Remarks on the Inversion of Hankel Matrices

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ABSTRACT

The paper continues the investigation into the links between algebraic system theory, more specifically the partial realization problem, and the problem of Hankel matrix inversion. The representation of the inverse of a Hankel matrix as a Bezoutiant and the Bezout equation play a principal role.

1. INTRODUCTION

In [6] the connection between the partial realization problem of system theory (see [11], [8]) and the inversion of Hankel matrices has been explored. The central idea was to re-prove a theorem of Lander [12] showing that the inverse of a Hankel matrix is a Bezoutiant. More precisely, suppose we are given a nonsingular Hankel matrix

$$H = \begin{pmatrix} g_1 & \cdots & g_n \\ \vdots & & \vdots \\ g_n & \cdots & g_{2n-1} \end{pmatrix}; \tag{1.1}$$

then any minimal rational extension, or equivalently, any minimal partial realization, is determined by the choice of $\xi = g_{2n}$. Let the extension be denoted by $g_\xi(z)$, and let

$$g_\xi(z) = \frac{p_\xi(z)}{q_\xi(z)} \tag{1.2}$$

with p_ξ and q_ξ coprime and q_ξ monic, and, by minimality, of degree n. We

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will reserve the use of p and q for the polynomials arising out of the choice $\xi = 0$.

Since p_ξ and q_ξ are coprime, there exist polynomials a and b, unique if we require additionally that $\deg a < \deg q$, such that

$$ap_\xi + bq_\xi = 1. \tag{1.3}$$

The theorem referred to before can now be stated.

THEOREM 1.1. *Let H be the Hankel matrix*

$$H = \begin{pmatrix} g_1 & \cdots & g_n \\ \vdots & & \vdots \\ g_n & \cdots & g_{2n-1} \end{pmatrix}, \tag{1.4}$$

which is assumed to be nonsingular. Let $g_\xi = p_\xi / q_\xi$ be any minimal extension of the sequence $g_1, \dots, g_{2n-1}$ with $\xi = g_{2n}$, p_ξ and q_ξ coprime, and q_ξ monic. Let a be the unique polynomial of degree $< n$ satisfying the Bezout equation (1.3). *Then we have*

$$H^{-1} = B(a, q_\xi), \tag{1.5}$$

i.e., the inverse is given by the Bezoutiant of the polynomials a and q_ξ.

Before passing on, a few remarks are in order. First, as has been shown in [6], if we set

$$q_\xi(z) = z^n + q_{n-1}(\xi) z^{n-1} + \cdots + q_0(\xi), \tag{1.6}$$

then the coefficients of q_ξ are given by the solution of the system of linear equations

$$\begin{pmatrix} g_1 & \cdots & g_n \\ \vdots & & \vdots \\ g_n & \cdots & g_{2n-1} \end{pmatrix} \begin{pmatrix} q_0 \\ \vdots \\ q_{n-1} \end{pmatrix} = - \begin{pmatrix} g_{n+1} \\ \vdots \\ g_{2n-1} \\ \xi \end{pmatrix}. \tag{1.7}$$

Thus it is clear that q_ξ depends linearly on ξ. Also note that while all results are stated in terms of Hankel matrices, they have an easy translation to the Toeplitz case.

We can state now the following.

THEOREM 1.2. *Given the Hankel matrix of* (1.4), *which is assumed to be nonsingular. Let* $g_\xi = p_\xi / q_\xi$ *be any minimal rational extension of the sequence* $g_1, \dots, g_{2n-1}$ *with* $\xi = g_{2n}$, p_ξ *and* q_ξ *coprime, and* q_ξ *monic. Let* $g = p/q$ *be the one corresponding to* $\xi = 0$. *Also let* a *be the unique polynomial of degree* $< n$ *satisfying the Bezout equation* (1.3). *Then*:

(i) *a and b are independent of ξ.*

(ii) $q_\xi(z) = q(z) - \xi a(z)$.

(iii) $p_\xi(z) = p(z) + \xi b(z)$.

