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On Duality in Some Problems of Geometric Control

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Abstract The paper focuses on the analysis of duality theory in the functional, or module theoretic, approach to geometric control. Various results, previously obtained, on the characterization of controlled and conditioned invariant subspaces are related by duality. The duality is not a simple using adjoint maps. The difficulties stem from the fact that we want all characterizations to be based on left matrix fractions. Such characterizations are close to autoregressive representations of behaviors. To obtain all characterizations to be based on left matrix fractions we have to recourse to a two step process involving isomorphisms of polynomial and rational models as well as the use of dual spaces. Doubly coprime factorizations play a significant role and help to illuminate the role of behaviors in this duality theory.

1. Introduction

Duality theory is one of the most powerful tools in the mathematical analysis of various problems. Linear systems theory is no exception. In the context of state space theory, this is usually expressed via matrix transpositions or Hermitian conjugation. Thus we have the dual concepts of controllability and observability with the corresponding Lyapunov and Riccati equations.

The difficulty with this approach stems from the fact that hardly ever is a system given in state space form. More often, it is given in terms of higher order differential, or difference, equations. The behavioral purists will argue that, in general, one should not prejudice himself by considering some of the manifest variables as inputs and the rest as outputs.

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Once we are in this more general, functional, setting, the question of duality becomes much more delicate. Addressing it is the principal purpose of this paper. In fact, one can see this paper as a sequel to Fuhrmann [8], where functional duality was initially addressed in the context of algebraic systems theory. In the close to a quarter century since the publication of that paper, much progress has been made, allowing us a deeper study of some duality related issues. In particular, behavior theory was initiated in Willems [30–33], and has provided a very basic definition of a linear system, as distinct from a plethora of possible representations of it. In fact, the very basic representation of behaviors, at least in discrete time, is a direct result of duality theory. Duality is also a basic tool in the pioneering paper on multidimensional systems, Oberst [26]. One should also mention in this context the recent paper, Yekutieli [35]. What makes the duality theory we discuss here more difficult is the fact that we are in a nonreflexive situation. Even in the Kalman based input/output approach to linear system, the spaces of past input and future output signals are very different. The same holds as far as behaviors are concerned, see Fuhrmann [10].

In the current paper, we assume that a system is given via matrix fraction descriptions, an assumption that is very close to behavioral theory. We will explore some objects, originating in geometric control theory, see Basile and Marro [2, 3] or Wonham [34], and give them functional characterizations. Although, conceptually, these objects are defined in state space terms, all characterizations are functional or module theoretic. The bridge between the two, seemingly very different, approaches is of course realization theory, more specifically the shift realization introduced in Fuhrmann [6, 7]. The use of shifts is also a basic tool in behavior theory. In fact, a behavior is defined as an appropriate linear, shift invariant space that is closed in an appropriate topology, see Willems [30] or Fuhrmann [10]. Thus, it is not very surprising that in a functional analysis of geometric control, behaviors enter in a very natural way.

The original intention was to develop duality theory in order to lift results, obtained in Fuhrmann [13, 14], relating to spectral assignability in controlled invariant subspaces to the context of quotient spaces modulo conditioned invariant subspaces, a basic ingredient of observer theory. However, it turned out that a direct study of the spectral assinability problem for observers is, in many respects, simpler than the original problem. These results, together with other material can be found in Fuhrmann and Trumpf [17]. Still, duality theory is worthy of study for its own sake, as it sheds more light on this whole circle of problems. Moreover, the development of this theory establishes very strong connections between geometric control and behaviors. The connections are so basic that it is hard to avoid behaviors even in problems which are, like observers for partial states, stated in input/output terms, see Fuhrmann [11].

One can argue about the relevance of geometric control at a time that H^∞-methods seem to be predominant. It is the author's contention that the results presented here in an algebraic context have their counterpart in a Hardy space setting. In particular, there is a striking similarity between the functional characterizations of controlled and conditioned invariant subspaces, using polynomial and rational models, on the one hand and the characterization of inner stabilizability and outer detectability subspaces using coinvariant subspaces, as given in Fuhrmann and Gombani [15], on the other. It is highly probable that there is still a lot to be done in filtering and observation of partial states, based on noisy observations, and the development of H^2-methods that reflect those of this paper may turn out to be very useful.

