

# Minimal Factorizations of Rational Matrix Functions in Terms of Polynomial Models

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## ABSTRACT

The problem of minimal factorization of rational bicausal matrices is considered, using polynomial models. The factors are given explicitly in terms of polynomial matrices.

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## 1. INTRODUCTION

Much has been written about the problem of minimal factorization of rational matrix functions, for example, [14, 12, 4, 13]. A comprehensive treatment of the problem is given in [2], using the concept of supporting projections, which relies strongly on systems and realizations.

In this paper, we attempt to simplify the solution of the problem by using polynomial models. The solution need not go into systems and realizations, but uses polynomial matrices, and the algebraic aspects of the system-theoretic interpretation are exhibited.

The factorization itself, whenever it exists, is given concretely in terms of polynomial matrices.

In Section 4 we consider a few particular cases, and in Section 5, symmetric matrices are treated.

It is well known that the transfer function of a series connection of two systems is the product of the two transfer functions. In particular, it was shown by Callier and Nahum [3] how minimality of a series connection of two systems is related to certain coprimeness conditions of the corresponding polynomial matrices. Our objective is to tackle the inverse problem, namely, when a transfer function can be factored minimally into the product of two

transfer functions. In this context minimality means that the McMillan degree of the transfer function  $W$ , denoted as  $\delta(W)$ , is equal to the sum of the McMillan degrees of the factors.

Factorization of polynomial matrices are related to invariant subspaces and  $(A, B)$ - or  $(C, A)$ -invariant subspaces. Here the work by Emre and Hautus [5], Antoulas [1], Fuhrmann and Willems [11], and Fuhrmann [8] is particularly relevant. In much the same way, one expects some such relation in the case of factorizations of transfer functions. As we shall see, this indeed turns out to be the case.

We shall assume throughout that  $W$  is a square proper matrix with constant term equal to the identity. We will call such a matrix a normalized bicausal isomorphism. In particular, the inverse of a normalized bicausal isomorphism is a normalized bicausal isomorphism.

Assume  $W = T^{-1}D$  is a left coprime factorization of  $W$  into nonsingular polynomial matrices  $T$  and  $D$ . Since  $W$  is normalized bicausal, it has the form  $W = I + T^{-1}U$ , where  $U = D - T$  is a polynomial matrix such that  $T^{-1}U$  is strictly proper,  $T$  and  $U$  being left coprime.

In  $F^m((z))$ , define  $\Pi_+$  and  $\Pi_-$  as the projections on the polynomial and strictly proper parts of a vector function respectively. Let  $T$  be a nonsingular  $m \times m$  polynomial matrix. We define a projection on  $F^m\{z\}$ :

$$\Pi_T f = T \Pi_- T^{-1} f$$

and  $X_T = \text{Im } \Pi_T$ . The restricted shift operator  $S_T$  operating in  $X_T$  is defined as  $S_T f = \Pi_T z f$ .

Let  $\Sigma = \{A, B, C, I\}$  be the associated minimal realization of  $W$  in the state space  $X = X_T$ , with  $F^m$  being the input-output space. Hence  $Af = S_T f$ ,  $B\xi = U\xi$ ,  $Cf = [T^{-1}f]_{-1}$  for  $f \in X_T$ ,  $\xi \in F^m$ .

For further details and notation, see [6].

By [2, Theorem 4.8], we have the following:

**THEOREM.**  *$W$  can be factored into two bicausal factors  $W = W_1 W_2$  minimally, i.e.  $\delta(W) = \delta(W_1) + \delta(W_2)$ , iff  $\Sigma$  can be represented as the series connection of two systems  $\Sigma_1, \Sigma_2$  (first  $\Sigma_2$ , then  $\Sigma_1$ ) such that if*

$$\Sigma_i = \{A_i, B_i, C_i, I\} \quad (i = 1, 2)$$

then

$$X = X_1 \oplus X_2,$$

and  $X_1$  is  $A$ -invariant whereas  $X_2$  is  $(A - BC)$ -invariant. In this case  $W_i$  is the transfer function of the system  $\Sigma_i$ .

We shall actually re-prove this theorem in terms of polynomial models.

## 2. PRELIMINARIES ON INVARIANT SUBSPACES

In terms of polynomial models,  $X_1$  is an  $A$ -invariant or  $S_T$ -invariant subspace of  $X_T$  iff it is of the form

$$X_1 = T_1 X_{T_2}$$

for some factorization  $T = T_1 T_2$  into nonsingular factors. Also,  $A^\# = A - BC$  is the state-space operator of the inverse system with transfer function  $W^{-1} = D^{-1}T$ , i.e.

$$A^\# = S_D = S_{T+U}.$$

A simple calculation confirms that indeed

$$S_{T+U}f = S_T f - U [T^{-1}f]_{-1} = (A - BC)f.$$

In particular, any  $S_D$ -invariant subspace is  $(A, B, C)$ -invariant, and hence both  $(A, B)$ - and  $(C, A)$ -invariant. We can actually prove that for the associated realization of any strictly proper transfer function  $G = T^{-1}U$  in the state space  $X_T$ ,  $V$  is an  $(A, B, C)$ -invariant subspace iff it is of the form  $EX_F$ , where  $EF = T + UK$  for any constant matrix  $K$  of appropriate size.

Also, since  $T^{-1}D$  is bicausal isomorphism,  $X_T$  and  $X_D$  are equal as sets, or as vector spaces, although they have different module structures (see [11]). Thus,  $X_2$  is an  $S_D$ -invariant subspace of  $X_T$  iff it is of the form

$$X_2 = D_1 X_{D_2}$$

for some factorization  $D = D_1 D_2$ .

Thus, the existence of a minimal factorization of  $W$  is equivalent to the existence of factorizations  $T = T_1 T_2$ ,  $D = D_1 D_2$  such that

$$X_T = T_1 X_{T_2} \oplus D_1 X_{D_2}. \tag{2.1}$$

In Theorem 3.3 we shall specify in terms of the polynomial factors when such decompositions exist. In this case, the factors  $W_1$  and  $W_2$  of  $W$  are the transfer functions of the restrictions and projections of the system  $\Sigma = (A, B, C)$  in the state spaces  $T_1X_{T_2}$  and  $D_1X_{D_2}$  respectively. We shall specify the factors  $W_1$  and  $W_2$  in Theorem 3.4.

Since we are dealing with geometric relations between subspaces that are in particular  $(C, A)$ -invariant, it is of interest to analyse the connection between these geometric properties and the arithmetic of the corresponding nonsingular polynomial matrices. This analysis is of interest on its own.

We begin by studying inclusion of  $(C, A)$ -invariant subspaces.

