (131)$$

(4) *The set of inner functions*

$$Q = \left(\begin{array}{c|c} A & B \\ \hline -B^* P^{-1} & I \end{array}\right)$$

such that $H_r(Q)\mathbf{P}_{H_+^2\pi_1} \subset H_r(Q_+)$. The ordering is as follows: $Q_\beta \leq Q_\alpha$ if there exist minimal realizations

$$Q_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline -B_i^* P_i^{-1} & I \end{array}\right)$$

for $i = \beta,\alpha$ and, as usual, $A_i P_i + P_i A_i^* + B_i B_i^* = 0$, with

$$A_\alpha = \begin{pmatrix} A_\beta & 0 \\ A_{\rho\beta} & A_\beta \end{pmatrix} \qquad B_\alpha\pi_1 = \begin{pmatrix} B_\beta\pi_1 \\ B_\rho\pi_1 \end{pmatrix}$$

and, with the obvious choice of dimensions for the zero matrices,

$$\begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & 0 \end{pmatrix} \leq P_\alpha^{-1} \qquad (132)$$

Remark: Observe that the first three statements do not involve realizations, whereas in the fourth statement the choice of realization cannot be preassigned. This has to do with the fact that the above set as a differential manifold is the product of r sphere of dimension $m - m_0$, and a sphere cannot be represented with a single chart. Since in our case the chart is determined by the realization of Q_+, this cannot be the same for all the inner functions in the set (see Baratchart and Gombani 1994 for details).

Note also that the inclusion (126) together with the injectivity of $H_r(Q)|\pi_1$ imply that the set of points in the right half plane where $\det Q$ vanishes is included in the set of points where $\det Q_+$ vanishes. In particular, $Q(s)^{-1}$ is analytic whenever $Q_+(s)^{-1}$ is analytic.

Proof: In view of Lemma 3 all the sets of the partial orderings coincide. We have to show now that the orderings are equivalent.

(1) ⇔ (4)

Since statement (4) makes use of a particular realization, we have to show first that such a realization can be derived for the functions in statement (1). Then the equivalence of the two statements will be proved for the particular realization.

Clearly, (127) forces $g\mathbf{P}_{H_r(Q_\beta)} = 0$ for $g \in H_r(Q_+)$ whenever $g\mathbf{P}_{H_r(Q_\alpha)} = 0$. Applying Lemma 4, we get the inclusion $H_r(Q_\beta)\mathbf{P}_{H_r(Q_+)} \subset H_r(Q_\alpha)\mathbf{P}_{H_r(Q_+)}$. Taking into account the equality $H_r(Q'')\pi_1 = H_r(Q'')P_{H_r(Q_+^e)}$, proved in Fuhrmann and Gombani (1998), it follows that if

$$Q_+ = \left(\begin{array}{c|c} A_+ & B_+ \\ \hline -B_+^* P_+^{-1} & I \end{array}\right)$$

is a minimal realization, then applying twice Proposition 8, the first time to Q_β and Q_α and the second time to Q_α and Q_+, we can find realizations

$$Q_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline -B_i^* P_i^{-1} & I \end{array}\right)$$

for $i = \beta, \alpha$ which have the following nested structure

$$\left.\begin{array}{ll} A_\alpha = \begin{pmatrix} A_\beta & 0 \\ A_{\rho\beta} & A_\rho \end{pmatrix} & B_\alpha \pi_1 = \begin{pmatrix} B_\beta \pi_1 \\ B_\rho \pi_1 \end{pmatrix} \\ A_+ = \begin{pmatrix} A_\alpha & 0 \\ A_{\sigma\alpha} & A_\sigma \end{pmatrix} & B_+ \pi_1 = \begin{pmatrix} B_\alpha \pi_1 \\ B_\sigma \pi_1 \end{pmatrix} \end{array}\right\} \tag{133}$$

