On Bezoutians, Van der Monde Matrices, and the Lienard-Chipart Stability Criterion

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ABSTRACT

We show how the Bezoutian can be reduced by congruence to diagonal form, with the congruence given by a Van der Monde matrix. This result is applied to obtain new proofs of some classical stability criteria and in particular of the Lienard-Chipart theorem.

1. INTRODUCTION

Given a pair $p(z), q(z)$ of polynomials with real coefficients, with $q(z)$ monic of degree n and p of degree $\leqslant n$, the Bezoutian $B(q,p)$ of q and p is the quadratic form (b_{ij}) defined through

$$B(q,p)(z,w)=\frac{q(z)p(w)-p(z)q(w)}{z-w}=\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}z^{i-1}w^{j-1}. \quad (1)$$

Clearly, B is a symmetric matrix.

Since its introduction in the last century it has played a central role in problems of polynomial coprimeness, location of polynomial zeroes, stability and control. It was used masterfully by Hermite (1856), and an account of

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many aspects of its use is given in the classic paper of Krein and Naimark (1936).

For a long time the use of Bezoutians, though a very efficient tool, was somewhat mysterious, as the conceptual basis for its use was missing. This resulted in many ad hoc computational proofs, and in many cases the working out of low dimensional cases was substituted for a proof.

The situation improved as a result of the theorem, proved in Fuhrmann (1981b), showing that the Bezoutian is a matrix representation of a polynomial module homomorphism with respect to a particular dual pair of bases. This used the polynomial models introduced by the first author in Fuhrmann (1976). It in turn was used to reprove some results on stability first derived in Datta (1978a).

The object of this paper is to reexamine the proof of the Lienard-Chipart stability criterion as given in Datta (1978a) and fill in some gaps. In particular we show how congruence by Van der Monde matrices reduces the Bezoutian to diagonal form. This form in turn is used to classify positive pairs of polynomials, as introduced in Krein and Naimark (1936) and Gantmacher (1959), and in turn some stability results. We conclude with an easy derivation of the Lienard-Chipart stability criterion.

The importance of the Lienard-Chipart stability criterion lies in the fact that the Hermite-Fujiwara matrix which has to be checked for positive definiteness is of size $n \times n$. There is a standard reduction, using elementary properties of Bezoutians, as in Krein and Naimark (1936) and Fuhrmann (1983), that reduces it to checking positive definiteness of two symmetric matrices of half the size. The Lienard-Chipart criterion makes a further reduction, apart from the trivial checking of the positivity of the polynomial coefficients, to the checking of only one of these matrices. Thus computationally it may be the most efficient stability criterion.

The paper is structured as follows. In Section 2 we review the definition of polynomial models and of duality in that context, and we study some bases and their duals. We apply this to the study of the Bezoutian. In the last section we apply the previous results to the analysis of some classical stability results. In particular we give a new proof of the Lienard-Chipart stability criterion. We hope our approach sheds some more light on this important area.

2. POLYNOMIAL MODELS

In what follows F denotes an arbitrary field, to be identified later with the real number field $\mathbf{R}$. By $F[z]$ we denote the ring of polynomials over F, by $F((z^{-1}))$ the set of truncated Laurent series in z^{-1}, and by $F[[z^{-1}]]$ and

$z^{-1}F[[z^{-1}]]$ the set of all formal power series in z^{-1} and the set of those power series with vanishing constant term respectively. Let π_+ and π_- be the projections of $F((z^{-1}))$ onto $F[z]$ and $z^{-1}F[[z^{-1}]]$ respectively. Since $F((z^{-1})) = F[z] \oplus z^{-1}F[[z^{-1}]]$, they are complementary projections. Also $z^{-1}F[[z^{-1}]]$ is isomorphic to $F((z^{-1}))/F[z]$, which is an $F[z]$-module with the module action given by

$$z \cdot h = S_- h = \pi_- zh. \tag{2}$$

Similarly we define

$$S_+ f = zf \qquad \text{for} \quad f \in F[z]. \tag{3}$$

Given a monic polynomial q of degree n, we define a projection π_q in $F[z]$ by

$$\pi_q f = q\pi_- q^{-1} f \qquad \text{for} \quad f \in F[z]. \tag{4}$$

We define the *polynomial model* associated with q as the space

$$X_q = \operatorname{Im} \pi_q \tag{5}$$

endowed with the module structure induced by the shift map defined through

$$S_q f = \pi_q S_+ f \qquad \text{for} \quad f \in X_q \tag{6}$$

A map Z in X_q commutes with S_q if and only if $Z = p(S_q)$ for some polynomial $p \in F[z]$, and $p(x_q)$ is invertible if and only if p and q are coprime. For this see Fuhrmann (1976). We define a pairing of elements of $F((z^{-1}))$ as follows: for