(iv) *Setting* $a(z) = a_0 + \cdots + a_{n-1}z^{n-1}$, *the* a_i *are solutions of the system of linear equations*

$$\begin{pmatrix} g_1 & \cdots & g_n \\ \vdots & & \vdots \\ g_n & \cdots & g_{2n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{1.8}$$

Defining the control basis polynomials corresponding to q_ξ *by*

$$e_i(\xi, z) = \pi_+ z^{-i} q_\xi = q_i(\xi) + q_{i+1}(\xi) z + \cdots + z^{n-i}, \tag{1.9}$$

$i = 1, \dots, n$, *and the polynomials* a_i *by*

$$a_i = \pi_+ z^{-i} a, \tag{1.10}$$

then:

(v) *The polynomial* p_ξ *has the representation*

$$p_\xi = \sum g_i e_i(\xi). \tag{1.11}$$

(vi) *The polynomial* b *of* (1.3) *has the representation*

$$b(z) = \sum g_i a_i(z), \tag{1.12}$$

i.e., b *also is independent of* ξ.

Proof. Let p_ξ, q_ξ be the polynomials in the coprime representation $g_\xi = p_\xi/q_\xi$ corresponding to the choice $g_{2n} = \xi$. Let a and b be the unique polynomials, assuming $\deg a < \deg q_\xi$, such that the Bezout equation

$$ap_\xi + bq_\xi = 1 \tag{1.13}$$

is satisfied. We rewrite this as

$$\frac{ap_\xi}{q_\xi} + b = \frac{1}{q_\xi}. \tag{1.14}$$

The right hand side has an expansion of the form

$$\frac{1}{q_\xi(z)} = \frac{i}{z^n} + \frac{\sigma_{n+1}}{z^{n+1}} + \cdots .$$

Equating nonnegative indexed coefficients in (1.14) we obtain

$$\begin{aligned} a_{n-1}g_1 &= -b_{n-2}, \\ a_{n-1}g_2 + a_{n-2}g_1 &= -b_{n-3}, \\ &\vdots \\ a_{n-1}g_{n-1} + \cdots + a_1 g_1 &= -b_0, \end{aligned}$$

or in matrix form

$$\begin{pmatrix} a_{n-1} & & 0 \\ \vdots & \ddots & \\ a_1 & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} g_0 \\ \vdots \\ g_{n-1} \end{pmatrix} = - \begin{pmatrix} b_{n-2} \\ \vdots \\ b_0 \end{pmatrix}. \tag{1.15}$$

Polynomially this means that

$$b(z) = \sum g_i a^i(z). \tag{1.16}$$

Equating coefficients of negative powers of z, we have

$$\begin{aligned} a_0 g_1 + \cdots + a_{n-1} g_n &= 0, \\ a_0 g_2 + \cdots + a_{n-1} g_{n+1} &= 0, \\ &\vdots \\ a_0 g_{n-1} + \cdots + a_{n-1} g_{2n-2} &= 0, \\ a_0 g_n + \cdots + a_{n-1} g_{2n-1} &= 1, \end{aligned}$$

or

$$\begin{pmatrix} g_1 & \cdots & g_n \\ \vdots & & \vdots \\ g_n & \cdots & g_{2n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{1.17}$$

Thus we use (1.17) to solve for the a_i and then (1.15) to calculate the b_i. In particular this proves that $a(z) = \sum a_i z^i$ and $b(z) = \sum b_i z^i$ are independent of the choice of $\xi = g_{2n}$, and we have proved statements (i), (iv), and (vi).

Now $g = p/q$ implies $p = qg$. In terms of coefficients we have

$$\begin{aligned} p_{n-1} &= g_1, \\ p_{n-2} &= g_2 + q_{n-1} g_1, \\ &\vdots \\ p_0 &= g_n + q_{n-1} g_{n-1} + \cdots + q_1 g_1, \end{aligned}$$

or

$$\begin{pmatrix} p_0 \\ \vdots \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} q_1 & \cdot & \cdot & \cdot & q_{n-1} & 1 \\ \vdots & & & \cdot & \cdot & \\ \vdots & & \cdot & \cdot & & \\ \vdots & \cdot & \cdot & & & \\ q_{n-1} & \cdot & & & & \\ 1 & & & & & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}. \tag{1.18}$$

This is equivalent to

$$p_\xi(z) = g_1 e_1(\xi, z) + \cdots + g_n e_n(\xi, z), \tag{1.19}$$

which proves (v).