Denote now by $P_{21}^{(i)}$ the solution to the equation

$$A_+ P_{21}^{(i)} + P_{21}^{(i)} A_i^* + B_+ B_i^* = 0 \tag{134}$$

Then, if $g \in H_r(Q_+)$, it can be represented as $g = \xi_g (sI - A_+)^{-1} B_+$, for a suitable constant vector ξ_g. From Proposition 5,

$$\begin{aligned} g\mathbf{P}_{H_r(Q_i)} &= \xi_g (sI - A_+)^{-1} B_+ \mathbf{P}_{H_r(Q_i)} \\ &= \xi_g P_{21}^{(i)} P_i^{-1} (sI - A_i)^{-1} B_i \end{aligned} \tag{135}$$

and

$$\begin{aligned} \|g\mathbf{P}_{H_r(Q_i)}\|^2 &= \frac{1}{2\pi} \int_{\mathbb{I}} \xi_g P_{21}^{(i)} P_i^{-1} (i\omega I - A_i)^{-1} \\ &\quad \times B_i B_i^* (-i\omega I - A_i^*)^{-1} P_i^{-1} (P_{21}^{(i)})^* \xi_g^* \, \mathrm{d}\omega \\ &= \xi_g P_{21}^{(i)} P_i^{-1} (P_{21}^{(i)})^* \xi_g^* \end{aligned}$$

This means that

$$\begin{aligned} \|g\mathbf{P}_{H_r(Q_\beta)}\| \le g\mathbf{P}_{H_r(Q_\alpha)}\| &\Leftrightarrow P_{21}^{(\beta)} P_\beta^{-1} (P_{21}^{(\beta)})^* \\ &\le P_{21}^{(\alpha)} P_\alpha^{-1} (P_{21}^{(\alpha)})^* \end{aligned} \tag{136}$$

If $A_\beta = A_\alpha$, then $B_\beta \pi_1 = B_\alpha \pi_1$ and therefore it is simple to verify that $P_{21}^{(\beta)} = P_{21}^{(\alpha)}$. Now, since $H_r(Q_+)|P_{H_r(Q_i)}$ is surjective, it follows that the $P_{21}^{(i)}$ have full column rank, i.e. are left invertible. Thus (136) is verified if and only if $P_\beta^{-1} \le P_\alpha^{-1}$.

If on the other hand the decomposition (133) is non-trivial, then the dimension of A_β is smaller than that of A_α and therefore, for a suitable partition, we have $P_{21}^{(\alpha)} = \begin{pmatrix} X & Y \end{pmatrix}$. We can rewrite equation (134) for $i = \alpha$ as

$$0 = A_+ P_{21}^{(\alpha)} + P_{21}^{(\alpha)} A_\alpha^* + B_+ B_\alpha^* \tag{137}$$

$$= A_+ \begin{pmatrix} X & 0 \end{pmatrix} + \begin{pmatrix} X & 0 \end{pmatrix} \begin{pmatrix} A_\beta^* & 0 \\ 0 & 0 \end{pmatrix} + B_+ \begin{pmatrix} B_\beta^* & 0 \end{pmatrix} \tag{138}$$

$$+ A_+ \begin{pmatrix} 0 & Y \end{pmatrix} + \begin{pmatrix} 0 & Y \end{pmatrix} \begin{pmatrix} 0 & A_{\rho\beta}^* \\ 0 & A_\rho^* \end{pmatrix} + B_+ \begin{pmatrix} 0 & B_\rho^* \end{pmatrix} \tag{139}$$

It is easily seen by inspection that $X = P_{21}^{(\beta)}$; remembering that, in view of Proposition 8, we can assume that P_α is block diagonal, i.e.

$$P_\alpha = \begin{pmatrix} P_\beta & 0 \\ 0 & P_\rho \end{pmatrix}$$

and observing, as above, that the matrix $P_{21}^{(\alpha)}$ has full column-rank, and therefore is left invertible, we conclude that the inequality

$$\begin{aligned} P_{21}^{(\alpha)} P_\alpha^{-1} (P_{21}^{(\alpha)})^* &= \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & P_\rho^{-1} \end{pmatrix} \begin{pmatrix} X^* \\ Y^* \end{pmatrix} \\ &\ge \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X^* \\ Y^* \end{pmatrix} \end{aligned}$$

holds if and only if

$$\begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & 0 \end{pmatrix} \le P_\alpha^{-1}$$

(2) $\Leftrightarrow$ (4)