**LEMMA 2.1.** *Let  $M_1, M_2$  be two  $(C, A)$ -invariant subspaces of  $X_T$  with the representations  $M_i = E_iX_{F_i}$ . Then  $M_1 \subset M_2$  if and only if  $E_1 = E_2Y$  for some polynomial matrix  $Y$ .*

*Proof.* Assume  $E_1 = E_2Y$  for some polynomial matrix  $Y$ . Then

$$M_1 = E_1X_{F_1} = E_2YX_{F_1} \subset E_2X_{YF_1} = E_2X_{F_2} = M_2.$$

Conversely, assume  $E_1X_{F_1} \subset E_2X_{F_2}$ . Then, since clearly  $E_2X_{F_2} \subset E_2F^p[z]$ , we have

$$E_1X_{F_1} \subset E_2F^p[z]. \quad (2.2)$$

From the inclusion  $M_1 \subset M_2$  it follows that  $M_1$  and  $M_2$  are compatible  $(C, A)$ -invariant subspaces. Thus there exist  $\bar{F}_1$  and  $\bar{F}_2$  such that

$$E_1\bar{F}_1 = E_2\bar{F}_2 = T'$$

with  $T^{-1}T'$  a bicausal isomorphism. Now

$$E_1X_{\bar{F}_1} = E_1X_{\bar{F}_1} \quad \text{and} \quad E_1\bar{F}_1F^p[z] \subset E_2F^p[z].$$

Taking this together with (2.1), we have the inclusion

$$E_1F^p[z] \subset E_2F^p[z],$$

which implies, by [6], that  $E_1 = E_2Y$ . ■

Next we study the intersection of  $(C, A)$ -invariant subspaces, which is again a  $(C, A)$ -invariant subspace.

**LEMMA 2.2.** *Let  $M_1, M_2 \subset X_T$  be  $(C, A)$ -invariant subspaces, and let  $M_i = E_i X_{F_i}$ ,  $i = 1, 2$ . Let  $M = M_1 \cap M_2$ . Then*

$$M = X_T \cap EF^p[z] \quad (2.3)$$

where  $E$  is the l.c.r.m. of  $E_1$  and  $E_2$ .

*Proof.* Let

$$f \in E_1 X_{F_1} \cap E_2 X_{F_2} \subset E_1 F^p[z] \cap E_2 F^p[z] = EF^p[z],$$

where  $E$  is the l.c.r.m. of  $E_1$  and  $E_2$ . But  $f$  also belongs to  $X_T$ , so  $f \in X_T \cap EF^p[z]$ .

Conversely, let  $E$  be any common right multiple of  $E_1$  and  $E_2$ . Then

$$E = E_i Y_i \text{ and } EF^p[z] \subset E_i F^p[z], \quad i = 1, 2.$$

In turn this implies

$$X_T \cap EF^p[z] = X_T \cap E_i F^p[z], \quad i = 1, 2,$$

and so

$$X_T \cap EF^p[z] = M_1 \cap M_2 = M.$$

In particular, if  $E$  is the l.c.r.m. of  $E_1$  and  $E_2$ , we have the equality (2.3). ■

Now the sum of two  $(C, A)$ -invariant subspaces is in general no longer  $(C, A)$ -invariant. However, there is a unique smallest  $(C, A)$ -invariant subspace containing this sum. This is characterized next.

**LEMMA 2.3.** *Let  $M_1, M_2 \subset X_T$  be  $(C, A)$ -invariant subspaces, and let*

$$M_i = E_i X_{F_i}.$$

*Let  $M$  be the smallest  $(C, A)$ -invariant subspace containing both  $M_1$  and  $M_2$ . Then  $M = X_T \cap EF^p[z]$ , where  $E$  is a g.c.l.d. of  $E_1$  and  $E_2$ .*

*Proof.* Let  $M$  be the smallest  $(C, A)$ -invariant subspace containing  $M_1 + M_2$ . Assume  $M = EX_F$ . Since  $M_1 \subset M$ , it follows, by Lemma 2.1, that  $E$

is a left divisor of  $E_1$ . Similarly  $E$  is a left divisor of  $E_2$ . So  $E$  is a common left divisor.

Let  $E'$  be any common left divisor of  $E_1$  and  $E_2$ . Thus  $E_1 = E'Y_1$ ,  $E_2 = E'Y_2$ , and so

$$E_1 X_{F_1} = E'Y_1 X_{F_1} \subset E'X_{Y_1 F_1} = X_T \cap E'F^p[z].$$

Analogously,

$$E_2 X_{F_2} \subset X_T \cap E'F^p[z].$$

But this implies that  $E'$  is a left divisor of  $E$ . Hence  $E$  is a g.c.l.d. of  $E_1$  and  $E_2$ .

Conversely, let  $E$  be a g.c.l.d. of  $E_1$  and  $E_2$ . Then  $X_T \cap EF^p[z]$  is  $(C, A)$ -invariant and has the representation  $EX_F$  for some  $F$ . Since  $E_i = EY_i$ , it follows that

$$M_i = X_T \cap E_i F^p[z] \subset X_T \cap EF^p[z] = M.$$

So  $M_1 + M_2 \subset M$ . Minimality follows by an argument as in the first part of the proof.  $\blacksquare$

### 3. MINIMAL FACTORIZATIONS IN TERMS OF POLYNOMIAL MODELS

In this section we shall prove some necessary and sufficient conditions for the existence of minimal factorizations of  $W$ , or equivalently, for the existence of a direct-sum decomposition of the state space of the form (2.1). We shall re-prove this equivalence in terms of polynomial models and polynomial matrices.

We first prove two lemmas.

**LEMMA 3.1.** *Let  $T = T_1 T_2$  and  $D = D_1 D_2$  be any factorizations of  $T$  and  $D$  respectively into nonsingular factors, and let*

$$E = T_1 \bar{D}_1 = D \bar{T}_1$$

*be a least common right multiple of  $T_1$  and  $D_1$ . Then*

$$T_1 X_{T_2} \cap D_1 X_{D_2} = X_T \cap EF^m[z].$$

*Proof.* We first note that by [6]

$$T_1 F^m[z] \cap D_1 F^m[z] = EF^m[z],$$

and of course

$$T_1 X_{T_2} \in D_1 X_{D_2} \subset X_T = X_D.$$

Hence

$$T_1 X_{T_2} \cap D_1 X_{D_2} \subset X_T \cap (T_1 F^m[z] \cap D_1 F^m[z]) = X_T \cap EF^m[z].$$

Conversely, if  $Ef \in X_T \cap EF^m[z]$ , then  $T^{-1}Ef$  is strictly proper, so that  $T^{-1}Ef = T_2^{-1}(T_1^{-1}E)f = T_1^{-1}\bar{D}_1 f$ . But  $T^{-1}\bar{D}_1 f$  strictly proper implies  $\bar{D}_1 f \in X_{T_2}$ . Thus,

$$T_1 \bar{D}_1 f = Ef \in T_1 X_{T_2}.$$

By the same considerations and bearing in mind that  $X_T = X_D$ , we have  $Ef \in D_1 X_{D_2}$ . Hence  $Ef \in T_1 X_{T_2} \cap D_1 X_{D_2}$ . ■

LEMMA 3.2. *With the same notation as in Lemma 3.1,  $X_T \cap EF^m[z] = \{0\}$  iff all the left Wiener-Hopf factorization indices of  $T^{-1}E$  are nonnegative.*

*Proof.* Let  $T^{-1}E = \Gamma\Delta V^{-1}$ , with  $\Gamma$  bicausal,  $\Delta = \text{diag}(z^{\kappa_1}, \dots, z^{\kappa_m})$ , and  $V$  unimodular, be a left W-H factorization of  $T^{-1}E$ .