In [6] it has been shown that

$$q(z)=(\det H)^{-1}\begin{vmatrix} g_1 & \cdots & g_n & 1 \\ \cdot & & \cdot & z \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ g_n & \cdots & g_{2n-1} & z^{n-1} \\ g_{n+1} & \cdots & \xi & z^n \end{vmatrix}. \tag{1.20}$$

So $q_\xi(z)=q(z)-\xi r(z)$, with

$$r(z)=\begin{vmatrix} g_1 & \cdots & g_{n-1} & 1 \\ \cdot & & \cdot & z \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ g_n & \cdots & g_{2n-2} & z^{n-1} \end{vmatrix}. \tag{1.21}$$

We will show that $r=a$. From the Bezout equations

$$ap+bq=1 \quad \text{and} \quad ap_\xi+bq_\xi=1$$

it follows by subtraction that

$$a(p_\xi-p)+b(q_\xi-q)=0.$$

Since $q_\xi=q-\xi r$ we must have

$$a(p_\xi-p)=\xi br.$$

The coprimeness of a and b implies a is a divisor of r. Setting $r=ad$, it follows that

$$p_\xi=p+\xi bd \tag{1.22}$$

and

$$q_\xi=q-\xi ad. \tag{1.23}$$

Now

$$g_\xi - g = \frac{p+\xi bd}{q-\xi ad} - \frac{p}{q} = \frac{\xi(ap+bq)d}{q(q-\xi ad)} = \frac{\xi d}{q(q-\xi ad)}.$$

Now g_ξ and g have expansions in powers of z^{-1} agreeing for the first $2n-1$ terms, and the next term has to be ξz^{-2n}. This implies $d=1$, and so (ii) and (iii) are proved. This completes the proof. ■

Incidentally, parts (ii) and (iii) of the theorem provide a nice parametrization of all solutions to the minimal partial realization problem arising out of a nonsingular Hankel matrix.

COROLLARY 1.3. *Given the nonsingular Hankel matrix* (1.1). *If* $g = p/q$ *is the minimal rational extension corresponding to the choice* $g_{2n} = 0$, *then the minimal rational extension corresponding to* $g_{2n} = \xi$ *is given by*

$$g_\xi = \frac{p+\xi b}{q-\xi a}, \tag{1.24}$$

where a, b *are the polynomials arising in the Bezout equation* (1.3).

This theorem explains why the particular minimal partial realization chosen does not influence the computation of the inverse, as of course it should not. Indeed, we see that

$$B(a,q_\xi) = B(a,q-\xi a) = B(a,q) - \xi B(a,a) = B(a,q)$$

and so

$$H^{-1} = B(a,q_\xi) = B(a,q). \tag{1.25}$$

Thus we are led to the main point of this paper. To invert the Hankel matrix H it suffices to know a and q_ξ, and hence it suffices to solve two equations of the form

$$\begin{pmatrix} g_1 & \cdots & g_n \\ \vdots & & \vdots \\ g_n & \cdots & g_{2n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \tag{1.26}$$

rather than n, as would be the case for the inversion of an arbitrary matrix. Of course there is nothing new in this observation, and it goes back at least to the work of Heinig and Rost [9]. In this connection we refer also to Fiedler [2]. However, the basic idea that to invert Hankel, or Toeplitz, matrices one needs usually to solve two equations of the form (1.26) goes back to the work of Gohberg and Semencul [7].

It is our purpose in the next section to give some easy derivations for other results of this type. We do this in a polynomial context, stressing the connections with the partial realization problem.

Most of the contents of this paper and of [6] have multivariable generalizations. An exposition of this will be forthcoming.

2. HANKEL MATRIX INVERSION: A POLYNOMIAL APPROACH

As we saw in the preceding section, to invert the Hankel matrix of (1.1) it suffices to know the polynomials a and q. Both have been shown to be derived as solutions to the system (1.26). We proceed now to give this solution a polynomial interpretation.

We use the theory of polynomial models freely. The little necessary background and notation for what follows can be found in [4–6].

Let X_q be the polynomial model defined by q, Z; $X_q \to X_q$ the module homomorphism defined by

$$Zf = \pi_q pf = p(S_q)f \qquad \text{for } f \text{ in } X_q. \tag{2.1}$$

As a consequence of the Bezout equation (1.3) it follows that Z^{-1} is given by

$$Z^{-1}f = \pi_q af = a(S_q)f \qquad \text{for } f \text{ in } X_q. \tag{2.2}$$

Also, the Bezoutiant of a and q is given as the matrix representation of Z^{-1} relative to the control basis and the standard basis of X_q, that is,

$$B(a,q) = [Z^{-1}]^{B_0}_{B_c}. \tag{2.3}$$

Thus, if we consider the system (1.26) and let

$$x(z) = \sum x_i z^i \quad \text{and} \quad r(z) = \sum r_i e_i(z),$$

then

$$x = H^{-1}\sum r_i e_i = Z^{-1}\sum r_i e_i$$
$$= \pi_q a \sum r_i e_i = \sum r_i \pi_q a e_i.$$

Let us consider some special cases first. Let

$$r_i = \begin{cases} 0, & i = 1,\dots,n-1, \\ 1, & i = n. \end{cases}$$

Then, since $e_n = 1$, the solution is given by

$$x = \pi_q a = a \tag{2.4}$$

as stated in Theorem 1.2(v).

