ELSEVIER

Systems & Control Letters 44 (2001) 127–134

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

On behavior homomorphisms and system equivalence

P.A. Fuhrmann [1,2]

Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva, Israel

Received 5 September 2000; received in revised form 3 May 2001

Abstract

In this paper we introduce the concept of a behavior homomorphisms and isomorphisms and use it to present a unified approach to the study of equivalence of different behavior representations concentrating on a special representation we call normalized ARMA (NARMA) representation. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Behaviors; Behavior homomorphisms; Behavior equivalence

1. Introduction

The object of this paper is to fill what seems to be a gap in the behavioral literature and that is the concepts of a behavior homomorphism and, more specifically, that of a behavior isomorphism. These seem to be basic objects and, once they are characterized, the study of the equivalence of different behavior representations is simplified. The question of equivalence is to find characterizations of two system representations which give rise to the same behavior. These problems are not new. The Kalman state space isomorphism, see [12], result is of this type. So is Rosenbrock's [17] notion of strict system equivalence for polynomial matrix descriptions (PMD) and its modification known as Fuhrmann system equivalence, see [3,11,15]. In the context of behaviors, of particular importance are the contributions of Hinrichsen and Prätzel-Wolters [8,9], Kuijper [13,14] and Schumacher [20]. In fact, part of the motivation for the present work is due to several, highly suggestive, formulas in Kuijper's thesis. More recently, in the context of multidimensional systems, [23] as well as [10] contain similar ideas.

The setting we chose to work in is that of discrete time systems. This setting is of importance in its own right, e.g. the application of behaviors in the area of coding theory, see [18]. Choosing to work in discrete time allows us to develop the theory over an arbitrary field F. We follow Kalman in choosing the vector space of truncated Laurent series $F^m((s^{-1}))$ to be the extended signal space. In the direct sum decomposition $F^m((s^{-1}))=F^m[s]\oplus s^{-1}F^m[[s^{-1}]]$ $F^m[s]$ is identified with the space of past trajectories whereas $s^{-1}F^m[[s^{-1}]]$ with the space of future ones. Choosing our time set to be $\mathbb{Z}_+$, the space $s^{-1}F^m[[s^{-1}]]$ becomes our signal space.

Following Willems [21,22], a discrete time behavior is a complete, linear, shift invariant subspace of $s^{-1}F^m[[s^{-1}]]$. For us completeness of a shift invariant subspace $\mathscr{V}$ of $s^{-1}F^m[[s^{-1}]]$ is equivalent to $({}^{\perp}\mathscr{V})^{\perp}=\mathscr{V}$. Here we use the characterization of $s^{-1}F^m[[s^{-1}]]$ as the dual space of $F^m[s]$. For the details we refer to a forthcoming paper, [7]. These behaviors have been shown by Willems to be those subspaces that admit an autoregressive, or kernel, representation of the form $\mathscr{B}=\operatorname{Ker}P(\sigma)$. The

[1] Earl Katz Family Chair in Algebraic System Theory.

[2] Partially supported by GIF under Grant No. gI-526-034.

E-mail address: paf@math.bgu.ac.il (P.A. Fuhrmann).

PII: S0167-6911(01)00133-5

polynomial matrix $P(s)$, under suitable minimality conditions namely that it is of full row rank, is uniquely defined up to a left unimodular factor. This solves the problem of equivalence of AR representations of a given behavior. However, AR representations are far from being the only representations possible. In fact there are quite a large number of representation classes of behaviors. In each such class we would like to derive a characterization of equivalence, where two representations are called equivalent if they represent the same behavior. Our aim is to use the notion of behavior isomorphism as a principal tool for the uniform derivation of equivalence characterizations. However, due to the space limitations, we will derive the equivalence characterization only for systems given in NARMA form. This characterization is closely related to Fuhrmann system equivalence, see [3], as well as to the analysis of state feedback in the PMD context, see [16]. Moreover, essentially all other equivalence characterizations follow easily from it. The full details of this derivation will appear elsewhere, see [7].

Given a behavior $\mathscr{B}$, it is natural to consider the map $\sigma^{\mathscr{B}}$ which is defined as the restriction of the (backward) shift σ to the behavior. Given two behaviors $\mathscr{B}_i$, $i=1,2$, a behavior homomorphism is defined to be a map $Z:\mathscr{B}_1 \rightarrow \mathscr{B}_2$ satisfying $Z\sigma^{\mathscr{B}_1}=\sigma^{\mathscr{B}_2}Z$. Thus behavior homomorphisms are intertwining maps and their analysis relate to the celebrated commutant lifting theorem of Sarason and Sz.-Nagy-Foias. Thus it is expected that the method presented in this paper will be found to be applicable in other contexts, most notably in the setting of Hardy spaces. Some of the relevant mathematics for this can be found in [5,6].