If, say  $\kappa_1 < 0$ , then

$$T^{-1}EV \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \Gamma \begin{pmatrix} z^{\kappa_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is strictly proper. Hence

$$EV \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in X_T \cap EF^m[z]$$

is nonzero.

Conversely, suppose all the  $\kappa_i$ 's are nonnegative. Note that if  $V$  is unimodular, then

$$EF^m[z] = EVF^m[z],$$

so we may as well take  $EV$  instead of  $E$  and assume the factorization  $T^{-1}E = \Gamma\Delta$ .

Assume  $0 \neq Ef \in X_T \cap EF^m[z]$ , so  $T^{-1}Ef$  is strictly proper. Thus,  $\Gamma T^{-1}Ef = \Delta f$  is also strictly proper. However, this cannot be, since  $f$  is a polynomial vector and  $\Delta$  a polynomial matrix. Thus  $f = 0$ . ■

**REMARK.** Since we assume  $T$  and  $D$  to be left coprime, then any left factors  $T_1$  and  $D_1$  of  $T$  and  $D$  respectively are left coprime.

We are now in a position to prove:

**THEOREM 3.3.** *Let  $W = T^{-1}D$  be a bicausal isomorphism, and let  $T = T_1T_2$ ,  $D = D_1D_2$  be factorizations of  $T$  and  $D$ . Then*

$$X_T = T_1X_{T_2} \oplus D_1X_{D_2}$$

*iff there exists a least common right multiple*

$$E = T_1\bar{D}_1 = D_1\bar{T}_1$$

*of  $T_1$  and  $D_1$  such that  $T^{-1}E$  is a bicausal isomorphism.*

*Proof.* Assume first that there exists a l.c.r.m. of  $T_1$  and  $D_1$  such that  $T^{-1}E$  is a bicausal isomorphism. Note that since  $T^{-1}D$  is a bicausal, then  $T^{-1}E$  is bicausal iff  $D^{-1}E$  is.

Now, if  $T^{-1}E$  is bicausal, then  $X_T = X_E$ , and by minimality of  $E$ ,  $\bar{D}_1$  and  $\bar{T}_1$  are right coprime. Also, by left coprimeness of  $T_1$  and  $D_1$ , the representations

$$E = T_1\bar{D}_1 = D_1\bar{T}_1$$

imply, by [6],  $X_T = X_E = T_1X_{\bar{D}_1} \oplus D_1X_{\bar{T}_1}$ . However,  $T^{-1}E = T_2^{-1}\bar{D}_1$  is bicausal, so  $X_{\bar{D}_1} = X_{T_2}$ , and by symmetry considerations,  $X_{\bar{T}_1} = X_{D_2}$ . Hence,  $X_T = T_1X_{T_2} \oplus D_1X_{D_2}$ .

Conversely, assume  $X_T = T_1X_{T_2} \oplus D_1X_{D_2}$ , and let  $E$  be any least common right multiple of  $T_1$  and  $D_1$ . Since  $T_1X_{T_2} \cap D_1X_{D_2} = \{0\}$  by Lemmas 3.1 and



3.2, all the left W-H indices of  $T^{-1}E$  are nonnegative. Moreover,

$$\dim X_{T_2} + \dim X_{D_2} = \dim X_T,$$

or  $\text{deg det } T_2 + \text{deg det } D_2 = \text{deg det } T_1 + \text{deg det } D_1 = \text{deg det } T = \text{deg det } D$ .

Let  $E = T_1 \bar{D}_1 = D_1 \bar{T}_1$ . By left coprimeness of  $T_1$  and  $D_1$  and right coprimeness of  $\bar{D}_1$  and  $\bar{T}_1$ , we have up to a constant factor

$$\det E = (\det T_1)(\det D_1),$$

so that  $\text{deg det } E = \text{deg det } T$ . Now, if  $T^{-1}E = \Gamma \Delta V^{-1}$  is a left W-H factorization with  $\Gamma$  bicausal, and  $V$  unimodular,  $\Delta = \text{diag}(z^{\kappa_1}, \dots, z^{\kappa_m})$ , the nonnegativity of all the  $\kappa_i$ 's and the equality  $\text{deg det } E = \text{deg det } T$  imply  $\kappa_i = 0$ ,  $i = 1, \dots, m$ . Hence  $T^{-1}E = \Gamma V^{-1}$ , or  $T^{-1}(EV) = \Gamma$ , so that the least common right multiple of  $T_1$  and  $D_1$ ,  $EV$ , satisfies the condition that  $T^{-1}(EV)$  is bicausal. ■

**REMARK.** The decomposition  $X_T = X_E = T_1 X_{\bar{D}_1} \oplus D_1 X_{\bar{T}_1}$  means that on  $X_T$  we can redefine the  $F[z]$ -module structure so that both  $T_1 X_{T_2}$  and  $D_1 X_{D_2}$  become submodules. Thus the invariant subspace  $T_1 X_{T_2}$  and the  $(C, A)$ -invariant subspace  $D_1 X_{D_2}$ , both regarded as  $(C, A)$ -invariant subspaces, are compatible, in the sense that for the same constant output injection matrix  $H$ , both are  $(A + HC)$ -invariant. In this case,  $A + HC$  is simply  $S_E$ .

This should come as no surprise, since we know from Wonham [15] that two  $(A, B)$ -invariant subspaces are compatible iff their intersection is also  $(A, B)$ -invariant. Hence, by duality, any two  $(C, A)$ -invariant subspaces are compatible iff their sum is also  $(C, A)$ -invariant. Incidentally, a direct proof of this fact is rather elusive.

With the above conditions we can give a concrete representation of the factors of  $W = W_1 W_2$  in terms of polynomial matrices whenever such a minimal factorization exists.

**THEOREM 3.4.**  $W = T^{-1}D$  has a minimal factorization  $W = W_1 W_2$  iff there exist factorizations  $T = T_1 T_2$ ,  $D = D_1 D_2$  such that

$$X_T = T_1 X_{T_2} \oplus D_1 X_{D_2}. \tag{3.1}$$

In this case,  $W_1 = T_2^{-1} \bar{D}_1$ ,  $W_2 = \bar{T}_1^{-1} D_2$ , where  $E = T_1 \bar{D}_1 = D_1 \bar{T}_1$  is a l.c.r.m. of  $T_1$  and  $D_1$  such that  $T^{-1}E$  is bicausal.