The paper is structured as follows. In Section 2 we introduce the basic spaces, a variety of shift operators and, in particular, the polynomial and rational models. We follow this by introducing and characterizing homomorphisms of polynomial and rational models and study their invertibility properties via the use of doubly coprime factorizations. This is followed up by introducing the basic duality relations, identifying the dual of a polynomial model with either a rational model or another polynomial model, depending on the choice of pairing. We end this section by introducing behaviors and behavior homomorphisms. Noting that behaviors are generalizations of rational models leads to a natural extension of all results concerning model homomorphisms to the behavioral context.

Section 3 is devoted to the basic objects of geometric control, namely controlled and conditioned invariant subspaces. We derive module theoretic and behavioral characterizations of these spaces, as well as characterizations based on factorization theory. For the spaces of output nulling and input containing subspaces we present the characterizations in the case the system is obtained via the shift realization from a left matrix fraction.

In Section 4, we develop the core of duality theory in the context of polynomial models. With a given left matrix fraction, we associate four polynomial models, grouped intwo pairs. In each pair, the shift realization based systems are related by isomorphism, whereas the two pairs relate by duality. This way we can get all characterizations to be given in terms of left matrix fractions. We trace the representation of the objects of geometric control through all realizations.

Finally, in Section 5, we study the spectral assignability problem in controlled invariant subspaces and the dual problem of spectral assignability for conditioned invariant subspaces using output injection. Both problems are solved by means of special polynomial matrix completions. We obtain precise constraints on the structure of invariant factors in the reduced and coreduced systems. The easier cases of reachability and observability subspaces are treated first and provide the basis for the extension to the general case.

2. Preliminaries

2.1. Functional Modules

We begin by giving a concise introduction to polynomial and rational models, first introduced in Fuhrmann [6]. Let $\mathbb{F}$ denote an arbitrary field. We will denote by $\mathbb{F}^m$ the space of all m-vectors with coordinates in $\mathbb{F}$. Let $\mathbb{F}((z^{-1}))^m$ be the space of vectorial truncated Laurent series and let π_+ and π_- denote the projections of $\mathbb{F}((z^{-1}))^m$ on $\mathbb{F}[z]^m$ and $z^{-1}\mathbb{F}[[z^{-1}]]^m$, the space of formal power series vanishing at infinity, respectively. The space $\mathbb{F}((z^{-1}))^m$ is endowed with a natural $\mathbb{F}[z]$-module structure, given by multiplication and $\mathbb{F}[z]^m$ is a submodule. In particular, $S : \mathbb{F}((z^{-1}))^m \longrightarrow \mathbb{F}((z^{-1}))^m$ is defined by

$$Sf(z) = zf(z). \tag{1}$$

As $\mathbb{F}[z]^m$ is a submodule, we can induce a module structure on it by restricting the module structure on $\mathbb{F}((z^{-1}))^m$. In particular, we define $S_+ : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^m$ by $S_+ = S|\,\mathbb{F}[z]^m$. We have, as $\mathbb{F}$-linear spaces, the direct sum representation

$$\mathbb{F}((z^{-1}))^m = \mathbb{F}[z]^m \oplus z^{-1}\mathbb{F}[[z^{-1}]]^m. \tag{2}$$

We denote by π_+ and π_- the projections of $\mathbb{F}\left(\left(z^{-1}\right)\right)^m$ on $\mathbb{F}[z]^m$ and $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m$, respectively. Clearly, π_+ and π_- are complementary projections. We can induce in the space $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m$ an $\mathbb{F}[z]$-module structure via the isomorphism $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m \simeq \mathbb{F}\left(\left(z^{-1}\right)\right)^m/\mathbb{F}[z]^m$. This $\mathbb{F}[z]$-module structure is equal to the one induced by the *left* or *backward shift operator* S_- or, for reasons of compatbility with behavioral theory usage, σ defined by

$$S_-h = \sigma h = \pi_- zh, \quad h \in z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m. \tag{3}$$