We recall that the approach to the study of equivalence in the setting of polynomial matrix descriptions of linear systems taken in [3] is based on the characterization of isomorphism of two polynomial models as derived in [2]. The derivation of this result is split into the characterization of all module homomorphisms of two polynomial models and, once this has been established, the characterization of invertibility conditions on the homomorphisms in terms of coprimeness conditions. Our aim in this paper is to adopt this philosophy and apply it to the study of behaviors. The principal insight is the fact that a behavior is a generalization of a rational model, see [2]. Thus the homomorphisms of rational models can be easily derived from the characterization of the homomorphism of polynomial models. This gives us a clue to the characterization of behavior homomorphisms which we derive in Section 3. Finally, we study in depth the behavior isomorphisms of two behaviors given in NARMA form. In [20], NARMA systems are denoted by AR/MA systems. The particular importance of NARMA systems stems from the fact that any behavior given in AR or ARMA representation can easily be put into NARMA form. The characterization of behavior homomorphisms and isomorphisms between two behaviors in NARMA representations will be the main topic of this paper. This is closely related to Fuhrmann strict system equivalence. This result allows the uniform derivation of behavior isomorphism results for most behavior representations. This is beyond the scope of the present paper. The full details of this approach will be presented in [7].

2. Preliminaries

Let F denote an arbitrary field. We will denote by F^m the space of all m-vectors with coordinates in F. Let π_+ and π_- denote the projections of $F^m((s^{-1}))$ the space of truncated Laurent series on $F^m[s]$ and $s^{-1}F^m[[s^{-1}]]$, the space of formal power series vanishing at infinity, respectively. Since

$$F^m((s^{-1}))=F^m[s]\oplus s^{-1}F^m[[s^{-1}]] \tag{1}$$

π_+ and π_- are complementary projections. Given a nonsingular polynomial matrix D in $F^{m\times m}[s]$ we define two projections π_D in $F^m[s]$ and π^D in $s^{-1}F^m[[s^{-1}]]$ by

$$\pi_D f=D\pi_-D^{-1}f \quad \text{for } f\in F^m[s], \tag{2}$$

$$\pi^D h=\pi_-D^{-1}\pi_+Dh \quad \text{for } h\in s^{-1}F^m[[s^{-1}]] \tag{3}$$

and define two linear subspaces of $F^m[s]$ and $s^{-1}F^m[[s^{-1}]]$ by

$$X_D=\operatorname{Im}\pi_D \tag{4}$$

and

$$X^D=\operatorname{Im}\pi^D. \tag{5}$$

In X_D we define a map S_D by

$$S_Df=\pi_D sf \quad \text{for } f\in X_D. \tag{6}$$

Thus X_D is has an $F[s]$ module structure given by $p\cdot f=p(S_D)f=\pi_D pf$. Similarly, we introduce in X^D a module structure, given by

$$S_Dh=\pi_- sh \quad \text{for } h\in X^D. \tag{7}$$

To conform with behavior notation we shall actually use σ^D for S^D.

For the context we are working in, that is the extended signal space $F^m((s^{-1}))$ a complete duality theory has been developed in [4]. Given $f,g \in F^m((s^{-1}))$ we define a pairing

$$[f,g] = \sum_{j=-\infty}^{\infty} [f_j, g_{-j-1}]. \tag{8}$$

It is clear that [,] is a bilinear form on $F^m((s^{-1})) \times F^m((s^{-1}))$. It is well defined as in the defining sum at most a finite number of terms are nonzero. Also this form is *nondegenerate* in the sense that $[f,g]=0$ for all $g \in F^m((s^{-1}))$ if and only if $f=0$.

Given a pair of polynomial matrices K_2, L_1 we say that there exists a *doubly unimodular embedding*, if there exist polynomial matrices K_1, L_2 such that

$$\begin{pmatrix} K_1(s) \\ K_2(s) \end{pmatrix} (L_1(s) \quad L_2(s)) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{9}$$

with both matrices on the left unimodular.

The following proposition gives a characterization of the existence of a doubly unimodular embedding.

Proposition 2.1. *Given a pair of polynomial matrices K_2, L_1. Then*

1. *There exists a doubly unimodular embedding, if and only if K_2 is left prime, L_1 right prime and*

$$\operatorname{Ker} K_2(s) = \operatorname{Im} L_1(s). \tag{10}$$

2. *There exists a doubly unimodular embedding for K_2 and L_1 if and only if there exists a doubly unimodular embedding for*

$$\begin{pmatrix} K_2(s) & 0 \\ 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} L_1(s) \\ 0 \end{pmatrix}.$$

3. *Given polynomial matrices satisfying*

$$N_2 M_1 = M_2 N_1, \tag{11}$$

with M_1, M_2 square and nonsingular. Then a doubly unimodular embedding for

$$(-N_2 \quad M_2), \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}$$

exists if and only if M_1, N_1 are right coprime and M_2, N_2 are left coprime.

Related results have been derived by Bisiacco and Valcher [1].

3. Behavior homomorphisms

As mentioned in the introduction, the principal insight to the analysis of behavior homomorphisms is the fact that a behavior is a generalization of a rational model, see [2]. This we try to explain in the following. Given a nonsingular polynomial matrix D, the rational model X^D is characterized by $X^D = \operatorname{Im} \pi^D$, with the projection $\pi^D : s^{-1}F^m[[s^{-1}]] \to s^{-1}F^m[[s^{-1}]]$ defined by $\pi^D h = \pi_- D^{-1} \pi_+ D h$. Clearly $\pi^D h = h$ if and only if Dh is a polynomial vector. Equivalently, using behavioral notation, if and only if $\pi_- Dh = D(\sigma)h = 0$, i.e.

$$X^D = \operatorname{Ker} D(\sigma). \tag{12}$$

So we see that rational models are identical to a subclass of behaviors, more specifically to the subclass of autonomous behaviors.

