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Systems & Control Letters 30 (1997) 217–223

SYSTEMS & CONTROL LETTERS

A homeomorphism between observable pairs and conditioned invariant subspaces [1]

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Received 23 December 1996; received in revised form 11 February 1997; accepted 11 February 1997

Abstract

A bijective correspondence between similarity classes of observable systems (C, A) and n-codimensional conditioned invariant subspaces of a pair $(\mathscr{C}, \mathscr{A})$ is constructed that leads to a homeomorphism of the spaces. This is applied to the parametrization of inner functions of fixed McMillan degree. Proofs using state space methods as well as using polynomial models are given. © 1997 Elsevier Science B.V.

Keywords: Parametrization problems; Conditioned invariant subspaces; Geometric control; Inner functions

1. Introduction

Controlled and conditioned invariant subspaces are crucial objects of geometric control theory, as developed by Wonham [12] and Basile and Marro [1]. Although the structure theory of such subspaces have been developed more than a decade ago by Fuhrmann and Willems [6], Fuhrmann [4], Emre and Hautus [2], Hinrichsen et al. [9], and others, surprisingly, many questions still remain open.

New interest in parametrization problems in geometric control theory arose through recent developments in spectral factorization theory, see Lindquist et al. [10]. Such work has led to the need of obtaining a deeper insight into the relation between various objects, such as invariant subspaces of an operator, conditioned invariant subspaces and observable pairs. The purpose of this paper is to add further understanding of parametrizing conditioned invariant subspaces by establishing a connection with the parametrization theory of observable pairs. In fact, we prove, using state space methods, that there is a homeomorphism between the set of similarity classes of k-dimensional observable pairs and of the set of k-codimensional conditioned invariant subspaces of an observable pair in Brunovsky canonical form. For reasons of simplicity, we focus on a rather special situation with generic Brunovsky indices. A complete theory for arbitrary Brunovsky indices is currently under development and is beyond the scope of this paper. With an eye towards generalizations to problems in a Hardy space context, it is also important to have a functional approach available as well. Thus, an independent proof of the existence of a bijection between tight, conditioned invariant subspaces and observable pairs is given using methods from the theory of polynomial models.

We believe that the results of this paper show only the very beginnings of a deeper theory that clarifies the links, so far only poorly understood, between various objects such as invariant subspaces of linear operators, conditioned invariant subspaces, inner functions,

* Corresponding author.
[1] Partially supported by GIF under Grant No. I 184.
[2] Earl Katz Family Chair in Algebraic System Theory.

PII S0167-6911(97)00013-3

spectral factorizations, Riccati equations and observable pairs.

We would like to thank the reviewers for helpful comments that led to an improvement of the paper. In particular, the present, more general, form of Theorem 2.1 together with the concept of a tight subspace has been suggested by one of the reviewers.

2. Observable pairs and conditioned invariant subspaces

For any integers $p, n \geqslant 1$ consider the observable pair $(\mathscr{C}, \mathscr{A}) \in \mathbb{R}^{p\times p(n+1)} \times \mathbb{R}^{p(n+1)\times p(n+1)}$ defined by

$$\mathscr{A} = \begin{pmatrix} 0 & \dots & 0 \\ I & \dots & . \\ . & \dots & . \\ . & \dots & . \\ 0 & I & 0 \end{pmatrix}, \qquad \mathscr{C} = (0 \ . \ . \ 0 \ I), \tag{1}$$

where I denotes the $p\times p$ identity matrix. Thus $(\mathscr{C}, \mathscr{A})$ is an observable canonical form with generic observability indices $v_i = n+1$, $i = 1, \dots, p$.

We consider now conditioned invariant subspaces $\mathscr{V} \subset \mathbb{R}^{p(n+1)}$, with respect to $(\mathscr{C}, \mathscr{A})$. Using the dual concept of the notion of a coasting subspace, see Willems [11], we refer to $\mathscr{V}$ as a *tight* subspace if

$$\mathscr{V} + \operatorname{Ker} \mathscr{C} = \mathbb{R}^{p(n+1)}$$

holds (i.e. if $\mathscr{V}$ is transversal to $\operatorname{Ker}\mathscr{C}$). The following lemma describes a class of codimension k-conditioned invariant subspaces with respect to $(\mathscr{C}, \mathscr{A})$.