We need now the following equality: let

$$Q = \left(\begin{array}{c|c} A & B \\ \hline -B^* P^{-1} & I \end{array}\right)$$

Then

$$\begin{aligned}
Q(s)^*Q(s) &= [I - B^*(\bar{s}I - A^*)^{-1}P^{-1}B] \\
&\quad \times [I - B^*P^{-1}(sI - A)^{-1}B] \qquad (140) \\
&= I + B^*(\bar{s}I - A^*)^{-1}P^{-1}BB^*P^{-1}(sI - A)^{-1}B \\
&\quad - B^*(\bar{s}I - A^*)^{-1}P^{-1}B - B^*P^{-1}(sI - A)^{-1}B \\
&= I + B^*(\bar{s}I - A^*)^{-1}P^{-1}BB^*P^{-1}(sI - A)^{-1}B \\
&\quad - B^*(\bar{s}I - A^*)^{-1}[P^{-1}(sI - A) \\
&\quad + (\bar{s}I - A^*)P^{-1}](sI - A)^{-1}B \\
&= I + B^*(\bar{s}I - A^*)^{-1}P^{-1}BB^*P^{-1}(sI - A)^{-1}B \\
&\quad - B^*(\bar{s}I - A^*)^{-1}[-P^{-1}A - A^*P^{-1}] \\
&\quad \times (sI - A)^{-1}B \\
&\quad - (s + \bar{s})B^*(\bar{s}I - A^*)^{-1}P^{-1}(sI - A)^{-1}B \\
&= I - (s + \bar{s})B^*(\bar{s}I - A^*)^{-1}P^{-1}(sI - A)^{-1}B
\end{aligned} \qquad (141)$$

In conclusion,

$$\left.\begin{aligned}
&\pi_1 Q_\beta(s)^* Q_\beta(s)\pi_1 \geq \pi_1(Q_\alpha(s))^* Q_\alpha(s)\pi_1 \\
&\Leftrightarrow \\
&(s + \bar{s})\pi_1 B_\beta^*(\bar{s}I - A_\beta^*)^{-1}P_\beta^{-1}(sI - A_\beta)^{-1}B_\beta\pi_1 \\
&\quad \leq (s + \bar{s})\pi_1 B_\alpha^*(\bar{s}I - A_\alpha^*)^{-1} \\
&\quad \times P_\alpha^{-1}(sI - A_\alpha)^{-1}B_\alpha\pi_1
\end{aligned}\right\} \qquad (142)$$

So, if $B_\beta\pi_1 = B_\alpha\pi_1$, also $A_\beta = A_\alpha$ and we are done, since (142) holds for all $s \in \mathbb{C}_+$ if and only if

$$P_\beta^{-1} \leq P_\alpha^{-1}$$

If the two inner functions have a different McMillan degree, from (140) we know from Proposition 8 that there exist realizations of

$$Q_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline -B_i^* P_i^{-1} & I \end{array}\right)$$

for $i = \beta, \alpha$ such that

$$A_\alpha = \begin{pmatrix} A_\beta & 0 \\ A_{\rho\beta} & A_\rho \end{pmatrix} \qquad B_\alpha\pi_1 = \begin{pmatrix} B_\beta\pi_1 \\ \tilde{B}_\rho\pi_1 \end{pmatrix}$$

Therefore the middle term of (142) can be written as

$$\begin{aligned}
&(s + \bar{s})\begin{pmatrix} \pi_1 B_\beta^* & \pi_1 B_\rho^* \end{pmatrix}(\bar{s}I - A_\alpha^*)^{-1}\begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\
&\qquad \times (sI - A_\alpha)^{-1}\begin{pmatrix} B_\beta\pi_1 \\ B_\rho\pi_1 \end{pmatrix}
\end{aligned}$$

and thus again (142) is equivalent to (132)

(3) $\Leftrightarrow$ (4)

For this we need another equality. Let

$$Q = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

Then, since Q is inner, we have $Q(s)^{-1} = Q^*(s) = Q(-\bar{s})^*$. In state space terms, this is equivalent to the representations

$$Q(s)^{-1} = \left(\begin{array}{c|c} A + BB^*P^{-1} & B \\ \hline B^*P^{-1} & I \end{array}\right) = \left(\begin{array}{c|c} -A^* & P^{-1}B \\ \hline B^* & I \end{array}\right)$$