$$f(z) = \sum_{j=-\infty}^{n_f} f_j z^j \tag{7}$$

and

$$g(z) = \sum_{j=-\infty}^{n_g} g_j z^j, \tag{8}$$

let

$$[f,g]=\sum_{j=-\infty}^{\infty} f_{-j-1}g_j. \tag{9}$$

Clearly, since both series are truncated, the sum in (2.9) is well defined. In terms of this pairing we can make the following identification (see Fuhrmann, 1981a). The dual of $F[z]$ as a linear space $z^{-1}F[[z^{-1}]]$. Now, given a nonzero polynomial q, the module X_q is isomorphic to $F[z]/qF[z]$. If we define, for a subset M of $F((z^{-1}))$, $M^{\perp}$ by

$$M^{\perp}=\left\{g\in F\left((z^{-1})\right)\middle|[f,g]=0 \text{ for all } f\in M\right\}, \tag{10}$$

then in particular $F[z]^{\perp}=F[z]$ and

$$(qF[z])^{\perp}=X^q=\left\{h\in z^{-1}F\left[[z^{-1}]\right]\mid qh\in F[z]\right\}.$$

Since, in general, $(X/M)^*\simeq M^{\perp}$, we have

$$X_q^*=(F[z]/qF[z])^*\simeq[qF[z]]^{\perp}=X^q. \tag{11}$$

But in turn we have $X^q\simeq X_q$ and so X_q^* can be identified with X_q. This can be made more concrete through the use of the bilinear form

$$\langle f,g\rangle=\left[q^{-1}f,g\right]. \tag{12}$$

Relative to this bilinear form we have the important relation

$$S_q^*=S_q. \tag{13}$$

Let X be a finite dimensional vector space over the field F, and let X^* be its dual space under the pairing $\langle\ ,\ \rangle$. Let $\{e_1,\dots,e_n\}$ be a basis for X; then the set of vectors $\{f_1,\dots,f_n\}$ in X^* is called the *dual basis* if

$$\langle e_i,f_j\rangle=d_{ij},\qquad 1\leqslant i,j\leqslant n. \tag{14}$$

Let X_q be the polynomial model associated with the polynomial $q(z)=z^n+q_{n-1}z^{n-1}+\cdots+q_0$. The elements of X_q are all polynomials of degree

$\leqslant n-1$. We consider the following very natural bases in X_q. The subset of X_q given by $B_{st} = \{f_1, \dots, f_n\}$, where

$$f_i(z) = z^{i-1}, \qquad i = 1, \dots, n, \tag{15}$$

is a basis for X_q. We will refer to this as the *standard basis*.

Given the polynomial q as above, we define

$$e_i(z) = \pi_+ z^{-i} q = q_i + q_{i+1} z + \cdots + z^{n-i}, \qquad i = 1, \dots, n, \tag{16}$$

and call the set $B_{co} = \{e_1, \dots, e_n\}$ the *control basis* of X_q.

The important fact about this pair of bases is that relative to the bilinear form $\langle \ , \ \rangle$ of (12) the standard and control bases are dual to each other. In particular, since $S_q^* = S_q$, we have $p(S_q)^* = p(S_q)$ and so $p(S_q)$ is a self-adjoint operator in the indefinite metric $\langle \ , \ \rangle$. Thus the matrix representation of $p(S_q)$ relative to a dual pair of bases is symmetric. In Fuhrmann (1981b) it has been shown that

$$B(q, p) = \left[p(S_q)\right]_{co}^{st}. \tag{17}$$

Thus the analysis of the Bezoutian is reduced to the study of the map $p(S_q)$, which is much easier.

We note, for later use, that

$$\left[S_q\right]_{st}^{st} = \begin{pmatrix} 0 & \cdots & 0 & -q_0 \\ 1 & & & -q_1 \\ & \ddots & & \vdots \\ & & 1 & -q_{n-1} \end{pmatrix} \tag{18}$$

and

$$C_q = \left[S_q\right]_{co}^{co} = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -q_0 & -q_1 & \cdots & -q_{n-1} \end{pmatrix}, \tag{19}$$

i.e., we obtain the companion matrices as matrix representations.