Lemma 2.1. *Let $(\mathscr{C}, \mathscr{A})$ be defined as above. For integers k, n let $(C, A) \in \mathbb{R}^{p\times k} \times \mathbb{R}^{k\times k}$ be an observable pair with partial observability matrix*

$$\mathscr{O}_n(C, A) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^n \end{pmatrix}.$$

Let $\operatorname{Im} \mathscr{O}_n(C, A)^\perp$ denote the orthogonal complement of $\operatorname{Im} \mathscr{O}_n$ in $\mathbb{R}^{p(n+1)}$ and let $v_1 \geqslant \cdots \geqslant v_p$ denote the observability indices of (C, A).

(a) *$\operatorname{Im} \mathscr{O}_n^\perp$ is a conditioned invariant subspace with respect to $(\mathscr{C}, \mathscr{A})$. It has codimension k if and only if $v_1 \leqslant n+1$.*
(b) *$\operatorname{Im} \mathscr{O}_n^\perp$ is a tight subspace if and only if $v_1 \leqslant n$.*
(c) *If $k \leqslant n$ then, $\operatorname{Im} \mathscr{O}_n^\perp$ is tight.*

Proof. (a) We have

$$x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \in \operatorname{Im} \mathscr{O}_n(C,A)^\perp \cap \operatorname{Ker} \mathscr{C}$$

$$\Leftrightarrow x = \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix} \perp \operatorname{Im} \mathscr{O}_n$$

$$\Leftrightarrow \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} \perp \operatorname{span} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

Thus,

$$\mathscr{A}x = \begin{pmatrix} 0 \\ x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} \perp \operatorname{span} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^n \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} \perp \operatorname{span} \begin{pmatrix} CA \\ \vdots \\ CA^n \end{pmatrix}.$$

Since

$$\operatorname{span} \begin{pmatrix} CA \\ \vdots \\ CA^n \end{pmatrix} \subset \operatorname{span} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix},$$

we obtain $x \in (\operatorname{Im} \mathscr{O}_n)^\perp \cap \operatorname{Ker} \mathscr{C} \Rightarrow \mathscr{A}x \in (\operatorname{Im} \mathscr{O}_n)^\perp$. Thus, the space is conditioned invariant. The condition on the observability indices implies (and is implied by) that it has codimension k.

(b) By duality, a subspace $\operatorname{Im} \mathscr{O}_n^\perp$ is tight if and only if

$$\operatorname{Im} \mathscr{O}_n(C, A) \cap \operatorname{Im} \mathscr{C}^{\mathrm{T}} = \{0\}.$$

By the special form of $\mathscr{C}$ this is equivalent to

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} CA^i \subset \operatorname{Ker} CA^n.$$

Equivalently, this means that the rows of CA^n are spanned by the rows of $C, CA, \dots, CA^{n-1}$, i.e. that $v_1 \leqslant n$.

(c) If $k \leqslant n$, then $\nu_1 \leqslant n$ holds and therefore $\operatorname{Im} \mathcal{O}_n^{\perp}$ is tight. □

We now show that every conditioned invariant subspace can be described via a partial observability matrix.

Lemma 2.2. *For every $(\mathscr{C}, \mathscr{A})$-invariant subspace $\mathscr{V} \subset \mathbb{R}^{p(n+1)}$ of codimension k there exists an observable pair $(C, A) \in \mathbb{R}^{p\times k} \times \mathbb{R}^{k\times k}$ with $\mathscr{V} = \operatorname{Im} \mathcal{O}_n(A, C)^{\perp}$.*