That these two realizations are isomorphic is a simple consequence of the fact that P is the solution of the Lyapunov equation $AP + PA^* + BB^* = 0$. We compute

$$\begin{aligned}
&Q(s)^{-1}Q(s)^{-*} \\
&= [I + B^*(sI + A^*)^{-1}P^{-1}B] \\
&\quad \times [I + B^*P^{-1}(\bar{s}I + A)^{-1}B] \\
&= I + B^*(sI + A^*)^{-1}P^{-1}BB^* \\
&\quad \times P^{-1}(\bar{s}I + A)^{-1}B \\
&\quad + B^*(sI + A^*)^{-1}P^{-1}B + B^*P^{-1}(\bar{s}I + A)^{-1}B \\
&= I + B^*(sI + A^*)^{-1}P^{-1}BB^*P^{-1}(\bar{s}I + A)^{-1}B \\
&\quad - B^*(sI + A^*)^{-1}[P^{-1}(\bar{s}I + A) + (sI + A^*)P^{-1}](\bar{s}I + A)^{-1}B \\
&= I + B^*(sI + A^*)^{-1}P^{-1}BB^*P^{-1}(\bar{s}I + A)^{-1}B \\
&\quad + B^*(sI + A^*)^{-1}[P^{-1}A + A^*P^{-1}](\bar{s}I + A)^{-1}B \\
&\quad + (s + \bar{s})B^*(sI + A^*)^{-1}P^{-1}(\bar{s}I + A)^{-1}B \\
&= I + (s + \bar{s})B^*(sI + A^*)^{-1}P^{-1}(\bar{s}I + A)^{-1}B \qquad (143)
\end{aligned}$$

We can now write an inequality similar to (142), for all $s \in \mathbb{C}_+$ for which both Q_β^{-1} and Q_α^{-1} are analytic

$$\left.\begin{aligned}&\pi_1 Q_\beta(s)^{-1} Q_\beta(s)^{-*}\pi_1 \le \pi_1 Q_\alpha(s)^{-1} Q_\alpha(s)^{-*}\pi_1\\ &\Leftrightarrow\\ &(s+\bar{s})\pi_1 B_\beta^*(sI+A_\beta^*)^{-1}P_\beta^{-1}(\bar{s}I+A_\beta)^{-1}B_\beta\\ &\le (s+\bar{s})B_\alpha^*(sI+A_\alpha^*)^{-1}\\ &\times P_\alpha^{-1}(\bar{s}I+A_\alpha)^{-1}B_\alpha\end{aligned}\right\} \tag{144}$$

Again, if $B_\beta\pi_1 = B_\alpha\pi_1$, also $A_\beta = A_\alpha$ and we are done, since (144) holds for all $s \in \mathbb{C}_+$ for which Q_β^{-1} and Q_α^{-1} are analytic if and only if

$$P_\beta^{-1} \le P_\alpha^{-1}$$

If the two inner functions have a different McMillan degree, from (143) we know from Proposition 8 that there exist realizations of

$$Q_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline -B_i^* P_i^{-1} & I \end{array}\right)$$

for $i = \beta, \alpha$ such that

$$A_\alpha = \begin{pmatrix} A_\beta & 0 \\ A_{\rho\beta} & A_\rho \end{pmatrix} \qquad B_\alpha\pi_1 = \begin{pmatrix} B_\beta\pi_1 \\ B_\rho\pi_1 \end{pmatrix}$$

Therefore, the middle term of (144) can be written as

$$(s+\bar{s})\begin{pmatrix}\pi_1 B_\beta^* & \pi_1 B_\rho^*\end{pmatrix}(sI+A_\alpha^*)^{-1}\begin{pmatrix} P_\beta^{-1} & 0 \\ 0 & 0\end{pmatrix} \times (\bar{s}I+A_\alpha)^{-1}\begin{pmatrix} B_\beta\pi_1 \\ B_\rho\pi_1\end{pmatrix}$$

and thus again (144) is equivalent to (132) and the proof is complete.

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