Next we specialize to the case that the polynomial q has n simple roots. Thus

$$q(z)=\prod_{i=1}^{n}(z-\lambda_i) \tag{20}$$

and $\lambda_i \neq \lambda_j$. Now, as $(S_q f)(z)=zf(z)-\rho q(z)$ for some ρ, it follows that α is an eigenvalue of S_q, and f an eigenfunction, if and only if $q(\alpha)=0$ and $f(z)=\rho q(z)/(z-\alpha)$, i.e., α is equal to one of the λ_i.

Clearly $\{p_j(z)=q(z)/(z-\lambda_i)\,|\,i=1,\dots,n\}$ is a set of n linearly independent functions in X_q and hence constitutes a basis. We call this the *spectral basis* and denote it by B_{sp}. Obviously

$$p_i(z)=\frac{q(z)}{z-\lambda_i}=\prod_{j\neq i}(z-\lambda_j). \tag{21}$$

Finally we introduce the *interpolation basis* B_{in} in X_q. Let $\pi_1,\dots,\pi_n\in X_q$ be defined by the requirement

$$\pi_i(\lambda_j)=\delta_{ij},\qquad i,j=1,\dots,n. \tag{22}$$

A simple calculation leads to

$$\pi_i(z)=\frac{p_i(z)}{\pi_i(\lambda_i)}. \tag{23}$$

Thus $B_{\mathrm{in}}=\{\pi_1,\dots,\pi_n\}$.

Now for an arbitrary polynomial f in X_q we have

$$\langle f,p_j\rangle=\left[q^{-1}f,p_j\right]=\left[f,q^{-1}p_j\right]=\left[f,(z-\alpha_j)^{-1}\right]=f(\alpha_j). \tag{24}$$

In particular

$$\langle \pi_i,p_j\rangle=\pi_i(\alpha_j)=\delta_{ij}. \tag{25}$$

So B_{in} is in fact the dual basis of B_{sp}.

The usage of the term *interpolation* is justified by the fact that $f(z)=\sum_{i=1}^{n} c_i \pi_i(z)$ is the unique polynomial solution, of degree $\leqslant n-1$, of the interpolation problem

$$f(\alpha_i)=c_i, \qquad i=1,\dots,n. \tag{26}$$

This is just the Lagrange interpolation problem.

Note that for every f in X_q we have the expansion

$$f(z)=\sum_{i=1}^{n} f(\lambda_i)\pi_i(z) \tag{27}$$

and in particular

$$z^k=\sum_{i=1}^{n} \lambda_i^k \pi_i(z). \tag{28}$$

We define the Van der Monde matrix $V=V(\lambda_1,\dots,\lambda_n)$ by

$$V(\lambda_1,\dots,\lambda_n)=\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}. \tag{29}$$

Now (28) can be written as

$$[I]_{\mathrm{st}}^{\mathrm{in}}=\tilde{V}(\lambda_1,\dots,\lambda_n). \tag{30}$$

Here $\tilde{A}$ denotes the transpose of A, and so, by duality,

$$[I]_{\mathrm{sp}}^{\mathrm{co}}=V(\lambda_1,\dots,\lambda_n). \tag{31}$$

We can state now the following theorem, proved in Lander (1974); see also Datta (1978b).

THEOREM 2.1. *Let $q(z)$ be a monic nth degree polynomial having n simple zeros $\lambda_1,\dots,\lambda_n$, and let p be a polynomial of degree $\leqslant n$. Then the*

Bezoutian $B(q, p)$ satisfies the following identity:

$$\tilde{V}B(q,p)V = R, \tag{32}$$

where R is the diagonal matrix $\operatorname{diag}(r_1,\ldots,r_n)$ *and*

$$r_i = p(\lambda_i)p_i(\lambda_i) = p(\lambda_i)q'(\lambda_i). \tag{33}$$

Proof. The trivial operator identity

$$Ip(S_q)I = p(S_q) \tag{34}$$

implies the matrix equality

$$[I]_{\mathrm{st}}^{\mathrm{in}}\left[p(S_q)\right]_{\mathrm{co}}^{\mathrm{st}}[I]_{\mathrm{sp}}^{\mathrm{co}} = \left[p(S_q)\right]_{\mathrm{sp}}^{\mathrm{in}}. \tag{35}$$