Proof. Let

$$L = \begin{pmatrix} L_0 \\ \vdots \\ L_n \end{pmatrix} \in \mathbb{R}^{p(n+1)\times k}$$

with $L_i \in \mathbb{R}^{p\times k}, \quad i = 0, \dots, n,$

be a rank k matrix such that $\mathscr{V}^{\perp} = \operatorname{Im} L$. Then $\mathscr{V}$ is conditioned invariant with respect to $(\mathscr{C}, \mathscr{A})$ if and only if the conditions $\mathscr{C}z = 0$ and $L^{\mathrm{T}}z = 0$ imply $L^{\mathrm{T}}\mathscr{A}z = 0$. Equivalently, there exist $A^{\mathrm{T}} \in \mathbb{R}^{k\times k}$ and $Y \in \mathbb{R}^{k\times p}$ such that

$$L^{\mathrm{T}}\mathscr{A} = A^{\mathrm{T}}L^{\mathrm{T}} + Y\mathscr{C}.$$

Written out in components this gives

$$L_{i+1}^{\mathrm{T}} = A^{\mathrm{T}}L_i^{\mathrm{T}}, \quad i = 0, \dots, n-1$$

$$Y = -A^{\mathrm{T}}L_n^{\mathrm{T}}.$$

Thus $L = \mathcal{O}_n(C, A)$, with $C := L_0$, and hence $\mathscr{V} = \operatorname{Im} \mathcal{O}_n(C, A)^{\perp}$. The observability of (C, A) follows from $rk\, \mathcal{O}_n(C, A) = \operatorname{codim} \mathscr{V} = k$. □

We prove now the main result of this section, which extends a previous result of Helmke [8].

Theorem 2.1. *There exists a bijective correspondence between tight conditioned invariant subspaces with respect to $(\mathscr{C}, \mathscr{A})$ of codimension k and similarity classes of k-dimensional observable pairs $(C, A) \in \mathbb{R}^{p\times k} \in \mathbb{R}^{k\times k}$ with largest observability index $\nu_1 \leqslant n$.*

Proof. Given any observable pair $(C, A) \in \mathbb{R}^{n\times k} \in \mathbb{R}^{p\times k}$ with $\nu_1 \leqslant n$ then Lemma 2.1 implies that $\operatorname{Im} \mathcal{O}_n(C, A)^{\perp}$ is a tight conditioned invariant subspace for $(\mathscr{C}, \mathscr{A})$ of codimension k. Moreover, by Lemma 2.2, any tight conditioned invariant subspace has such a representation. It remains to show that the pair (C, A) is (up to similarity) uniquely determined by the tight subspace $\operatorname{Im} \mathcal{O}_n(C, A)^{\perp}$. Let $(C_1, A_1), (C_2, A_2)$ be observable with observability indices $\leqslant n$ and $\operatorname{Im} \mathcal{O}_n(C_1, A_1)^{\perp} = \operatorname{Im} \mathcal{O}_n(C_2, A_2)^{\perp}$. W.l.o.g. we can assume that $\mathcal{O}_n(C_1, A_1) = \mathcal{O}_n(C_2, A_2)$ and hence $C_1 = C_2$. It follows that the observability indices of (C_1, A_1) and (C_2, A_2) coincide and A_1 has the same minimal polynomial as A_2. Thus, from $\mathcal{O}_n(C_1, A_1) = \mathcal{O}_n(C_2, A_2)$ we obtain $A_1 = A_2$ and hence the result. □

The above proof actually shows that there is a homeomorphism between observable pairs and conditioned invariant subspaces. More precisely, without elaborating on the details, we have the following result.

Corollary 2.1. *Let $k \leqslant n$. Let $\Sigma_{k,p}(\mathbb{R})$ denote the orbit space of k-dimensional observable pairs and let $\mathscr{X}_k(\mathscr{C}, \mathscr{A})$ denote the set of conditioned invariant subspaces of codimension k. Then the map*

$$R : \Sigma_{k,p}(\mathbb{R}) \to \mathscr{X}_k(\mathscr{C}, \mathscr{A}),$$

$$[C, A]_{\sigma} \mapsto \operatorname{Im} \mathcal{O}_n(C, A)^{\perp}$$

is a homeomorphism.