*Proof.* If there exist factorizations of  $T$  and  $D$  such that (3.1) holds, then by Theorem 3.3, there exists a polynomial matrix  $E$  such that  $E$  is a least common right multiple of  $T_1$  and  $D_1$  and  $T^{-1}E$  is bicausal. Suppose  $E = T_1\bar{D}_1 = D_1\bar{T}_1$ , so we have, by coprimeness, that  $\det T_1 = \det \bar{T}_1$ . Hence,  $W = T^{-1}D = (T^{-1}E)(E^{-1}D) = (T_2^{-1}\bar{D}_1)(\bar{T}_1^{-1}D_2) = W_1W_2$  and  $\delta(W) = \delta(T^{-1}D) = \deg \det T = \deg \det T_2 + \deg \det \bar{T}_1 = \delta(T_2^{-1}\bar{D}_1) + \delta(\bar{T}_1^{-1}D_2)$ , so the factorization is minimal and  $W_1$  and  $W_2$  are bicausal isomorphisms.

Conversely, suppose  $W = \bar{W}_1\bar{W}_2$  is a minimal factorization. Let

$$W_1 = T_2^{-1}\bar{D}_1, \quad W_2 = \bar{T}_1^{-1}D_2 \quad (3.2)$$

be left coprime factorizations of  $W_1$  and  $W_2$ . By minimality of the factorization  $W = (T_2^{-1}\bar{D}_1)(\bar{T}_1^{-1}D_2)$ ,  $\bar{D}_1$  and  $\bar{T}_1$  must be right coprime.

Let

$$\bar{D}_1\bar{T}_1^{-1} = T_1^{-1}D_1 \quad (3.3)$$

with  $T_1$  and  $D_1$  left coprime. This factorization is determined up to a common left unimodular factor. It follows that  $\deg \det \bar{D}_1 = \deg \det D_1 = \deg \det T_2$  and that  $\deg \det \bar{T}_1 = \deg \det T_1 = \deg \det D_2$ , and by minimality  $\delta(W) = \deg \det T = \delta(W_1) + \delta(W_2) = \deg \det T_2 + \deg \det \bar{T}_1 = \deg \det T_2 + \deg \det T_1$ , and by the same considerations  $\deg \det D = \deg \det D_1 + \deg \det D_2$ .

Hence the equality  $W = T^{-1}D = (T_1T_2)^{-1}(D_1D_2)$  and left coprimeness of  $T$  and  $D$  imply that the matrices  $T_1T_2$  and  $D_1D_2$  are also left coprime. This, however implies that up to a left unimodular factor  $V$  we have  $VT = T_1T_2$ ,  $VD = D_1D_2$ .

Since we could have started out the the factorization  $W = (VT)^{-1}(VD)$  in the first place, we can conclude without loss of generality that  $T = T_1T_2$  and  $D = D_1D_2$ . Thus  $T_1$  and  $D_1$  are left factors of  $T$  and  $D$  respectively.

Now (3.3) implies that  $\bar{T}_1\bar{D}_1 = D_1\bar{T}_1 =: E$ , and by right coprimeness of  $\bar{D}_1$  and  $\bar{T}_1$ ,  $E$  is a least common right multiple of  $T_1$  and  $D_1$ . Also,

$$T^{-1}E = T^{-1}T_1\bar{D}_1 = T_2^{-1}\bar{D}_1 = W_1$$

is bicausal by assumption. Thus the conditions of Theorem 3.3 are fulfilled, and we have a direct-sum decomposition  $X_T = T_1X_{T_2} \oplus D_1X_{D_2}$ , where  $T = T_1T_2$ ,  $D = D_1D_2$ , so that  $T_1X_{T_2}$  is  $S_T$ -invariant and  $D_1X_{D_2}$  is  $S_D$ -invariant. ■

## REMARKS.

(1) Note that  $W$  can be factored minimally into  $W = W_1 W_2$  iff  $W^{-1}$  can be factored  $W^{-1} = W_2^{-1} W_1^{-1}$ . Thus, since  $W^{-1} = D^{-1} T$ , we have symmetry of roles for  $T$  and  $D$ .

(2) Since  $E$  is unique up to a right unimodular factor, then so is the factorization of  $W$  associated with the particular decomposition of the state space.

## 4. EXAMPLES AND SOME PARTICULAR CASES

In this section we shall work out some particular cases and examples. The following is well known [4, 14, 2]:

**LEMMA 4.1.** *Suppose  $\delta(W) = \dim X_T = n$ . If  $X_T$  can be decomposed into a direct sum of  $n$  one-dimensional  $S_T$ -invariant subspaces, or equivalently, if  $A$  can be diagonalized, then  $W$  admits a minimal factorization  $W = W_1 \cdots W_n$  such that  $\delta(W_i) = 1$  and the poles of the  $W_i$ 's can be arranged in any order.*

*In particular, if  $S_T$  has  $n$  distinct eigenvalues, then there exists a minimal factorization of  $W$  into first-order factors.*

*By symmetry of roles, if  $A^\#$ , or  $S_D$  can be diagonalized, we have the same result.*

**REMARK.** Given any transfer function  $W$ , there is always an output-injection equivalent transfer function  $W'$  with any spectrum. Hence by output injection, we can always get (say)  $n$  distinct eigenvalues, and hence factorization into first-order factors.

**LEMMA 4.2.** *Suppose  $\delta(W) = \dim X_T = 2$ , and suppose neither  $S_T$  nor  $S_D$  can be diagonalized. Let  $\lambda$  be the eigenvalue of  $S_T$ , and  $\mu$  of  $S_D$ . Then  $W$  can be factored iff either*

$$\text{Ker } T(\lambda) \cap \text{Ker } D(\mu) = \{0\}$$

or, for some

$$0 \neq \xi \in \text{Ker } T(\lambda) \cap \text{Ker } D(\mu),$$

we have

$$\frac{1}{z-\lambda} T\xi - \frac{1}{z-\mu} D\xi = 0.$$

*Proof.* By the assumptions, the only  $S_T$ - or  $S_D$ -invariant subspaces are the respective one-dimensional eigenspaces, say  $M_1$  and  $M_2$ . So either  $M_1 = M_2$ , in which case there is no direct-sum decomposition, or  $X_T = M_1 \oplus M_2$ .

Equivalently, there is no decomposition, hence no factorization of  $W$  iff  $S_T$  and  $S_D$  have a common eigenvector.