As $S_q p_i = \lambda_i p_i$, if follows that

$$p(S_q)p_i = p(\lambda_i)p_i = p(\lambda_i)p_i(\lambda_i)\pi_i. \tag{36}$$

Now $p_i(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$, but $q'(z) = \sum_{i=1}^{n}\prod_{j\neq i}(z-\lambda_j)$ and hence

$$q'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j) = p_i(\lambda_i). \tag{37}$$

Thus

$$R = \left[p(S_q)\right]_{\mathrm{sp}}^{\mathrm{in}}, \tag{38}$$

and the result follows. ■

Incidentally the representation (31) of $V(\lambda_1,\ldots,\lambda_n)$ is another proof of the nonsingularity of the Van der Monde matrix.

Another corollary is a diagonalization, by similarity, of the companion matrices.

COROLLARY 2.1. *Let q be as in Theorem* 2.1, C_q *the companion matrix of* (19). *Then for* $V = V(\lambda_1, \dots, \lambda_n)$ *we have*

$$V^{-1}C_qV = \Lambda \tag{39}$$

and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

Proof. From the equality $S_qI = IS_q$ we get

$$[S_q]_{\mathrm{co}}^{\mathrm{co}}[I]_{\mathrm{sp}}^{\mathrm{co}} = [I]_{\mathrm{sp}}^{\mathrm{co}}[S_q]_{\mathrm{sp}}^{\mathrm{sp}}, \tag{40}$$

or

$$C_qV = V\Lambda. \quad \blacksquare \tag{41}$$

3. STABILITY CRITERIA

In this section we state a theorem that covers most classical stability criteria related to quadratic forms. Thus we naturally omit Hurwitz type criteria as well as those approached via Liapunov theory. We will not prove all parts of the theorem, but concentrate on those parts which can be simplified via our results on the Bezoutian.

We begin by introducing some notation. For a monic polynomial

$$q(z) = z^n + q_{n-1}z^{n-1} + \cdots + q_0 \tag{42}$$

of degree n we define the polynomial q_* by

$$q_*(z) = q(-z). \tag{43}$$

Let us define q_+ and q_- through

$$q_+(z) = \sum q_{2j}z^j \tag{44}$$

and

$$q_-(z) = \sum q_{2j+1}z^j. \tag{45}$$

Then

$$q(z) = q_+(z^2) + zq_-(z^2) \tag{46}$$

and

$$q_*(z) = q_+(z^2) - zq_-(z^2). \tag{47}$$

Let $B = B(q, q_*) = (b_{ij})$ be the Bezoutian of q and q_*. We define the Hermite-Fujiwara form $H = (h_{ij})$ by

$$h_{ij} = (-1)^i b_{ij}. \tag{48}$$

Following Krein and Naimark (1936), we say that a pair of real polynomials $q(z)$ and $p(z)$ with $q(z)$ monic of degree m and real simple zeros

$$\lambda_1 < \cdots < \lambda_m \tag{49}$$

and $p(z)$ of degree m or $m-1$ with positive leading coefficient and real zeros

$$\mu_1 < \mu_2 < \cdots < \mu_{m-1} \qquad \text{if} \quad \deg p = m-1 \tag{50}$$

and

$$\mu_1 < \mu_2 < \cdots < \mu_m \qquad \text{if} \quad \deg p = m \tag{51}$$

are a *real pair* if the zeros satisfy

$$\begin{aligned} \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{m-1} < \lambda_m \qquad &\text{if} \quad \deg p = m-1, \\ \mu_1 < \lambda_1 < \mu_2 < \cdots < \beta_m < \lambda_m \qquad &\text{if} \quad \deg p = m. \end{aligned} \tag{52}$$

We say that q and p form a positive pair if they form a real pair and in addition $\lambda_m < 0$.

We say that a polynomial p is *stable* or *Hurwitz* if all its zeros are in the open left half plane Π_-.

The following theorem sums up the Bezoutian related stability criteria. For a proof of the Hurwitz determinantal conditions in the spirit of this paper we refer to Helmke and Fuhrmann [1989].