One can extend the preceding construction to the case of conditioned invariant subspaces of arbitrary dimension. Also, one can use the previous results to determine topological properties of the space of conditioned invariant subspaces of fixed codimension. This will be done elsewhere. For example, it follows from the above theorem that the space of conditioned invariant subspaces of a fixed codimension is connected.

3. The polynomial model approach

We present now a polynomial approach to the proof of the previous results. One of its advantages is that the method of proof works for an arbitrary field. As in the state space approach, duality plays an important underlying role. In polynomial terms, we have two ways of dealing with duality. One is to use transposition of polynomial matrices and dual operators, requiring the simultaneous use of both reachable and observable pairs. The other way, which we adopt, is to use both row and column polynomial models. In case we deal with row vectors, the operators will always be written on the right.

Specifically, given an observable pair (C, A), with C, A being $p \times n$ and $n \times n$ matrices respectively, we consider the coprime factorizations

$$C(zI - A)^{-1} = T(z)^{-1}H(z). \tag{2}$$

With this factorization, we associate the polynomial model X_T^{c}, the superscript denotes that we are dealing with spaces of column vector polynomials, as the state space. Define the pair (C_T, A_T), with $f \in X_T^{\mathrm{c}}$, by

$$\begin{aligned} A_T f &= S_T f = \pi_T z f, \\ C_T f &= (T^{-1} f)_{-1}. \end{aligned} \tag{3}$$

We refer to Fuhrmann [3,4] for the relevant details of the shift realization. The pair (C_T, A_T) is necessarily observable and is similar to the pair (C, A), the similarity given by the map $\phi : \mathbb{R}^n \to X_T^{\mathrm{c}}$ defined by

$$\phi(x) = H(z)x.$$

Completely analogously, given a coobservable pair (A, C), i.e. an observable pair when using row vector spaces, we take the coprime factorizations

$$(zI - A)^{-1}C = H(z)T(z)^{-1}. \tag{4}$$

With this factorization, we associate the polynomial model X_T^{r} as the state space. Define the pair (C_T, A_T), with $f \in X_T^{\mathrm{r}}$, by

$$\begin{aligned} f A_T &= f S_T = f z \pi_T, \\ f C_T &= (f T^{-1})_{-1}. \end{aligned} \tag{5}$$

Using the representation (5) of an observable pair in terms of a polynomial model, a complete characterization of conditioned invariant subspaces has been obtained in Fuhrmann [4, Theorem 3.6]. In fact, using X_D^{r} as the state space and the pair (C_D, A_D) defined as in (5), a subspace $\mathscr{V} \subset X_D^{\mathrm{r}}$ is conditioned invariant, if and only if $\mathscr{V} = X_D^{\mathrm{r}} \cap M$ with $M \subset \mathbb{R}_r^p[z]$ a submodule. Now any such submodule has a representation of the form $M = \mathbb{R}_r^p[z]T$ for some $p \times p$ polynomial matrix T. The representation of conditioned invariant subspaces in the form $\mathscr{V} = X_D^{\mathrm{r}} \cap \mathbb{R}_r^p[z]T$ does not specify the polynomial matrix T uniquely. In fact, a necessary and sufficient condition for the uniqueness of the representations is that the observability indices of any reduction of (C, A) to $\mathscr{V}$ are all positive. This will be proved in Proposition 3.2. To give a polynomial proof of Theorem 2.1, in the case where $k \leqslant n$, we proceed by establishing some preliminary results which are of independent interest. The straightforward proof of the first result is omitted.

Proposition 3.1. *Let (C, A) be an observable pair with the observability indices $\nu_1 \geqslant \cdots \geqslant \nu_p$, and let $\mathscr{V}$ be a conditioned invariant subspace. Let J be an output injection map such that $(A + JC)\mathscr{V} \subset \mathscr{V}$. Then the restricted pair (A_1, C_1) acting in the state space $\mathscr{V}$ and defined by*

$$\begin{aligned} A_1 &= (A + JC)|\mathscr{V}, \\ C_1 &= C|\mathscr{V}, \end{aligned}$$

is observable and its observability indices $\lambda_1 \geqslant \cdots \geqslant \lambda_p$ satisfy $\lambda_i \leqslant \nu_i$.