Next let

$$r_i = \begin{cases} 0, & i = 2,\dots,n, \\ 1, & i = 1. \end{cases}$$

Then the solution x is given by

$$x = \Pi_q a e_1 = \frac{\pi_q a[q(z) - q(0)]}{z}.$$

Thus for some polynomial σ, the solution x is given by

$$x(z) = \frac{a(z)[q(z) - q(0)]}{z} - q(z)\sigma(z)$$
$$= q(z)\left[\frac{a(z) - a(0)}{z} - \sigma(z)\right] + \frac{q(z)a(0) - a(z)q(0)}{z}.$$

Since $\deg x < \deg q$, it follows that

$$\sigma(z) = \frac{a(z) - a(0)}{z}$$

and hence

$$x(z)=\frac{q(z)a(0)-a(z)q(0)}{z}. \tag{2.5}$$

So in case $a(0)\neq 0$, we set

$$y(z)=a(0)^{-1}zx(z)=q(z)-a(0)^{-1}q(0)a(z); \tag{2.6}$$

it follows that

$$B(a,y)=B\big(a,q-a(0)^{-1}q(0)a\big)=B(a,q). \tag{2.7}$$

Also note that (2.5) implies

$$x_{n-1}=a_0. \tag{2.8}$$

Thus we can state the following result, which in view of the representation results for Bezoutiants obtained by Pták [13] and Fuhrmann [6], is equivalent to the Gohberg-Semencul theorem.

THEOREM 2.1. *Let a and x be the solutions of the system* (1.26) *with the right hand side being given respectively by* $(0,\dots,0,1)^{\sim}$ *and* $(1,0,\dots,0)^{\sim}$*. Then if* $x_{n-1}\neq 0$*, the Hankel matrix H is invertible and*

$$H^{-1}=B(a,y). \tag{2.9}$$

Proof. We saw already that, assuming H is nonsingular, the inverse of H is the Bezoutiant of a and y. To complete the proof of the theorem we will show that the assumptions imply the invertibility of H.

The existence of a solution with the right hand side $(0,\dots,0,1)^{\sim}$ implies that the McMillan degree of a minimal rational extension of $g_1,\dots,g_{2n-1}$ is at least n. We will show that it has an extension of McMillan degree n. To this end let us define $q(z)$ by

$$q(z)=x_{n-1}^{-1}zx(z) \tag{2.10}$$

and $a(z)$ a before. Define the polynomial p by

$$p(z)=\sum g_i e_i(z), \tag{2.11}$$

where e_i is the control basis determined by q. Let $G=p/q=\Sigma G_i z^{-i}$. We will show that G is an extension of $g_1,\dots,g_{2n-1}$.

Since $\{e_1,\dots,e_n\}$ and $\{1,\dots,z^{n-1}\}$ are dual bases relative to the pairing introduced in [3],

$$\langle f,g\rangle=[q^{-1}f,g], \tag{2.12}$$

it follows that for $i=1,\dots,n$

$$G_i=[G,z^{i-1}]=[q^{-1}p,z^{i-1}]=\left\langle \sum g_j e_j, z^{i-1}\right\rangle=g_i.$$

Now as $p=Gq$, we have, equating coefficients,

$$G_{n+i}+q_{n-1}G_{n+i-1}+\cdots+q_1G_{i+1}=0. \tag{2.13}$$

Equation (1.26) with the right hand side $(1,0,\dots,0)\tilde{}$ can be rewritten, multiplying both sides by x_{n-1}^{-1}, as

$$\begin{pmatrix} g_1 & \cdots & g_n \\ \vdots & & \vdots \\ g_n & \cdots & g_{2n-1}\end{pmatrix}\begin{pmatrix} q_1 \\ \vdots \\ 1\end{pmatrix}=\begin{pmatrix} x_{n-1}^{-1} \\ 0 \\ \vdots \\ 0\end{pmatrix}$$