Now a rational model X^D is related to a polynomial model X_D via a multiplication map $\rho_D h = Dh$ for $h \in s^{-1}F^m[[s^{-1}]]$. In fact we have $\rho_D S^D = S_D \rho_D$. Thus the isomorphism of two polynomial models can be translated into the isomorphism of the corresponding rational models. So let us consider two nonsingular polynomial matrices D_1, D_2. If $Z : X^{D_1} \to X^{D_2}$ is an $F[s]$-module homomorphism, then $\bar{Z} : X_{D_1} \to X_{D_2}$ defined by $\bar{Z} f = D_2 Z D_1^{-1} f$, with $f \in X_{D_1}$ is given by $\bar{Z} f = \pi_{D_2} U f$ and the intertwining relation $UD_1 = D_2 V$ holds for some polynomial matrices U, V. Now, for $h \in X^{D_1}$, we have

$$\begin{aligned} Zh &= D_2^{-1} \bar{Z} D_1 h = D_2^{-1} \pi_{D_2} U D_1 h \\ &= D_2^{-1} D_2 \pi_- D_2^{-1} U D_1 h = \pi_- V h \\ &= V(\sigma) h \end{aligned}$$

or $Zh = V(\sigma)h$, with $UD_1 = D_2 V$ holding.

The invertibility properties of Z are the same as for $\bar{Z}$. Hence, using the results of Fuhrmann [2], Z is injective if and only if V, D_1 are right coprime and Z is surjective if and only if U, D_2 are left coprime.

With applications to behavior theory in mind, we want to extend the previous theorems concerning homomorphisms of polynomial and rational models and their invertibility properties. Note that for the case of a nonsingular polynomial matrix D, the polynomial model X_D is isomorphic to the quotient module $F^m[s]/DF^m[s]$, and this quotient module is a torsion module. Similarly, the rational model X^D is a torsion submodule of $s^{-1}F^m[[s^{-1}]]$. We generalize these results by dropping the nonsingularity assumptions.

An $F[s]$-submodule L of $F^p[s]$ has a representation of the form $L = MF^m[s]$, with $M \in F^{p\times m}[s]$. If we assume that M has full column rank, then M is uniquely defined up to a right unimodular factor. Given a $p \times m$ polynomial matrix $M(s)$ and $f \in F^p[s]$, we shall denote by $[f]_M$ the equivalence class of f in the quotient module $F^p[s]/MF^m[s]$. We denote by π_M the canonical projection of $F^p[s]$ onto $F^p[s]/MF^m[s]$, i.e. $\pi_M f = [f]_M$. We define the shift operator $S_M : F^p[s]/MF^m[s] \to F^p[s]/MF^m[s]$ by

$$S_M[f]_M = [sf]_M. \tag{13}$$

Thus we have the following generalization of Theorem 4.5 in [2].

Theorem 3.1. *Let $M \in F^{p\times m}[s]$ and $\bar{M} \in F^{\bar{p}\times\bar{m}}[s]$ be full column rank.*

Then $Z : F^p[s]/M(s)F^m[s] \to F^{\bar{p}}[s]/\bar{M}(s)F^{\bar{m}}[s]$ is an $F[s]$-homomorphism if and only if there exist $U \in F^{\bar{p}\times p}[s]$ and $V \in F^{\bar{m}\times m}[s]$ such that

$$UM = \bar{M}V \tag{14}$$

and

$$Z[f]_M = [Uf]_{\bar{M}}. \tag{15}$$

Proof. If Z is defined as above, then we have

$$ZS_M[f]_M = Z[sf]_M = [Usf]_{\bar{M}}$$
$$= [sUf]_{\bar{M}} = S_M[Uf]_{\bar{M}} = S_{\bar{M}}Z[f]_M,$$

i.e. Z is an $F[s]$-homomorphism.

Define the map $Z_1 : F^p[s] \to F^p[s]/\bar{M}(s)F^m[s]$ by $Z_1 f = Z[f]_M$, for $f \in F^p[s]$. Clearly

$$Z_1S_+f = Z_1[sf]_M = [Usf]_{\bar{M}}$$
$$= [sUf]_{\bar{M}} = S_M[Uf]_{\bar{M}} = S_MZ_1f,$$

i.e.

$$Z_1S_+ = S_MZ_1. \tag{16}$$

Let $e_1, \ldots, e_p$ be the standard basis elements in F^p. Let $Z_1e_i = [u_i]_{\bar{M}}$ with $u_i \in F^m[s]$. The u_i are fixed but not uniquely determined. Let U be the $p \times m$ polynomial matrix whose columns are the u_i. It is easy to check that by defining $\bar{Z} : F^p[s] \to F^p[s]$ via $\bar{Z}f = Uf$ for $f \in F^p[s]$ we have obtained (15).