Proposition 3.2. *Let D be a $p \times p$ nonsingular polynomial matrix and let the pair (C_D, A_D) be the pair acting in X_D^{r} defined by (5). Let $\mathscr{V} = X_D^{\mathrm{r}} \cap \mathbb{R}^p[z]T$ be a conditioned invariant subspace of X_D^{r} having the observability indices of the restricted system given by $\lambda_1 \geqslant \cdots \geqslant \lambda_p$. If $\lambda_i > 0$ for all i then the polynomial matrix T is invertible and uniquely defined up to a left unimodular factor.*

Proof. Assume $\lambda_1 \geqslant \cdots \geqslant \lambda_p > 0$. $\mathscr{V}$ has another representation of the form $\mathscr{V} = X_{S_1}^{\mathrm{r}} T_1 = X_D^{\mathrm{r}} \cap \mathbb{R}^p[z]T_1$, with $S_1 T_1 D^{-1}$ biproper. Without loss of generality, we can assume that S_1 is column proper with column indices $\lambda_1 \geqslant \cdots \geqslant \lambda_p$. Now, by assumption,

$$\mathscr{V} = X_{S_1}^{\mathrm{r}} T_1 = X_D^{\mathrm{r}} \cap \mathbb{R}^p[z]T_1 = X_D^{\mathrm{r}} \cap \mathbb{R}^p[z]T. \tag{6}$$

Since all λ_i are positive, $X_{S_1}^{\mathrm{r}}$ contains all constant polynomials. The previous equality implies therefore that $T_1(z) = E(z)T(z)$ for some polynomial matrix E. Thus, necessarily, both T and E are nonsingular. Defining $D_1 = S_1 T_1$, we have $D_1 = S_1 T_1 = S_1 E T$. Therefore, we have

$$\begin{aligned} \mathscr{V} = X_D^{\mathrm{r}} \cap \mathbb{R}^p[z]T &= X_{S_1}^{\mathrm{r}} T_1 = X_{S_1}^{\mathrm{r}} E T \\ &\subset X_{S_1 E}^{\mathrm{r}} T = X_D^{\mathrm{r}} \cap \mathbb{R}^p[z]T. \end{aligned}$$

Thus, we must have equality throughout and hence E is necessarily unimodular. □

We consider the special case of the realization (5) for the case of the $p \times p$ polynomial matrix (of scalar type) $\Delta(z) = z^{n+1}I_p$. We will denote in the rest of this section by (C_Δ, A_Δ) the observable pair in X_Δ^{r} associated with the polynomial matrix Δ via the realization (5). Let $e_1, \ldots, e_p$ be the basis of the standard unit row vectors in $\mathbb{R}^p$. Clearly, the matrix pair defined in (1) is just a matrix representation of (C_Δ, A_Δ) with respect to the standard basis of X_Δ given by $\mathscr{B} = \{e_1, \dots, e_p, ze_1, \dots, ze_p, \dots, z^n e_1, \dots, z^n e_p\}$. We define the Toeplitz operator $\mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}}$, with symbol $T\Delta^{-1}$, acting in $\mathbb{R}^p_r[z]$, by

$$f\mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}} = fT\Delta^{-1}\pi_+. \tag{7}$$

Lemma 3.1. *Let $\Delta(z) = z^{n+1}I_p$ and T a $p \times p$ nonsingular polynomial matrix. Let $\mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}}$ be the Toeplitz operator defined in* (7). *Then we have*

$$\dim \operatorname{Ker} \mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}} = \dim X_\Delta \cap \mathbb{R}^p[z]\,T. \tag{8}$$

Specifically, the map $\psi : \operatorname{Ker} \mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}} \to X_\Delta \cap \mathbb{R}^p[z]\,T$ defined by

$$\psi(p) = pT$$

is a bijective linear map between these spaces.