THEOREM 3.1. *Let $q(z)$ be monic of degree n. Then the following statements are equivalent:*

(i) *$q(z)$ is a stable, or Hurwitz, polynomial.*
(ii) *The Hermite-Fujiwara form is positive definite.*
(iii) *The two Bezoutians $B(q_+, q_-)$ and $B(zq_-, q_+)$ are positive definite.*
(iv) *The polynomials q_+ and q_- form a positive pair.*
(v) *The Bezoutian $B(q_+, q_-)$ is positive definite and all q_i are positive.*

REMARK 1. The equivalence of (i) and (v) is referred to as the Lienard-Chipart theorem.

Proof. The equivalence of (i) and (ii) is classical and goes back to Hermite; see Krein and Naimark (1936) and Gantmacher (1959). Also it is well known [see Krein and Naimark (1936), Datta (1978a), Fuhrmann (1981a)] that the Hermite-Fujiwara form is isomorphic to the direct sum of the forms $B(q_+, q_-)$ and $B(zq_-, q_+)$. So conditions (ii) and (iii) are equivalent.

Next we show the equivalence of conditions (iii) and (iv). Let C_{q+}, C_{q-} be the companion matrices associated with q_+ and q_- respectively. Note that when $n = 2m+1$, i.e. n is odd, we have $\deg q_+ \leqslant \deg q_- = m$, and when $n = 2m$, i.e. n is even, then $\deg q_- < \deg q_+ = m$. Also we observe that the following relation holds:

$$B(q_+, q_-)C_{q+} = \tilde{C}_{q+} B(q_+, q_-). \tag{53}$$

This follows trivially from $q_-(S_{q+})S_{q+} = S_{q+}q_-(S_{q+})$ and taking matrix representations as follows:

$$\left[p(S_{q+})\right]^{\text{st}}_{\text{co}}\left[S_{q+}\right]^{\text{co}}_{\text{co}} = \left[S_{q+}\right]^{\text{st}}_{\text{st}}\left[p(S_{q+})\right]^{\text{st}}_{\text{co}}. \tag{54}$$

Let (ξ, η) be the usual inner product on $\mathbf{R}^m$ or $\mathbf{C}^m$. Since $B(q_+, q_-)$ is symmetric, we can introduce a new inner product in $\mathbf{R}^m$ by letting

$$[\xi, \eta]_B = (B(q_+, q_-)\xi, \eta). \tag{55}$$

This is in general an indefinite inner product and is definite whenever the

Bezoutian $B(q_+, q_-)$ is a positive definite form. Clearly the relation (53) means that

$$[C_{q+}, \xi, \eta]_B = [\xi, C_{q+}, \eta]_B, \tag{56}$$

i.e. that C_{q+} is a self-adjoint map in the $B(q_+, q_-)$ metric. In particular, if $B(q_+, q_-) > 0$ it follows that the spectrum of C_{q+}, $\sigma(C_{q+})$, is real. This means that all zeros of $q_+(z)$ are real. The same holds true for $q_-(z)$. Moreover the zeros of C_{q+} and C_{q-} are simple, as both C_{q+} and C_{q-} are cyclic matrices.

Assume now (iii) holds, i.e., $B(q_+, q_-)$ and $B(zq_-, q_+)$ are both positive definite. By our previous remarks the zeros of q_+ and q_- are all real and simple. Let the zeros of q_+ be ordered as

$$\lambda_1 < \cdots < \lambda_m, \tag{57}$$

and the zeros of q_- as

$$\mu_1 < \mu_2 < \cdots \lambda\mu_{m-1} \quad (<\mu_m), \tag{58}$$

the last inequality holding if $n = 2m + 1$.

Now $B(q_+, q_-)$ is, by the result of Section 2, congruent to the diagonal matrix $\operatorname{diag}(r_1, \dots, r_m)$ with

$$r_i = q_-(\lambda_i) q_+'(\lambda_i). \tag{59}$$

In the same way $B(zq_-, q_+) = -B(q_+, zq_-)$ is congruent to $\operatorname{diag}(s_1, \dots, s_m)$ with

$$s_i = -\lambda_i q_-(\lambda_i) q_+'(\lambda_i). \tag{60}$$

We consider now two cases.