The last $n-1$ equations imply

$$g_{n+i}+q_{n-1}g_{n+i-1}+\cdots+q_1g_{i+1}=0 \tag{2.14}$$

for $i=1,\dots,n-1$. Since $G_i=g_i$ for $i=1,\dots,n$, the equalities extend the range $i=1,\dots,2n-1$. Thus G is a minimal rational extension of McMillan degree n. Therefore the Hankel matrix H is necessarily nonsingular. ∎

Note that in view of the representation result of Pták [13] (see also [6]), Equation (2.9) can also be rewritten as

$$H^{-1}=B(a,q)=\{a(\tilde{S})q\#(S)-q(\tilde{S})a\#(S)\}J,$$

where

$$J=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$S=\begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & & \cdot & \cdot \\ \cdot & & & & & \cdot & 0 & \cdot \\ \cdot & & & & & & 1 & 0 \\ \cdot & & & & & & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}.$$

Thus in this case the inverse of a Hankel matrix is determined by two of its columns, the first and the last, provided the extra condition $x_{n-1} \neq 0$ is satisfied. This is not always the case: see Iohvidov [10]. Ben-Artzi and Shalom [14] have shown that three columns of H^{-1} are always enough for its reconstruction. We give an alternative proof of their result. This uses the underlying equation for the control basis, namely

$$S_q e_i = e_{i-1} - q_{i-1} e_n, \qquad i = 1, \dots, n. \tag{2.15}$$

where e_{-1} is defined as the zero polynomial.

Let $(y_0, \dots, y_{n-1})^{\sim}$ and $(w_0, \dots, w_{n-1})^{\sim}$ be the solutions of the system (1.26) with the right hand side being $(\delta_{1,i}, \dots, \delta_{n,i})^{\sim}$ and $(\delta_{1,i-1}, \dots, \delta_{n,i-1})^{\sim}$ respectively. Set

$$y(z) = \sum y_i z^i, \qquad w(z) = \sum w_i z^i.$$

Then clearly

$$y = \pi_q a e_i \tag{2.16}$$

and

$$w = \pi_q a e_{i-1}. \tag{2.17}$$

We define now a polynomial h by

$$\begin{aligned} h(z) &= zy(z) - w(z) \\ &= za(S_q)e_i - a(S_q)e_{i-1}. \end{aligned} \tag{2.18}$$

Now in general, for f in X_q,

$$S_q f = zf(z) - q(z)\delta_f, \tag{2.19}$$

where δ_f is a constant depending linearly on f. Using this we compute

$$\begin{aligned} h(z) &= za(S_q)e_i - a(S_q)e_{i-1} \\ &= S_q a(S_q)e_i + q(z)\delta - a(S_q)e_{i-1} \\ &= a(S_q)\left[S_q e_i - e_{i-1}\right] + q(z)\delta \\ &= -q_{i-1}a(S_q)e_n + q(z)\delta \\ &= q(z)\delta - q_{i-1}a(z) \end{aligned}$$

as

$$a(S_q)e_n = \pi_q a e_n = \pi_q a \cdot 1 = a.$$

It remains to evaluate δ. From (2.19) it follows that δ is the coefficient of z^{n-1} of the polynomial $a(S_q)e_i = y$. So $\delta = y_{n-1}$. However, this can be evaluated in terms of the coefficients of a. Indeed,

$$a(S_q)e_i = \sum a_k S_q^k e_i. \tag{2.20}$$

It is quite easy to see that $S_q^k e_i$ has degree less than $n-1$ unless $k = i-1$. In that case it is monic of degree $n-1$. Thus from (2.20) it follows that the z^{n-1} coefficient of $a(S_q)e_i$ is a_{i-1}. Hence

$$\delta = y_{n-1} = a_{i-1}. \tag{2.21}$$

Thus we have the representation

$$h(z) = a_{i-1}q(z) - q_{i-1}a(z).$$

Clearly, if $a_{i-1} \neq 0$, then we can define a polynomial f by

$$f = a_{i-1}^{-1}h = q - a_{i-1}^{-1}q_{i-1}a, \tag{2.22}$$

and then

$$B(a, f) = B(a, q - a_{i-1}^{-1}q_{i-1}a) = B(a, q).$$

We can summarize this as follows.