Proof. Let $p \in \operatorname{Ker} \mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}}$, i.e. $p\mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}} = pT\Delta^{-1}\pi_+ = 0$. Setting $f = pT$, we get $f \in X_\Delta$ as well as $f \in \mathbb{R}^p[z]\,T$, that is $f \in X^{\mathrm{r}}_\Delta \cap \mathbb{R}^p[z]\,T$. If T is nonsingular, this argument is reversible. This shows that the two spaces are isomorphic. □

Lemma 3.2. *Given $\Delta(z) = z^{n+1}I_p$ and the observable pair $(C, A) \in \Sigma_{p,k}(\mathbb{R})$ with the observability indices $\nu_1 \geqslant \dots \geqslant \nu_p$. Then, with the nonsingular polynomial matrix T defined by* (2), *we have*

$$\operatorname{codim} X_\Delta \cap \mathbb{R}^p[z]\,T = \deg\det T \tag{9}$$

if and only if $\nu_1 \leqslant n + 1$.

Proof. Assume first that $\nu_1 \leqslant n+1$. We note that the observability indices of (C, A), i.e. $\nu_1 \geqslant \dots \geqslant \nu_p$ with $\sum_{i=1}^p \nu_i = k$, are equal to the row indices of T. Furthermore $k = \deg\det T$. Clearly $\nu_1 \leqslant k$. Let $T(z) = U(z)\Delta_T(z)\Gamma(z)$ be a right Wiener–Hopf factorization at infinity, see Fuhrmann and Willems [5], of T with $\Delta_T(z) = \operatorname{diag}(z^{\nu_1}, \dots, z^{\nu_p})$. This implies the right Wiener–Hopf factorization

$$T\Delta^{-1} = U(z)\operatorname{diag}(z^{\nu_1-n-1}, \dots, z^{\nu_p-n-1})\,\Gamma(z).$$

By our assumptions, we have for all indices i, the inequality $\nu_i - n - 1 \leqslant 0$. Since the dimension of the kernel of a Toeplitz operator is equal to minus the sum of all negative, and equivalently nonpositive, factorization indices, this shows that

$$\dim \operatorname{Ker} \mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}} = -\sum_{i=1}^p (\nu_i - n - 1) = p(n+1) - \sum_{i=1}^p \nu_i.$$

Therefore, we also have $\dim X_\Delta \cap \mathbb{R}^p[z]\,T = p(n+1) - \sum_{i=1}^p \nu_i$. Since $\dim X_\Delta = p(n+1)$, we obtain $\operatorname{codim} X_\Delta \cap \mathbb{R}^p[z]\,T = \sum_{i=1}^p \nu_i = k = \deg\det T$.

Assume now the codimension formula (9) holds. Assume also that

$$\nu_1 \geqslant \dots \geqslant \nu_j > n+1 \geqslant \nu_{j+1} \geqslant \dots \geqslant \nu_p.$$

As before, we have

$$\dim \operatorname{Ker} \mathscr{T}^{\mathrm{r}}_{T\Delta^{-1}} = -\sum_{i=j+1}^p (\nu_i - n - 1) = (n+1)(p-j) - \sum_{i=j+1}^p \nu_i.$$

This implies that

$$\operatorname{codim} X_\Delta \cap \mathbb{R}^p[z]\,T = (n+1)j + \sum_{i=j+1}^p \nu_i < \sum_{i=1}^p \nu_i,$$

which contradicts the codimension formula. □

The next proposition treats the uniqueness issue of Proposition 3.2 in state space terms and it turns out that it relates to the tightness of the corresponding conditioned invariant subspace.