Case I: $n = 2m + 1$. In this case q_+ and q_- are both of degree m. By our assumptions $B(q_+, q_-)$ and $B(zq_-, q_+)$ are both positive definite. This means that for all $i = 1, \dots, m$ we have $r_i > 0$ and $s_i > 0$. Comparing (59) and (60), we get $\lambda_i < 0$. Moreover, from (59) it follows that $q_-(z)$ has different signs at neighboring zeros λ_i of q_+. So the zeros of q_+ and q_- interlace. Now $\mu_m > \lambda_m$ is impossible, as

$$r_m = (\lambda_m - \mu_1) \cdots (\lambda_m - \mu_m)(\lambda_m - \lambda_1) \cdots (\lambda_m - \lambda_{m-1}), \tag{61}$$

and the condition $r_m > 0$ would be violated for all the factors, but $\lambda_m - \mu_m$ are positive. So we get

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{m-1} < \lambda_m < 0. \tag{62}$$

Case II: $n = 2m$. In this case $\deg q_+ = m$ and $\deg q_- = m - 1$. By the same reasoning as before, we get $\lambda_i < 0$ and the zeros of q_- interlace those of q_+. Since q_- has only $m - 1$ zeros, we must have

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{m-1} < \lambda_m < 0. \tag{63}$$

Thus (iii) implies (iv).

Conversely, assume q_+ and q_- form a positive pair. In particular the leading coefficient of q_- is positive. Let

$$r_i = q_-(\lambda_i) q_+'(\lambda_i). \tag{64}$$

We distinguish between two cases.

Case I: $n = 2m + 1$. In this case

$$r_i = \prod_{j=1}^{i} (\lambda_i - \mu_j) \cdot \prod_{k=1}^{i-1} (\lambda_i - \lambda_k) \cdots \prod_{j=i+1}^{m} (\lambda_i - \mu_j) \cdot \prod_{k=i+1}^{m} (\lambda_i - \lambda_k). \tag{65}$$

The factors in the first bracket are all positive, and in the second all negative. However, there are an even number of negative factors, which implies $r_i > 0$.

Case II: $n = 2m$. In the same way

$$r_i = \prod_{j=1}^{i-1} (\lambda_i - \mu_j) \cdot \prod_{k=1}^{i-1} (\lambda_i - \lambda_k) \cdots \prod_{j=i} (\lambda_i - \mu_j) \cdot \prod_{k=i+1}^{m} (\lambda_i - \lambda_k). \tag{66}$$

The number of negative is now $(m - 1 - i) + [m - (i + 1)] = 2(m - i - 1)$. Hence once again we have $r_i > 0$. By our congruence result we get $B(q_+, q_-) > 0$. In the same way we get $s_i = -\lambda_i r_i > 0$, and so also $B(zq_-, q_+) > 0$, (iv) implies (iii).

That (iii) implies (v) is trivial, for we assume q to be monic, and since all λ_i are negative, clearly all coefficients q_i of q are positive.

Assume now $B(q_+, q_-) > 0$ and all q_i are positive. By our identification of the Bezoutian as a matrix representation we have

$$B(q_+, zq_-) = \left[S_{q+}q_-(S_{q+})\right]_{\mathrm{co}}^{\mathrm{st}} = \left[S_{q+}\right]_{\mathrm{st}}^{\mathrm{st}}\left[q_-(S_{q+})\right]_{\mathrm{co}}^{\mathrm{st}}$$

$$= \left[q_-(S_{q+})\right]_{\mathrm{co}}^{\mathrm{st}}\left[S_{q+}\right]_{\mathrm{co}}^{\mathrm{co}}, \tag{67}$$

or

$$B(q_+, zq_-) = B(q_+, q_-)C_{q+} = \tilde{C}_{q+}B(q_+, q_-). \tag{68}$$

This equality implies first that C_{q+} is self-adjoint in the $B(q_+, q_-)$ metric. Hence all zeros of q_+ are real. But, by our assumption, all the coefficients of q, and therefore also all coefficients of q_+, are positive. In particular $q_+(z) > 0$ for $z \geqslant 0$. Thus all zeros of q_+ are real and negative, and so C_{q+} is a negative operator in the $B(q_+, q_-)$ metric.

Let $\xi \in \mathbf{R}^m$ be an arbitrary nonzero constant vector. Then

$$\left(B(zq_-, q_+)\xi, \xi\right) = -\left(\tilde{C}_{q+}B(q_+, q_-)\xi, \xi\right) = -\left[\xi, C_{q+}\xi\right]$$

$$= -\left[C_{q+}\xi, \xi\right] > 0. \tag{69}$$

Thus $B(zq_-, q_+)$ is positive definite, and we have proved that (v) implies (iii). ∎

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