Finally, since we have $Z\pi_M = \pi_{\bar{M}}\bar{Z}$, it follows that $\bar{Z}\,\mathrm{Ker}\,\pi_M \subset \mathrm{Ker}\,\pi_{\bar{M}}$, or $\bar{Z}M(s)F^{p\times m}[s] \subset F^{p\times m}[s]$. This, by a standard argument, implies the existence of a polynomial matrix V for which (14) holds. □

Given a $p\times m$ polynomial matrix M, then $\mathrm{Ker}\,M(\sigma)$ is a submodule of $s^{-1}F^m[[s^{-1}]]$. We define the restricted shift map $\sigma^M : \mathrm{Ker}\,M(\sigma) \to \mathrm{Ker}\,M(\sigma)$ by $\sigma^M = \sigma \mid \mathrm{Ker}\,M(\sigma)$ or $\sigma^M h = \pi_- sh = \sigma h$. By a judicious use of duality we can state the analogue, in the rational model setting, of Theorem 3.1.

Theorem 3.2. *Let $M \in F^{p\times m}[s]$ and $\bar{M} \in F^{\bar{p}\times\bar{m}}[s]$ be of full row rank. Then $\mathrm{Ker}\,M(\sigma)$ is an $F[s]$-submodule of $s^{-1}F^m[[s^{-1}]]$ and $\mathrm{Ker}\,\bar{M}(\sigma)$ is an $F[s]$-submodule of $s^{-1}F^{\bar{m}}[[s^{-1}]]$. Moreover Z: $\mathrm{Ker}\,M(\sigma) \to \mathrm{Ker}\,\bar{M}(\sigma)$ is an $F[s]$-homomorphism, i.e. satisfies $Z\sigma^M = \sigma^{\bar{M}}Z$, if and only if there exist $U \in F^{\bar{p}\times p}[s]$ and V in $F^{\bar{m}\times m}[s]$ such that*

$$U(s)M(s) = \bar{M}(s)V(s) \tag{17}$$

and

$$Zh = V(\sigma)h, \quad h \in \mathrm{Ker}\,M(\sigma). \tag{18}$$

Proof. Let $h \in \mathrm{Ker}\,M(\sigma)$. Then $M(\sigma)(\sigma h) = \sigma(M(\sigma)h) = 0$, i.e. $\sigma h \in \mathrm{Ker}\,M(\sigma)$ which shows that it is a submodule. Similarly for $\mathrm{Ker}\,\bar{M}(\sigma)$.

Let Z be defined by (18), with (17) holding. Then, for $h \in \mathrm{Ker}\,M(\sigma)$, $\bar{M}(\sigma)Zh = \bar{M}(\sigma)(V(\sigma)h) = U(\sigma)(M(\sigma)h) = 0$, i.e. $Zh \in \mathrm{Ker}\,\bar{M}(\sigma)$. Moreover, we compute

$$Z\sigma^M h = V(\sigma)\sigma h = \sigma V(\sigma)h = \sigma^{\bar{M}}Zh,$$

that is Z is an $F[s]$-homomorphism.

Conversely, assume Z: $\mathrm{Ker}\,M(\sigma) \to \mathrm{Ker}\,\bar{M}(\sigma)$ is an $F[s]$-homomorphism. For a linear space X and a subspace $\mathscr{V} \subset X$, we have the isomorphism $\mathscr{V}^* \simeq X^*/\mathscr{V}^\perp$. We note that

$$(\mathrm{Ker}\,M(\sigma))^\perp = \tilde{M}(s)F^p[s]$$

and this leads to

$$(\mathrm{Ker}\,M(\sigma))^* = F^m[s]/\tilde{M}F^p[s]. \tag{19}$$

The identity $Z\sigma^M = \sigma^{\bar{M}}Z$ leads to $Z^*S_{\tilde{\bar{M}}} = S_{\tilde{M}}Z^*$, that is Z^* is an $F[s]$-module homomorphism. By Theorem 3.1, there exist polynomial matrices $U \in F^{\bar{p}\times p}$ and $V \in F^{\bar{m}\times m}$, satisfying $\tilde{V}\tilde{\bar{M}} = \tilde{M}\tilde{U}$, which is equivalent to (17), and for which

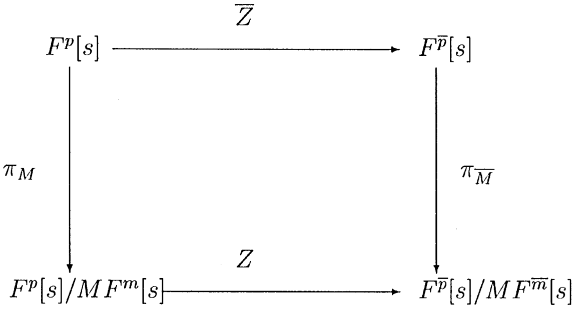
$$Z^*[f]_{\tilde{\bar{M}}} = [f]_{\tilde{M}}.$$

We can easily check now that necessarily Z: $\mathrm{Ker}\,M(\sigma) \to \mathrm{Ker}\,\bar{M}(\sigma)$ is given by (18). □

Both Theorem 3.1 and Theorem 3.2 have an interpretation as lifting homomorphism results. Theorem 3.1 can be restated as follows.