Proposition 3.3. *Let $\Delta(z) = z^{n+1}I_p$ and let*

$$\mathscr{V} = X_\Delta \cap \mathbb{R}^p[z]\,T \tag{10}$$

be a conditioned invariant subspace. Let $\nu_1 \geqslant \dots \geqslant \nu_p$ be the observability indices associated with $T(z)$ and let $\lambda_1 \geqslant \dots \geqslant \lambda_p$ be the observability indices of the system (C_Δ, A_Δ) reduced to $\mathscr{V}$. Then the following conditions are equivalent:

1. *$\mathscr{V}$ is a tight subspace of X_Δ.*
2. *We have $\nu_1 \leqslant n$.*
3. *We have $\lambda_p > 0$.*
4. *The representation* (10) *of $\mathscr{V}$ is unique up to a left unimodular factor for T.*

Proof. Assume the representation of $\mathscr{V}$ is unique in the above sense. Then, by Proposition 3.2, we have $\lambda_i > 0$, for $i = 1, \dots, p$. By results in Fuhrmann [4], there exists a nonsingular polynomial matrix S such that $ST = D$ and $\Delta^{-1}D = \Gamma$ is biproper. Let

$T = U_T \operatorname{diag}(z^{\nu_1},\ldots,z^{\nu_p})\Gamma_T$ be a right Wiener–Hopf factorization with U_T unimodular and γ_T biproper. Similarly, there exists a left Wiener–Hopf factorization for S of the form $S = \Gamma_S \operatorname{diag}(z^{\lambda_p},\ldots,z^{\lambda_1})U_S$. Since T is only determined up to a left unimodular factor, we may assume, without loss of generality, that $U_S U_T = I$. Thus, it follows that $\operatorname{diag}(z^{\nu_1+\lambda_p-n-1},\ldots,z^{\nu_p+\lambda_1-n-1}) = \Gamma_S^{-1}\Gamma\Gamma_T^{-1}$. This equality implies $\Gamma = \Gamma_S\Gamma_T$ and $\nu_i+\lambda_{p-i+1} = n+1$, for $i = 1,\ldots,p$. This shows that $\lambda_{p-i+1} > 0$ is equivalent to $\nu_i \leqslant n$.

Next, assume $\nu_i \leqslant n$. We show that $\mathscr{V} = X_\Delta \cap \mathbb{R}^p[z]\,T$ is tight. Clearly, for $\Delta(z) = z^{n+1}I_p$, we have $\operatorname{Ker}\mathscr{C} = \{x(z) \in \mathbb{R}^p[z] \mid \deg x < n\}$. Now, without loss of generality, we can assume $T(z)$ to be row proper with row indices equal to $\nu_1 \geqslant \cdots \geqslant \nu_p$ and the highest coefficient matrix of T to be a permutation matrix. Thus, the ith row of $T(z)$, denoted by $t_i(z)$ has degree ν_i. We consider now the polynomial vectors $v_i(z) = z^{\lambda_{p-i+1}-1}t_i(z)$. Clearly, the v_i are in $\mathscr{V}$, have degree n and their highest coefficients are the row unit vectors. This implies that $\mathscr{V}$ is tight.

Conversely, assume $\mathscr{V}$ is tight. Thus, there exist in $\mathscr{V}$ p vectors of degree n whose highest coefficients are linearly independent. This shows that necessarily the ν_i, i.e. the degrees of t_i, are all $\leqslant n$. □

We can give now a polynomial proof of Theorem 2.1. We point out that the theorem is valid over arbitrary fields.

Proof. For $k \leqslant n$, we construct a map from the orbit space $\Sigma_{p,k}(\mathbb{R})$ of observable pairs, with C and A being $p\times k$ and $k \times k$ matrices, respectively, into the set of conditioned invariant subspaces of the observable pair associated with Δ. This map is constructed by taking the coprime factorization (2) and mapping (C,A) to the conditioned invariant subspace $\mathscr{V} = X_\Delta \cap \mathbb{R}^p[z]\,T$. Although T is defined only up to a left unimodular factor, the subspace $\mathscr{V}$ is uniquely determined by the similarity class of (C,A). By Lemma 3.2, we have $\operatorname{codim}\mathscr{V} = \deg\det T = k$.