THEOREM 2.2 (Ben-Artzi and Shalom). *Given the Hankel matrix H of (1.1), let $(a_0, \dots, a_{n-1})\tilde{}$, $(y_0, \dots, y_{n-1})\tilde{}$, and $(w_0, \dots, w_{n-1})\tilde{}$ be the solutions of the system (1.26) with the right hand side being $(0, \dots, 0, 1)\tilde{}$, $(\delta_{1,i}, \dots, \delta_{n,i})\tilde{}$, and $(\delta_{1,i-1}, \dots, \delta_{n,i-1})\tilde{}$. If $a_{i-1} \neq 0$, then the inverse of H can be written in terms of these columns.*

Next we pass to the analysis of a rather general case. Again we consider the system (1.26), with the right hand sides being given by $(\alpha_1, \dots, \alpha_n)\tilde{}$ and $(\beta_1, \dots, \beta_n)\tilde{}$. Let the respective solutions be given by $(y_0, \dots, y_{n-1})\tilde{}$ and $(w_0, \dots, w_{n-1})\tilde{}$. Set

$$y(z) = \sum y_i z^i \quad \text{and} \quad w(z) = \sum w_i z^i. \tag{2.23}$$

Then, by the same reasoning applied before, we have

$$y(z) = a(S_q)\sum \alpha_i e_i(z)$$

and

$$w(z) = a(S_q)\sum \beta_i e_i(z).$$

Let us define the polynomial $f(z)$ by

$$f(z) = zy(z) - w(z). \tag{2.24}$$

Then

$$\begin{aligned} f(z) &= za(S_q)\sum \alpha_i e_i(z) - a(S_q)\sum \beta_i e_i(z) \\ &= S_q a(S_q)\sum \alpha_i e_i(z) + q(z)\delta - a(S_q)\sum \beta_i e_i(z) \\ &= a(S_q)\sum \left[\alpha_i (S_q e_i)(z) - \beta_i e_i(z)\right] + q(z)\delta \\ &= a(S_q)\sum \left[\alpha_i (e_{i-1}(z) - q_{i-1}e_n(z)) - \beta_i e_i(z)\right] + q(z)\delta \\ &= q(z)\delta + a(S_q)\left[\sum (\alpha_i - \beta_{i-1})e_i(z) - \left\{\beta_n - \sum \alpha_i q_{i-1}\right\} e_n(z)\right] \\ &= q(z)\delta - \left[\beta_n - \sum \alpha_i q_{i-1}\right] a(z) + a(S_q)\sum (\alpha_i - \beta_{i-1})e_i(z). \end{aligned}$$

Now δ is the coefficient of z^{n-1} of the polynomial $\Sigma\alpha_i a(S_q)e_i$, and hence

$$\delta = \sum \alpha_i a_{i-1}. \tag{2.25}$$

Thus we can state the following theorem.

THEOREM 2.3. *Given the Hankel matrix* (1.1). *Let y and w be the polynomials corresponding to the solutions of the system* (1.26) *with the right hand sides $(\alpha_1,\dots,\alpha_n)^{\sim}$ and $(\beta_1,\dots,\beta_n)^{\sim}$ respectively. Let a be the polynomial corresponding to the solution of Equation* (1.8). *If $\Sigma\alpha_i a_{i-1} \neq 0$ and $\beta_{i-1} = \alpha_i$ for $i = 2,\dots,n$, then H is invertible and its inverse given by*

$$H^{-1} = B(a, \delta^{-1}f). \tag{2.26}$$

Proof. Follows from the fact that

$$B(a,\delta^{-1}f) = B\left(a, q - \delta^{-1}\left[\beta_n - \sum \alpha_i q_{i-1}\right] a\right) = B(a,q).$$

That H is nonsingular follows by reasoning similar to that employed in the proof of Theorem 2.1. ■

If, for the previous polynomial y, we define the polynomial f by

$$f(z) = zy(z) - y(z), \tag{2.27}$$

then the previous computation yields

$$f(z) = q(z)\delta - \left[\alpha_n - \sum \alpha_i q_{i-1}\right] a(z) + a(S_q)\sum (\alpha_i - \alpha_{i-1})e_i(z).$$

THEOREM 2.4 (Ben-Artzi and Shalom). *Given the Hankel matrix* (1.1). *Let y be the polynomial corresponding to the solution of the system* (1.26) *with the right hand side* $(\alpha_1,\dots,\alpha_n)^{\sim}$ *Let a be the polynomial corresponding to the solution of Equation* (1.8). *If $y_{n-1}\neq 0$ and $\alpha_{i-1}=\alpha_i$ for $i=1,\dots,n-1$, then H is invertible and its inverse given by*

$$H^{-1}=B\left(a,y_{n-1}^{-1}f\right). \tag{2.28}$$

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