Theorem 3.3. *Let $M \in F^{p\times m}[s]$ and $\bar{M} \in F^{\bar{p}\times\bar{m}}[s]$ have full column rank.*

Then any $F[s]$-homomorphism $Z : F^p[s]/M(s) F^m[s] \to F^{\bar{p}}[s]/\bar{M}(s)F^{\bar{m}}[s]$ can be lifted to an $F[s]$-homomorphism $\bar{Z} : F^p[s] \to F^{\bar{p}}[s]$ such that the following diagram is commutative:



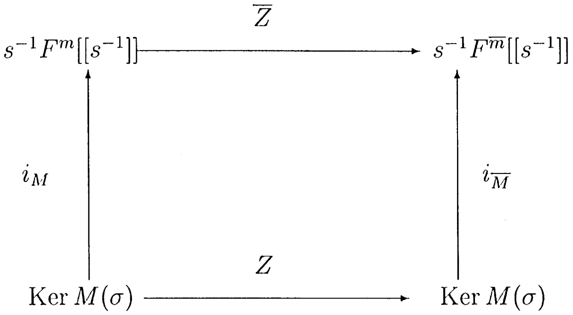
Here π_M is the canonical projection defined by

$$\pi_M f = [f]_M, \quad f \in F^p[s]. \tag{20}$$

Proof. Define $\bar{Z}f = Uf$ for $f \in F^p[s]$. □

In the same way, Theorem 3.2 can be restated as follows.

Theorem 3.4. *Let $M \in F^{p\times m}[s]$ and $\bar{M} \in F^{\bar{p}\times\bar{m}}[s]$ be of full row rank. Then any $F[s]$-homomorphism, Z: $\operatorname{Ker} M(\sigma) \to \operatorname{Ker} \bar{M}(\sigma)$ can be lifted to an $F[s]$-homomorphism $\bar{Z} : s^{-1}F^m[[s^{-1}]] \to s^{-1}F^{\bar{m}}[[s^{-1}]]$ such that the following diagram is commutative*:



Here i_M and $i_{\bar{M}}$ are the natural embedding maps.

Proof. Define $\bar{Z} = V(\sigma)$. □

The next theorem gives a characterization of the invertibility properties of the module homomorphisms introduced in Theorem 3.1. Due to the fact that we are dealing with rectangular polynomial matrices, there is an asymmetry between the conditions of injectivity and surjectivity.

Theorem 3.5. *Let $M \in F^{p\times m}[s]$ and $\bar{M} \in F^{\bar{p}\times\bar{m}}[s]$ be of full column rank. Let $Z : F^p[s]/M(s)F^m[s] \to F^{\bar{p}}[s]/\bar{M}(s)F^{\bar{m}}[s]$ be an $F[s]$-homomorphism defined by* (15) *with* (14) *holding for some $\bar{U} \in F^{\bar{p}\times p}[s]$ and $U \in F^{\bar{m}\times m}[s]$. Then*

1. *Z is injective if and only if U, M are right coprime and*

$$\operatorname{Ker}\,(-\bar{U}(s) \quad \bar{M}(s)) = \operatorname{Im}\begin{pmatrix} M(s) \\ U(s) \end{pmatrix}. \tag{21}$$

2. *Z is surjective if and only if $\bar{U}, \bar{M}$ are left coprime.*
3. *Z as defined above is the zero map if and only if, for some appropriately sized polynomial matrix $V(s)$, we have*

$$\bar{U}(s) = \bar{M}(s)V(s). \tag{22}$$

4. *Z is invertible if and only if there exists a doubly unimodular embedding*

$$\begin{pmatrix} \bar{X} & -\bar{Y} \\ -\bar{U} & \bar{M} \end{pmatrix}\begin{pmatrix} M & Y \\ U & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{23}$$

of $(-\bar{U}(s)\ \bar{M}(s))$ and $\begin{pmatrix} M(s) \\ U(s) \end{pmatrix}$.

5. *If Z is invertible, then in terms of the doubly unimodular embedding* (23), *$Z^{-1} : F^{\bar{p}}[s]/\bar{M}(s) F^{\bar{m}}[s] \to F^p[s]/M(s)F^m[s]$ is given by*

$$Z^{-1}[g]_{\bar{M}} = -\,[Yg]_M. \tag{24}$$

For the next theorem as well as the analysis of behavior equivalence, we shall need the following standard proposition, see [21] or [19].

Proposition 3.1. *Let $P \in F^{g\times q}[s]$ and $M \in F^{k\times q}[s]$. Define $P(\sigma) : s^{-1}F^q[[s^{-1}]] \to s^{-1}F^g[[s^{-1}]]$ by*

$$P(\sigma)h = \pi_- Ph = \mathscr{T}_P h. \tag{25}$$

Then

$$\operatorname{Ker} P(\sigma) \subset \operatorname{Ker} M(\sigma) \tag{26}$$

if and only if

$$M(s) = A(s)P(s) \tag{27}$$

for some $A \in F^{k\times g}[s]$.

It is an easy corollary that if $M_i(s)$ are full row rank polynomial matrices, then $\operatorname{Ker} M_1(\sigma) = \operatorname{Ker} M_2(\sigma)$ if and only if, for some unimodular matrix $U(s)$, we have $M_2(s) = U(s)M_1(s)$. This also settles the problem of equivalence on the level of autoregressive representations.

Next we discuss the invertibility properties of behavior homomorphisms.