Now, by [6],  $f \in X_T$  is an eigenvector of  $S_T$  iff it is of the form

$$f = \frac{1}{z-\lambda} T\xi_f$$

for some  $\xi_f \in \text{Ker } T(\lambda) \subset F^m$  and  $\xi_f = \Pi_+ T^{-1}zf$ . Also note that for any scalar constant  $\alpha$ ,  $\xi_{\alpha f} = \alpha\xi_f$ .

Furthermore,

$$\begin{aligned} \Pi_+ D^{-1}zf &= \Pi_+ (D^{-1}T)T^{-1}zf \\ &= \Pi_+ (I - D^{-1}U)T^{-1}zf = \Pi_+ T^{-1}zf. \end{aligned}$$

The last equality holds because  $D^{-1}U$  is strictly proper and  $T^{-1}zf$  is proper. Thus  $f$  is a common eigenvector iff

$$f = \frac{1}{z-\lambda} T\xi_f = \frac{1}{z-\mu} D\xi_f$$

and  $0 \neq \xi_f \in \text{Ker } T(\lambda) \cap \text{Ker } D(\mu)$ . ■

We conclude this section with some simple examples, the first of which illustrates the results of Section 3:

EXAMPLE 4.3. Let

$$W = \begin{pmatrix} \left( \frac{z+1}{z} \right)^2 & -\frac{z+1}{z^2} \\ 0 & \frac{z+1}{z} \end{pmatrix} = T^{-1}D$$

with

$$T = \begin{pmatrix} z^2 & z \\ 0 & z \end{pmatrix}, \quad D = \begin{pmatrix} (z+1)^2 & 0 \\ 0 & z+1 \end{pmatrix}.$$

Let

$$T = T_1 T_2 =: \begin{pmatrix} z & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

and

$$D = D_1 D_2 = \begin{pmatrix} z+1 & 0 \\ 0 & z+1 \end{pmatrix} \begin{pmatrix} z+1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now

$$X_T = \text{Sp} \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

whereas

$$X_{T_2} = \text{Sp} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad X_{D_2} = \text{Sp} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

So

$$T_1 X_{T_2} = \text{Sp} \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad D_1 X_{D_2} = \text{Sp} \left\{ \begin{pmatrix} z+1 \\ 0 \end{pmatrix} \right\}.$$

Thus,  $X_T = T_1 X_{T_2} \oplus D_1 X_{D_2}$ . Let

$$\begin{aligned} E &= \begin{pmatrix} z & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z+1 & 0 \\ 0 & z+1 \end{pmatrix} = \begin{pmatrix} z+1 & 0 \\ 0 & z+1 \end{pmatrix} \begin{pmatrix} z & 1 \\ 0 & 1 \end{pmatrix} \\ &= T_1 \bar{D}_1 = D_1 \bar{T}_1 \end{aligned}$$

be a least common right multiple of  $T_1$  and  $D_1$  with

$$T^{-1}E = \begin{pmatrix} z/(z+1) & 0 \\ 0 & z/(z+1) \end{pmatrix}$$

bicausal. Thus  $W$  has a factorization:

$$\begin{aligned} W &= (T_2^{-1}\bar{D}_1)(\bar{T}_1^{-1}D_2) \\ &= \left[ \begin{pmatrix} 1/z & 0 \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} z+1 & 0 \\ 0 & z+1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1/z & -1/z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z+1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\ &= W_1W_2, \end{aligned}$$

with  $\delta(W) = 3$ ,  $\delta(W_1) = 2$ ,  $\delta(W_2) = 1$ .

The next two examples illustrate Lemma 4.2.

**EXAMPLE 4.4.** This example is very similar in structure to the one given by Dewilde and Vandewalle [4], where there was no factorization.

Let

$$T = \begin{pmatrix} z & 0 \\ 1 & z \end{pmatrix}, \quad D = \begin{pmatrix} z-1 & 0 \\ 1 & z-1 \end{pmatrix}.$$

In this case, the eigenvalues of  $T$  and  $D$  respectively are  $\lambda = 0$ ,  $\mu = 1$ . Now

$$\text{Ker } T(0) = \text{Sp} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \text{Ker } D(1)$$

and

$$\frac{1}{z-0}T \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{z-1}D \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

So, by Lemma 4.2, there is no factorization of  $W = T^{-1}D$ .

**EXAMPLE 4.5.** Let

$$T = \begin{pmatrix} z^2 & 0 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} (z-1)^2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Although

$$\text{Ker } T(0) = \text{Ker } D(1) = \text{Sp} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$

we have

$$\frac{1}{z}T\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{z-1}D\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0,$$

so there is a factorization of  $W$ , since  $S_T$  and  $S_D$  have no common eigenvector.

Let

$$T = T_1 T_2 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 1 & 1 \end{pmatrix},$$

$$D = D_1 D_2 = \begin{pmatrix} z-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z-1 & 0 \\ 1 & 1 \end{pmatrix}.$$

So

$$T_1 X_{T_2} = \text{Sp}\left\{\begin{pmatrix} z \\ 0 \end{pmatrix}\right\} \quad \text{and} \quad D_1 X_{D_2} = \text{Sp}\left\{\begin{pmatrix} z-1 \\ 0 \end{pmatrix}\right\}$$

and

$$X_T = \left\{ \begin{pmatrix} az+b \\ 0 \end{pmatrix} \right\} = T_1 X_{T_2} \oplus D_1 X_{D_2},$$

and we have the following factorization:

$$W = \begin{pmatrix} \left(\frac{z-1}{z}\right)^2 & 0 \\ \frac{2z-1}{z^2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{z-1}{z} & 0 \\ \frac{1}{z} & 1 \end{pmatrix} \begin{pmatrix} \frac{z-1}{z} & 0 \\ \frac{1}{z} & 1 \end{pmatrix}$$

$$= W_1 W_2, \quad \text{with} \quad W_1 = W_2 = T_2^{-1} \bar{D}_1 = \bar{T}_1^{-1} D_2,$$

where

$$E = T_1 \bar{D}_1 = D_1 \bar{T}_1$$

$$= \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z-1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} z-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 1 & 1 \end{pmatrix}$$

is such that  $T^{-1}E$  is bicausal.

## 5. FACTORIZATION OF SYMMETRIC MATRICES

In this section we specialize the analysis of Section 3 to the case of a real symmetric transfer function. This work is motivated by Ran [13], whose tools in the analysis are those of Bart, Gohberg, and Kaashoek [2], whereas here we apply again polynomial methods. In particular we shall see how the symmetry of  $W$  is reflected in the factors.

A complete analysis in terms of supporting projections is given in [13].

Assume throughout this section that  $W = \tilde{W}$ , where  $\tilde{\phantom{x}}$  denotes transpose. As before, assume  $W = T^{-1}D = I + T^{-1}U$ . So  $D = T + U$ ,  $T^{-1}U$  is strictly proper, and  $T$  and  $D$  left coprime.