Conversely, assume $\mathscr{V}$ is a conditioned invariant subspace of X_Δ satisfying $\operatorname{codim}\mathscr{V} = k \leqslant n$. We shall construct an observable pair (C_1,A_1) that maps to $\mathscr{V}$ under the previously defined map. Clearly, we have $\mathscr{V} = X_\Delta \cap \mathbb{R}^p[z]\,T$ for some polynomial matrix T. By Lemma 3.2, T is nonsingular with row indices satisfying $n+1 > k \geqslant \nu_1 \geqslant \cdots \geqslant \nu_p$. This implies, by Lemma 3.3, that $\lambda_i > 0$ for all i and hence that the above representation determines T uniquely up to a left unimodular factor.

Now we apply the shift realization, as in (3), to associate with T an observable pair (C,A) with observability indices $\nu_1 \geqslant \cdots \geqslant \nu_p$. To be precise, let us consider the polynomial model space X_T^{c}, which has dimension k. Let the $p\times k$ matrix $H(z)$ be a basis matrix for X_T^{c}, that is its columns form a basis. This assumption on H implies in particular that T and H are left coprime. Since $T^{-1}H$ is strictly proper, it has a minimal realization of the form $C(zI-A)^{-1}B$. Since H is a basis matrix, the map from $\mathbb{R}^k$ into X_T^{c} given by $x \mapsto H(z)x$ is bijective. By the state space isomorphism theorem, B is a nonsingular matrix. Thus,

$$\begin{aligned} C(zI-A)^{-1}B &= CBB^{-1}(zI-A)^{-1}B \\ &= (CB)(zI - B^{-1}AB)^{-1} \\ &= C_1(zI-A_1)^{-1}. \end{aligned}$$

Clearly, as $C_1(zI-A_1)^{-1} = T^{-1}H$, the observable pair (C_1,A_1) maps to the conditioned invariant subspace $\mathscr{V} = X_\Delta \cap \mathbb{R}^p_r[z]\,T$. □

Theorem 2.1 can be easily extended to other contexts. Recall that, given a co-observable pair (A,C), a subspace $\mathscr{V}$ of the state space is *outer detectable* if there exists an output injection map H such that $\mathscr{V}$ is $A+CH$ invariant and $(A+CH)|_{X/\mathscr{V}}$ is stable. We will state without proof one such extension. Its proof is based on the following characterization of outer detectable subspaces.

Proposition 3.4. *Let D be a $p\times p$ nonsingular polynomial matrix. Then, with respect to the pair (A_D, C_D) defined in* (5), *a subspace $\mathscr{V}$ of the state space $X_{D_r}^{\mathrm{r}}$ is outer antidetectable if and only if*

$$\mathscr{V} = X_{D_r}^{\mathrm{r}} \cap \mathbb{R}^m[s]E_- \tag{11}$$

for some stable polynomial matrix E_-.

Theorem 3.1. *Let $\Delta(z) = z^{n+1}I_p$. There is a bijective correspondence between outer detectable subspaces with respect to the pair $(\mathscr{C},\mathscr{A})$, defined as in* (5) *which have codimension $k \leqslant n$ and similarity classes of k-dimensional observable, stable pairs $(C,A) \in \mathbb{R}^{p\times k}\times\mathbb{R}^{k\times k}$.*

Now an observable, stable pair $(C,A) \in \mathbb{R}^{p\times k}\times \mathbb{R}^{k\times k}$ determines an inner function U of McMillan degree k which is unique up to a constant right unitary factors. In fact, choosing the normalization $U(\infty) = I$, it is given by

$$U(z) = \left(\begin{array}{c|c} A & -ZC^* \\ \hline C & I \end{array}\right),$$

where Z is the unique positive-definite solution of the homogeneous Riccati equation $ZA^* + AZ + ZC^*CZ = 0$. Thus, we can state:

Corollary 3.1. *There is a bijective correspondence between outer detectable subspaces with respect to $(\mathscr{C}, \mathscr{A})$ of codimension $k \leqslant n$ and equivalence classes of $p \times p$ inner functions in the right half plane of McMillan degree k modulo right multiplication by constant unitary matrices.*

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