Theorem 3.6. *Given two full row rank polynomial matrices $M \in F^{p\times m}[s]$, $\bar{M} \in F^{\bar{p}\times\bar{m}}[s]$ describing the behaviors $\mathscr{B} = \operatorname{Ker} M(\sigma)$ and $\bar{\mathscr{B}} = \operatorname{Ker} \bar{M}(\sigma)$, respectively. Let $\bar{U}, U$ be appropriately sized polynomial matrices satisfying*

$$\bar{U}(s)M(s) = \bar{M}(s)U(s) \tag{28}$$

and let $Z: \operatorname{Ker} M(\sigma) \to \operatorname{Ker} \bar{M}(\sigma)$ be defined by

$$Zh = U(\sigma)h = \pi_- Uh, \quad h \in \operatorname{Ker} M(\sigma). \tag{29}$$

Then

1. *Z is injective if and only if M, U are right coprime.*
2. *Z is surjective if and only if $\bar{U}, \bar{M}$ are left coprime and*

$$\operatorname{Ker}(-\bar{U}(s) \quad \bar{M}(s)) = \operatorname{Im}\begin{pmatrix} M(s) \\ U(s) \end{pmatrix}. \tag{30}$$

3. *Z as defined above is the zero map if and only if, for some appropriately sized polynomial matrix $L(s)$, we have*

$$U(s) = L(s)M(s). \tag{31}$$

4. *Z defined in (29) is invertible if and only if there exists a doubly unimodular embedding*

$$\begin{pmatrix} \bar{X} & -\bar{Y} \\ -\bar{U} & \bar{M} \end{pmatrix}\begin{pmatrix} M & Y \\ U & X \end{pmatrix} = \begin{pmatrix} M & Y \\ U & X \end{pmatrix} \times \begin{pmatrix} \bar{X} & -\bar{Y} \\ -\bar{U} & \bar{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{32}$$

   of $(-\bar{U}(s) \ \bar{M}(s))$ and $\begin{pmatrix} M(s) \\ U(s) \end{pmatrix}$.
5. *If Z is invertible, then in terms of the doubly unimodular embedding (9), its inverse $Z^{-1}: \operatorname{Ker} \bar{M}(\sigma) \to \operatorname{Ker} M(\sigma)$ is given by*

$$Z^{-1} = -\bar{Y}(\sigma). \tag{33}$$

The proof can be given either directly or by using Theorem 3.5 and duality. For this the fact that a behavior is a closed shift invariant subspace is critical. The full details will be given in [7].

4. Equivalence

A behavior $\mathscr{B}$ has a *normalized ARMA representation*, or *NARMA representation*, if it satisfies

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi, \tag{34}$$

for $M_1 \in F^{r\times m}[s]$, $M_2 \in F^{q\times m}[s]$. We will assume that M_1 has full row rank and M_1, M_2 are right coprime. These two conditions imply that the representation (34) is minimal.

As stated in the introduction, the importance of NARMA representations arises out of the fact that most behavior representations can be either interpreted as be in NARMA form or easily be rewritten in this form. Thus an ARMA representation of a behavior in the form $P(\sigma)w = Q(\sigma)\xi$ can be rewritten as

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} P(\sigma) & -Q(\sigma) \\ I & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{35}$$

Given two NARMA representations

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi, \qquad \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \bar{M}_1(\sigma) \\ \bar{M}_2(\sigma) \end{pmatrix} \xi \tag{36}$$

with the behaviors $\mathscr{B}$ and $\bar{\mathscr{B}}$ respectively, we say that the representations are *NARMA equivalent* if there exists polynomial matrices $\bar{U}, V, \bar{X}$ of appropriate size such that

$$\begin{pmatrix} \bar{U}(s) & 0 \\ -\bar{X}(s) & I \end{pmatrix}\begin{pmatrix} M_1(s) \\ M_2(s) \end{pmatrix} = \begin{pmatrix} \bar{M}_1(s) \\ \bar{M}_2(s) \end{pmatrix} V(s), \tag{37}$$

$\bar{U}, \bar{M}_1$ are left coprime and $\begin{pmatrix} M_1(s) \\ M_2(s) \end{pmatrix}$, V right coprime and

$$\operatorname{Ker}\begin{pmatrix} \bar{U}(s) & 0 & \bar{M}_1(s) \\ -\bar{X}(s) & I & \bar{M}_2(s) \end{pmatrix} = \operatorname{Im}\begin{pmatrix} M_1(s) \\ M_2(s) \\ -V(s) \end{pmatrix} \tag{38}$$

holds, i.e. there exists a doubly unimodular embedding of the polynomial matrices

$$\begin{pmatrix} \bar{U}(s) & 0 & \bar{M}_1(s) \\ -\bar{X}(s) & I & \bar{M}_2(s) \end{pmatrix}, \begin{pmatrix} M_1(s) \\ M_2(s) \\ -V(s) \end{pmatrix}. \tag{39}$$

Using a somewhat lengthy computation, it can be shown that NARMA equivalence is indeed a bonafide equivalence relation.

Theorem 4.1. *Given two behaviors in minimal NARMA representations*

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \tag{40}$$

and

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \bar{M}_1(\sigma) \\ \bar{M}_2(\sigma) \end{pmatrix} \xi. \tag{41}$$

Then $\mathscr{B} = \bar{\mathscr{B}}$ if and only if the two representations are NARMA equivalent.