$W = \tilde{W}$  implies

$$\begin{aligned} T^{-1}D &= \tilde{D}\tilde{T}^{-1}, \\ T^{-1}U &= \tilde{U}\tilde{T}^{-1}. \end{aligned} \tag{5.1}$$

Since  $T^{-1}D$  is bicausal,  $X_T = X_D$  (equality as sets). This, however, is not the case for  $X_{\tilde{D}}$  and  $X_{\tilde{T}}$ . But they are isomorphic as linear spaces (not as modules). This isomorphism is related to the following: Let  $G \in F^{m \times m}((z^{-1}))$ . Then  $T_G: F^m[z] \rightarrow F^m[z]$  is the Toeplitz operator defined by

$$T_G f = \Pi_+ Gf, \quad f \in F^m[z].$$

In particular, if  $EF^{-1}$  is a polynomial matrix fraction representation of a bicausal isomorphism, then

$$\Pi_E T_{EF^{-1}} \text{ restricted to } X_F$$

is an isomorphism acting from  $X_F$  to  $X_E$ , whose inverse is given by

$$\Pi_F T_{FE^{-1}} \text{ restricted to } X_E.$$

Since we can assume without loss of generality that  $E^{-1}$  is proper, in which case so is  $F^{-1}$ , then the isomorphisms above are simply

$$T_{EF^{-1}} \text{ restricted to } X_F, \quad T_{FE^{-1}} \text{ restricted to } X_E.$$

Now (5.1) implies that  $D\tilde{T} = \tilde{T}D$  with  $T, D$  left coprime and  $\tilde{T}, \tilde{D}$  right coprime; hence by [6] there exists a module isomorphism

$$Z_1: X_{\tilde{T}} \rightarrow X_T$$



given by  $Z_1 f = \Pi_T Df$ ,  $f \in X_{\hat{T}}$ . Note that

$$Z_1 f = \Pi_T Df = T \Pi_- T^{-1} Df = T \Pi_- (I - T^{-1} U) f = T \Pi_- T^{-1} U f = \Pi_T U f.$$

$Z_1$  is an intertwining map in the sense that

$$Z_1 S_{\hat{T}} = S_T Z_1.$$

By symmetry of roles, we also have

$$Z_2: X_{\hat{D}} \rightarrow X_D, \quad Z_2 f = \Pi_D T f,$$

$f \in X_D$  a module isomorphism, and

$$Z_2 f = -\Pi_D U f.$$

**DEFINITION.** Let  $f \in X_T$ ,  $g \in X_{\hat{T}}$ . Define  $\langle f, g \rangle = [T^{-1} f, g]$ , where for  $h, k \in F^m((z^{-1}))$ ,  $h = \sum h_i z^{-i}$ ,  $k = \sum k_i z^{-i}$  we define

$$[h, k] = \sum (h_i, k_{-i-1}).$$

Under the pairing  $\langle \cdot, \cdot \rangle$ , we can identify the dual space of  $X_T$  with  $X_{\hat{T}}$ . Also, under this scalar product,  $Z_1$  is a self-adjoint system isomorphism.

**LEMMA 5.1.** *The following diagram is commutative:*

$$\begin{array}{ccc} X_{\hat{T}} & \xrightarrow{Z_1} & X_T \\ R = T_{\hat{T}\hat{D}}^{-1} \uparrow & & \uparrow \text{Id} = R^* \\ X_{\hat{D}} & \xrightarrow{-Z_2} & X_D \end{array}$$

or

$$-Z_2 = Z_1 T_{\hat{T}\hat{D}}^{-1}$$

Notice that both  $Z_1$  and  $Z_2$  are self-adjoint maps, and since, by [6],

$$T_{\hat{T}\hat{D}}^* = \text{Id},$$

it follows that

$$-Z_2 = R^*Z_1R,$$

i.e.,  $-Z_2$  and  $Z_1$  are congruent.

*Proof.* Let  $f \in X_{\tilde{D}}$ . Then

$$\begin{aligned} Z_1 T_{\tilde{T}\tilde{D}^{-1}} f &= \Pi_T D \Pi_+ T D^{-1} f = \Pi_T D (1 - \Pi_-) \tilde{T} \tilde{D}^{-1} f \\ &= \Pi_T D \tilde{T} \tilde{D}^{-1} f - \Pi_T D \Pi_- T D^{-1} f = \Pi_T T \tilde{D} \tilde{D}^{-1} f - \Pi_T D \Pi_- D^{-1} T f \\ &= \Pi_T T f - \Pi_T D \Pi_- (I - D^{-1} U) f = 0 + \Pi_T D \Pi_- D^{-1} U f \\ &= T \Pi_- T^{-1} D \Pi_- D^{-1} U f = \Pi_D U f, \end{aligned}$$

the last equality holding because  $\Pi_- D^{-1} U f$  is strictly proper and  $T^{-1} D$  proper, whence the first  $\Pi_-$  can be canceled and so can  $T$  and  $T^{-1}$ . ■

The congruence of  $Z_1$  and  $-Z_2$  is related to generalized Bezoutians. The generalized Bezoutian of the quadrupole  $(D, \tilde{T}, T, \tilde{D})$  is defined as

$$B_1(z, w) = \frac{D(z) \tilde{T}(w) - T(z) \tilde{D}(w)}{z - w}.$$

Since  $D\tilde{T} = T\tilde{D}$ ,  $B_1(z, w)$  is a polynomial matrix in  $z$  and  $w$ ; hence it has a representation as

$$B_1(z, w) = \sum \sum B_{ij} z^{i-1} w^{j-1}.$$

It has been shown in Fuhrmann [9, Theorem 4.2] that the Bezoutian  $B_1$  can be identified as a matrix representation of the intertwining map  $Z_1$ , so its signature is equal to the signature of  $Z_1$ , i.e.,

$$\sigma(B_1) = \sigma(Z_1).$$

Now the Bezoutian associated with the quadruple  $(T, \tilde{D}, D, \tilde{T})$  is

$$B_2(z, w) = \frac{T(z) \tilde{D}(w) - D(z) \tilde{T}(w)}{z - w},$$

and obviously,

$$B_1 = -B_2, \quad \text{whence} \quad \sigma(B_1) = -\sigma(B_2).$$

By symmetry,  $B_2$  can be identified as a matrix representation of  $Z_2$ , so its signature is equal to that of  $Z_2$ . In other words,

$$\sigma(Z_1) = -\sigma(Z_2).$$

This is not surprising, since by Theorem 5.3 of [9], the congruency of  $-Z_2$  and  $Z_1$ , i.e.  $-Z_2 = R^*Z_1R$ , means that  $-Z_2$  and  $Z_1$  have the same rank and signature.

We now turn to orthogonality and orthogonal complements. Since we are identifying the dual space of  $X_T$  with  $X_{\tilde{T}}$ , the orthogonal complement of a subspace of  $X_T$  is a subspace of  $X_{\tilde{T}}$ .

**LEMMA 5.2.** *Let  $T = T_1T_2$ ,  $D = D_1D_2$ . Then*

- (a)  $(T_1X_{T_2})^\perp = \tilde{T}_2X_{\tilde{T}_1}$ ,
- (b)  $(D_1X_{D_2})^\perp = T_{\tilde{T}\tilde{D}^{-1}}(\tilde{D}_2X_{\tilde{D}_1})$ ,

where the orthogonal complements in each case are subspaces of  $X_{\tilde{T}}$ .

*Proof.* Part (a) has been proven in Fuhrmann [8, 9].

For part (b), since the dimensions are complementary, it suffices to prove orthogonality. Let  $f \in X_{D_2}$ ,  $g \in X_{\tilde{D}_1}$ . Then

$$\begin{aligned} \langle D_1f, T_{\tilde{T}\tilde{D}^{-1}}\tilde{D}_2g \rangle &= [T^{-1}D_1f, \Pi_+ \tilde{T}\tilde{D}^{-1}\tilde{D}_2g] \\ &= [T^{-1}D_1f, \tilde{T}\tilde{D}^{-1}D_2g] = [D_1f, \tilde{D}_1^{-1}g] \\ &= [f, g] = 0. \end{aligned} \quad \blacksquare$$

We now turn to decompositions and factorizations.

Assume  $W = W_1W_2$  is a minimal factorization corresponding to a direct-sum decomposition

$$X_T = T_1X_{T_2} \oplus D_1X_{D_2}.$$

This induces a decomposition of  $X_{\tilde{T}}$

$$X_{\tilde{T}} = (T_1X_{T_2})^\perp \oplus (D_1X_{D_2})^\perp,$$

or, by Lemma 5.2,

$$X_{\tilde{T}} = \tilde{T}_2 X_{\tilde{T}_1} \oplus T_{\tilde{T}\tilde{D}^{-1}}(\tilde{D}_2 X_{\tilde{D}_1}).$$

Now,  $\tilde{T}_2 X_{\tilde{T}_1}$  is an  $S_{\tilde{T}}$ -invariant subspace, and  $\tilde{D}_2 X_{\tilde{D}_1}$  is  $S_{\tilde{D}}$ -invariant. (But  $T_{\tilde{T}\tilde{D}^{-1}}(\tilde{D}_2 X_{\tilde{D}_1})$  is not necessarily  $S_{\tilde{T}}$ -invariant.) If we apply the module isomorphism  $Z_1: X_{\tilde{T}} \rightarrow X_T$ , then

$$\begin{aligned} X_T &= Z_1(\tilde{T}_2 X_{\tilde{T}_1}) \oplus Z_1 T_{\tilde{T}\tilde{D}^{-1}}(\tilde{D}_2 X_{\tilde{D}_1}) \\ &= Z_1(\tilde{T}_2 X_{\tilde{T}_1}) \oplus Z_2(\tilde{D}_2 X_{\tilde{D}_1}) \quad \text{by Lemma 5.1.} \end{aligned}$$

Since  $Z_1$  and  $Z_2$  are intertwining maps, they preserve invariant subspaces; hence

$$Z_1(\tilde{T}_2 X_{\tilde{T}_1}) = T_1' X_{T_2'} \quad \text{and} \quad Z_2(\tilde{D}_2 X_{\tilde{D}_1}) = D_1' X_{D_2'}$$

for some other factorizations  $T = T_1' T_2'$ ,  $D = D_1' D_2'$  and obviously, the direct-sum decomposition

$$X_T = T_1' X_{T_2'} \oplus D_1' X_{D_2'}$$

corresponds to the factorization  $W = \tilde{W} = \tilde{W}_2 \tilde{W}_1$ , since  $Z_1$  is a system isomorphism, so it preserves transfer functions.

The next lemma specifies what  $T_1'$  and  $D_1'$  are, in terms of the first factorizations  $T = T_1 T_2$ ,  $D = D_1 D_2$ .

LEMMA 5.3. *Let*

$$X_T = T_1 X_{T_2} \oplus D_1 X_{D_2} \quad \text{and} \quad X_T = T_1' X_{T_2'} \oplus D_1' X_{D_2'}$$

*be the direct-sum decompositions corresponding to the factorizations  $W = W_1 W_2$  and  $W = \tilde{W}_2 \tilde{W}_1$  respectively. Then  $T_1'$  is a greatest common left divisor of  $T$  and  $D\tilde{T}_2$ .*

*Symmetrically,  $D_1'$  is a g.c.l.d. of  $D$  and  $T\tilde{D}_2$ .*

*Proof.* By construction, we have

$$Z_1(\tilde{T}_2 X_{\tilde{T}_1}) = T_1' X_{T_2'}$$

This means

$$\dim X_{\tilde{T}_1} = \dim X_{T'_2}$$

or, equivalently,

$$\deg \det T'_1 = \deg \det T_2$$

and

$$Z_1(\tilde{T}_2 X_{\tilde{T}_1}) \subset T'_1 X_{T'_2}.$$

Let  $f \in X_{T_1}$ , so

$$\begin{aligned} Z_1(\tilde{T}_2 f) &= \Pi_T D \tilde{T}_2 f = T \Pi_- T^{-1} D \tilde{T}_2 f = T \Pi_- \tilde{D} \tilde{T}^{-1} \tilde{T}_2 f \\ &= T \Pi_- \tilde{D} \tilde{T}_1^{-1} f \in T'_1 X_{T'_2}, \end{aligned}$$

or

$$T'_1{}^{-1} T \Pi_- \tilde{D} \tilde{T}_1^{-1} f = T'_2 \Pi_- \tilde{D} \tilde{T}_1^{-1} f \in X_{T'_2},$$

which implies that the last term is a polynomial. Hence

$$0 = \Pi_- T'_2 \Pi_- \tilde{D} \tilde{T}_1^{-1} f = \Pi_- T'_2 \tilde{D} \tilde{T}_1^{-1} f \quad \forall f \in X_{\tilde{T}_1}.$$

Now, since any  $g \in F^m[z]$  can be decomposed into

$$g = f + \tilde{T}_1 h$$

with

$$f \in X_{\tilde{T}_1}, \quad h \in F^m[z]$$

and  $\Pi_- T'_2 \tilde{D} \tilde{T}_1^{-1}(\tilde{T}_1 h) = 0$  for any  $h$ , we have

$$\Pi_- T'_2 \tilde{D} \tilde{T}_1^{-1} g = 0 \quad \text{for any polynomial vector } g.$$

This implies that  $T_2\tilde{D}\tilde{T}_1^{-1} = N$  is a polynomial matrix. To simplify:  $N = T_2\tilde{D}\tilde{T}_1^{-1} = T_2\tilde{D}\tilde{T}^{-1}T_1^{-1} = T_2T^{-1}D\tilde{T}_2 = T_1'^{-1}D\tilde{T}_2$ ; hence,

$$T_1'N = D\tilde{T}_2 \quad (5.2)$$

so  $T_1'$  is a common left factor of  $T$  and  $D\tilde{T}_2$ . Maximality follows from the fact that

$$\deg \det T_1' = \deg \det T_2$$

and by left coprimeness of  $T_1'$  and  $D$ .