Proof. Assume first that the representations are NARMA equivalent. Let $(N_1(s)\ N_2(s))$ and $(\bar{N}_1(s)\ \bar{N}_2(s))$ be left prime polynomial matrices for which

$$\operatorname{Ker}(N_1(s) \quad N_2(s)) = \operatorname{Im}\begin{pmatrix} M_1(s) \\ M_2(s) \end{pmatrix},$$

$$\operatorname{Ker}(\bar{N}_1(s) \quad \bar{N}_2(s)) = \operatorname{Im}\begin{pmatrix} \bar{M}_1(s) \\ \bar{M}_2(s) \end{pmatrix}.$$

By Lemma 3.15 in [13], we have $\mathscr{B} = \operatorname{Ker} N_2(\sigma)$ and $\bar{\mathscr{B}} = \operatorname{Ker} \bar{N}_2(\sigma)$. We compute

$$\begin{aligned} 0 &= (\bar{N}_1(s) \quad \bar{N}_2(s)) \begin{pmatrix} \bar{M}_1(s) \\ \bar{M}_2(s) \end{pmatrix} V \\ &= (\bar{N}_1(s) \quad \bar{N}_2(s)) \begin{pmatrix} U(s) & 0 \\ -X(s) & I \end{pmatrix} \begin{pmatrix} M_1(s) \\ M_2(s) \end{pmatrix} \\ &= ((\bar{N}_1(s)U(s) - \bar{N}_2(s)X(s)) \quad \bar{N}_2(s)) \begin{pmatrix} M_1(s) \\ M_2(s) \end{pmatrix}, \end{aligned}$$

i.e. $\operatorname{Ker}((\bar{N}_1(s)U(s) - \bar{N}_2(s)X(s))\ \bar{N}_2(s)) \supset \operatorname{Ker}(N_1(s)\ N_2(s))$. By Proposition 3.1, there exists a polynomial matrix $L(s)$ for which

$$\begin{aligned} &(\bar{N}_1(s)U(s) - \bar{N}_2(s)X(s) \quad \bar{N}_2(s)) \\ &\quad = L(s)(N_1(s) \quad N_2(s)). \end{aligned}$$

This implies the equality $\operatorname{Ker} \bar{N}_2(\sigma) \supset \operatorname{Ker} N_2(\sigma)$ or $\bar{\mathscr{B}} \supset \mathscr{B}$. Since NARMA equivalence is an equivalence relation, and in particular a symmetric relation, the equality $\bar{\mathscr{B}} = \mathscr{B}$ follows.

Conversely, assume the behaviors $\mathscr{B}$ and $\bar{\mathscr{B}}$ are equal. Clearly we have

$$\mathscr{B} = M_2(\sigma)\operatorname{Ker} M_1(\sigma) = \bar{M}_2(\sigma)\operatorname{Ker} \bar{M}_1(\sigma).$$

The right coprimeness of M_1, M_2 implies that $M_2(\sigma) \mid \operatorname{Ker} M_1(\sigma)$ is injective and so $M_2(\sigma)$ as a map from $\operatorname{Ker} M_1(\sigma)$ onto $\mathscr{B}$ is bijective. Moreover, it is an $F[s]$-homomorphism. In the same way $\bar{M}_2(\sigma) \mid \operatorname{Ker} \bar{M}_1(\sigma)\colon \operatorname{Ker} \bar{M}_1(\sigma) \to \bar{\mathscr{B}}$ is a behavior isomorphism. We define now a map $Z\colon \operatorname{Ker} M_1(\sigma) \to \operatorname{Ker} \bar{M}_1(\sigma)$ by

$$Zh = \bar{M}_2(\sigma)^{-1} M_2(\sigma) h, \quad h \in \operatorname{Ker} M_1(\sigma). \tag{42}$$

Clearly Z is an $F[s]$-isomorphism, i.e. satisfies $Z\sigma^{M_1} = \sigma^{\bar{M}_1} Z$ and is invertible. Since $M_1(s), \bar{M}_1(s)$ have both full row rank, we can apply Theorem 3.2 to conclude the existence of appropriately sized polynomial matrices U and V for which $U, \bar{M}_1$ are left coprime, M_1, V are right coprime, they satisfy the following equality:

$$\operatorname{Ker}(U(s) \quad \bar{M}_1(s)) = \operatorname{Im}\begin{pmatrix} M_1(s) \\ -V(s) \end{pmatrix} \tag{43}$$

in terms of which $Z = V(\sigma)$. Note that the previous conditions are equivalent to the existence of a doubly unimodular embedding of

$$(U(s) \quad \bar{M}_1(s)), \begin{pmatrix} M_1(s) \\ -V(s) \end{pmatrix}.$$

Thus we have $\bar{M}_2(\sigma)^{-1} M_2(\sigma) h = V(\sigma) h$ for all $h \in \operatorname{Ker} M_1(\sigma)$. So

$$\operatorname{Ker}(M_2(\sigma) - \bar{M}_2(\sigma)V(\sigma) \supset \operatorname{Ker} M_1(\sigma).$$

By Proposition 3.1, we conclude the existence of a polynomial matrix $X(s)$ such that

$$M_2(s) - \bar{M}_2(s)V(s) = X(s)M_1(s). \tag{44}$$

The equalities (43) and (44), taken together, imply

$$\begin{pmatrix} U(s) & 0 & \bar{M}_1(s) \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} M_1(s) \\ 0 \\ -V(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It remains to show that there exists a doubly unimodular embedding for

$$\begin{pmatrix} U(s) & 0 & \bar{M}_1(s) \\ -X(s) & I & \bar{M}_2(s) \end{pmatrix}, \begin{pmatrix} M_1(s) \\ M_2(s) \\ -V(s) \end{pmatrix}.$$