If  $T_1'S$  is a common left factor, we have

$$T_1'SM = D\tilde{T}_2 \quad \text{for some polynomial matrix } M.$$

Therefore, by equality of determinantal degrees,  $SM$  and  $D$  are right coprime, so  $M$  and  $D$  are right coprime. Also,  $T_1'S$  and  $D$  are left coprime, since  $T_1'S$  is a left factor of  $T$ . This implies  $\deg \det T_1'S = \deg \det T_2 = \deg \det T_1'$ , so  $S$  is unimodular. ■

**COROLLARY.** *With the above notation, we have the following factorization of  $W$ :*

$$T^{-1}D = W = T_2'^{-1}N\tilde{T}_2^{-1},$$

following from (5.2), and  $T_2'$  has the same determinantal degree as  $T_1$  (or  $\tilde{T}_1$ ) by construction, so this is a “minimal” factorization. In particular,  $T_2'$  has the same spectrum as  $T_1$ .

We conclude with an alternative proof of a theorem of Ran [13] about factorizations of the form

$$W = \tilde{W}_2W_3W_2.$$

We need two lemmas.

The first lemma is a general result about inclusion of invariant subspaces, which follows directly from Lemma 2.1.

**LEMMA 5.4.** *Suppose  $T$  has two factorizations*

$$T = T_1T_2 = T_1'T_2'.$$

Then

$$T_1'X_{T_2'} \subset T_1X_{T_2}$$

if for some polynomial matrix  $T_3$ ,

$$T_1' = T_1T_3.$$

LEMMA 5.5. Suppose  $X_T = T_1X_{T_2} \oplus D_1X_{D_2} = T_1'X_{T_2'} \oplus D_1'X_{D_2'}$  are two direct-sum representations such that

$$T = T_1T_2 = T_1'T_2', \quad D = D_1D_2 = D_1'D_2',$$

and suppose

$$T_1'X_{T_2'} \subset T_1X_{T_2}.$$

Let  $E$  be the l.c.r.m. of  $T_1$  and  $D_1$  such that  $T^{-1}E$  is bicausal, and let  $E'$  be the corresponding matrix for  $T_1'$  and  $D_1'$ . Then  $T_1X_{T_2}$  is invariant under both  $S_E$  and  $S_{E'}$ .

*Proof.* The invariance under  $S_E$  is due to construction. Let  $E' = T_1'\bar{D}_1' = D_1'\bar{T}_1'$ . By the previous lemma,

$$T_1' = T_1T_3;$$

hence

$$E' = T_1T_3\bar{D}_1',$$

and  $T^{-1}E' = T_2^{-1}(T_3\bar{D}_1')$  is bicausal. This implies that  $X_{T_2} = X_{T_3\bar{D}_1'}$ .

Thus, the  $S_{E'}$ -invariant subspace  $T_1X_{T_3\bar{D}_1'}$  is actually equal to  $T_1X_{T_2}$ . ■

COROLLARY. If  $T_1'X_{T_2'} \subset T_1X_{T_2}$  and  $D_1X_{D_2} \subset D_1'X_{D_2'}$ , then the  $(C, A)$ -invariant subspace

$$V = T_1X_{T_2} \cap D_1'X_{D_2'}$$

is invariant under both  $S_E$  and  $S_{E'}$ .

This means that the pair of subspaces  $T_1'X_{T_2'}$  and  $V$ , whose (direct) sum is  $T_1X_{T_2}$ , are compatible  $(C, A)$ -invariant subspaces, and so is the pair  $V$  and  $D_1X_{D_2}$ .

**THEOREM 5.6.** *Let  $W$  be a symmetric normalized bicausal isomorphism, and suppose  $W = W_1W_2$  is a minimal factorization corresponding to the direct-sum decomposition*

$$X_T = T_1X_{T_2} \oplus D_1X_{D_2}$$

and  $W = \tilde{W}_2\tilde{W}_1$  corresponds to

$$X_T = T_1'X_{T_2'} \oplus D_1'X_{D_2'}$$

Then  $W$  admits a minimal factorization of the form

$$W = \tilde{W}_2W_3W_2$$

iff  $T_1'X_{T_2'} \subset T_1X_{T_2}$  and  $D_1X_{D_2} \subset D_1'X_{D_2'}$ .

*Proof.* Recall that by Theorem 3.3,  $W_1 = T_2^{-1}\bar{D}_1 = T^{-1}E$ , with  $E = T_1\bar{D}_1 = D_1\bar{T}_1$  being a l.c.r.m. of  $T_1$  and  $D_1$  such that  $T^{-1}E$  is bicausal.  $W_1$  is the transfer function of the induced system operating in the state space  $T_1X_{T_2}$ , and  $\tilde{W}_2$  is the transfer function of the system induced in  $T_1'X_{T_2'}$ .

Now  $W_1$  has a minimal factorization iff  $X_{T_2}$  can be decomposed into a direct sum of an  $S_{T_2}$ -invariant and an  $S_{D_1}$ -invariant subspace. Equivalently,  $T_1X_{T_2}$  can be decomposed into an  $S_T$ -invariant subspace and an  $S_E$ -invariant subspace.

Thus if  $W = \tilde{W}_2W_3W_2 = W_1W_2$  is a minimal factorization, then  $W_1 = \tilde{W}_2W_3$  is a minimal factorization. This implies

$$T_1'X_{T_2'} \subset T_1X_{T_2}.$$

By the same considerations, since  $\tilde{W}_1$  is the transfer function of the system operating in  $D_1'X_{D_2'}$ , then the minimal factorization  $\tilde{W}_1 = \tilde{W}_3\tilde{W}_2$  implies  $D_1X_{D_2} \subset D_1'X_{D_2'}$ .

Conversely, if  $T_1'X_{T_2'} \subset T_1X_{T_2}$  and  $D_1X_{D_2} \subset D_1'X_{D_2'}$ , then by Lemma 5.5,

$$V = T_1X_{T_2} \cap D_1'X_{D_2'},$$

regarded as a subspace of  $T_1X_{T_2}$ , is the necessary  $S_E$ -invariant complement to



$T_1 X_{T_2}$  yielding the minimal factorization  $W_1 = \tilde{W}_2 W_3$  corresponding to the direct-sum decomposition

$$T_1 X_{T_2} = T_1 X_{T_2} \oplus V.$$

Hence  $W = W_1 W_2 = \tilde{W}_2 W_3 W_2$  is a minimal factorization. ■

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