First, we note that there exists a doubly unimodular embedding for

$$\begin{pmatrix} U(s) & 0 & \bar{M}_1(s) \\ 0 & I & 0 \end{pmatrix}, \begin{pmatrix} M_1(s) \\ 0 \\ -V(s) \end{pmatrix}.$$

This follows from Lemma 2.1 and the fact that there exists a doubly unimodular embedding of $(U(s)\ \bar{M}_1(s))$, $\begin{pmatrix} M_1(s) \\ -V(s) \end{pmatrix}$. We note that

$$\begin{pmatrix} I & 0 & 0 \\ -X(s) & I & \bar{M}_2(s) \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ X(s) & I & -\bar{M}_2(s) \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

with both matrices unimodular. Now

$$\begin{pmatrix} U(s) & 0 & \bar{M}_1(s) \\ -X(s) & I & \bar{M}_2(s) \end{pmatrix} = \begin{pmatrix} U(s) & 0 & \bar{M}_1(s) \\ 0 & I & 0 \end{pmatrix} \times \begin{pmatrix} I & 0 & 0 \\ -X(s) & I & \bar{M}_2(s) \\ 0 & 0 & I \end{pmatrix}$$

and, using Eq. (44), we have

$$\begin{pmatrix} I & 0 & 0 \\ X(s) & I & -\bar{M}_2(s) \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} M_1(s) \\ 0 \\ -V(s) \end{pmatrix} = \begin{pmatrix} M_1(s) \\ X(s)M_1(s) + \bar{M}_2(s)V(s) \\ -V(s) \end{pmatrix} = \begin{pmatrix} M_1(s) \\ M_2(s) \\ -V(s) \end{pmatrix}$$

and (38) follows. □

Acknowledgements

The author is thankful to an anonymous referee who pointed out a mistake in the original formulation of Theorem 3.6, saving the author an unnecessary embarassment.

References

[1] M. Bisiacco, M.E. Valcher, Behavior decompositions and two-sided diophantine equations, Automatica, to appear.

[2] P.A. Fuhrmann, Algebraic system theory: an analyst's point of view, J. Franklin Inst. 301 (1976) 521–540.

[3] P.A. Fuhrmann, On strict system equivalence and similarity, Int. J. Control 25 (1977) 5–10.

[4] P.A. Fuhrmann, Duality in polynomial models with some applications to geometric control theory, IEEE Trans. Aut. Control AC-26 (1981) 284–295.

[5] P.A. Fuhrmann, Linear Systems and Operators in Hilbert Space, McGraw-Hill, New York, 1981.

[6] P.A. Fuhrmann, A duality theory for robust control and model reduction, Linear Algebra Appl. 203–204 (1994) 471–578.

[7] P.A. Fuhrmann, A study of behaviors, Linear Algebra Appl., submitted for publication.

[8] D. Hinrichsen, D. Prätzel-Wolters, Module theoretic characterization of system equivalence in time domain, in: Algebraic and Geometric Methods in Linear System Theory, A.M.S. Lectures in Applied Mathematics, Vol. 18, 1980, pp. 137–156.

[9] D. Hinrichsen, D. Prätzel-Wolters, Solution modules and system equivalence, Int. J. Control 32 (1980) 777–802.

[10] M. Hou, A.C. Pugh, G.E. Hayton, Generalized transfer functions and input/output equivalence, Int. J. Control 68 (1997) 1163–1178.

[11] T. Kailath, Linear Systems, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[12] R.E. Kalman, P. Falb, M. Arbib, Topics in Mathematical System Theory, McGraw-Hill, New York, 1969.

[13] M. Kuijper, First order representations of linear systems, Ph.D. Thesis, Catholic University of Brabant, 1992.

[14] M. Kuijper, First order Representations of Linear Systems, Birkhauser, Basel, 1994.

[15] A.B. Özgüler, Linear Multichannel Control, Prentice-Hall, Englewood Cliffs, NJ, 1994.

[16] D. Prätzel-Wolters, Feedback morphism between linear systems—a unified approach to state space systems, transfer functions and system matrices, Ph.D. Dissertation, University of Bremen, 1981.

[17] H.H. Rosenbrock, State-space and Multivariable Theory, Wiley, New York, 1970.

[18] J. Rosenthal, Connections between linear systems and convolutional codes, 2000.

[19] J.M. Schumacher, Transformations of linear systems under external equivalence, Linear Algebra Appl. 102 (1988) 1–34.

[20] J.M. Schumacher, Linear system representations, in: H. Nijmeijer, J.M.S. Schumacher (Eds.), Three Decades of Mathematical System Theory. A Collection of Surveys at the Occasion of the Fiftieth Birthday of Jan C. Willems, Lecture Notes in Control and Information Sciences, Vol. 135, Springer, Berlin, 1989, pp. 382–408.

[21] J.C. Willems, From time series to linear systems. Part I: finite-dimensional linear time invariant systems, Automatica 22 (1986) 561–580.

[22] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Trans. Automat. Control AC-36 (1991) 259–294.

[23] E. Zerz, On strict system equivalence for multidimensional systems, Int. J. Control 73 (6) (2000) 495–504.