Any $\mathbb{F}[z]$-submodule $\mathcal{M} \subset \mathbb{F}[z]^m$ has a representation $\mathcal{M} = M(z)\mathbb{F}[z]^k$ for some $m \times k$ polynomial matrix. If we require M to have full column rank, then $M(z)$ is uniquely determined up to a right unimodular factor. Given a $p \times m$ polynomial matrix $R(z)$, the set $\mathcal{M} = \{f \in \mathbb{F}[z]^m \,|\, R(z)f(z) = 0\}$ is a submodule, hence has a representation $\mathcal{M} = M(z)\mathbb{F}[z]^k$, with $M(z)$ having full column rank. We call M a *minimal right annihilator* or *MRA* for short. Similarly, given a $p \times m$ polynomial matrix $R(z)$, we say M is a *minimal left annihilator*, or *MLA* for short, if $\widetilde{M}$ is a *MRA* of $\widetilde{R}$. Here $\widetilde{R}$ denotes the transpose of the polynomial matrix R.

Given a $p \times m$, full row rank polynomial matrix $V(z)$, then $\mathcal{E} = \{Vf \,|\, f \in \mathbb{F}[z]^m\}$ is an $\mathbb{F}[z]$-submodule of $\mathbb{F}[z]^p$, moreover, it is a full submodule, i.e. given by $\mathcal{E} = E\mathbb{F}[z]^p$ with E nonsingular. Since for each constant vector $\xi \in \mathbb{F}[z]^m$, we have $V(z)\xi \in \mathcal{E}$, there exists a unique polynomial matrix $V'(z)$ for which

$$V(z) = E(z)V'(z). \tag{4}$$

We refer to Equation (4) as an *internal/external factorization*.

Any full column rank polynomial matrix H has an essentially unique factorization of the form $H = H_1H_0$ with H_1 right prime and H_0 nonsingular. We call such a factorization an *external/internal factorization*.

The terminology has been chosen in analogy with inner/outer factorizations used in operator theory.

2.2. Polynomial and Rational Models

Given a nonsingular polynomial matrix D in $\mathbb{F}[z]^{m\times m}$ we define two projections π_D acting in $\mathbb{F}[z]^m$ and π^D acting in $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m$ by

$$\pi_D f = D\pi_- D^{-1} f \quad for \;\; f \in \mathbb{F}[z]^m \tag{5}$$

$$\pi^D h = \pi_- D^{-1}\pi_+ Dh \quad for \;\; h \in z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m \tag{6}$$

and define two linear subspaces, of $\mathbb{F}[z]^m$ and $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m$, respectively, by

$$X_D = \operatorname{Im}\pi_D \tag{7}$$

and

$$X^D = \operatorname{Im}\pi^D. \tag{8}$$

In X_D we define the shift S_D by

$$S_D f = \pi_D zf. \tag{9}$$

With the $\mathbb{F}[z]$-module structure induced by S_D we have, noting that

$$\operatorname{Ker}\pi_D = D\mathbb{F}[z]^m, \tag{10}$$

the module isomorphism

$$X_D \simeq \mathbb{F}[z]^m \big/ D\mathbb{F}[z]^m. \tag{11}$$

Moreover, we have the direct sum, as linear spaces,

$$\mathbb{F}[z]^m = X_D \oplus D\mathbb{F}[z]^m. \tag{12}$$

Similarly, in X^D we define a map S^D by

$$S^D = S_- \left| X^D \right. . \tag{13}$$

We refer to X_D as a *polynomial model* whereas to X^D as a *rational model*. A subspace $\mathcal{V} \subset X_D$ is a submodule, or, equivalently, an S_D-invariant subspace if and only if it has a representation of the form $\mathcal{V} = D_1 X_{D_2}$, for some factorization $D = D_1 D_2$, with the factors nonsingular.

It should be noted that the isomorphism (11) is not canonical. The quotient module $\mathbb{F}[z]^m \big/ D\mathbb{F}[z]^m$ depends only on the submodule $D\mathbb{F}[z]^m$ and not directly on $D(z)$. Indeed, for any unimodular polynomial matrix U, we have $D\mathbb{F}[z]^m = DU\mathbb{F}[z]^m$. Thus, using the isomorphism (11), we are moving from a categorical object to a noncategorical one. Such a decision has its advantages and disadvantages. The disadvantage is obvious, we lose the generality of the quotient module $\mathbb{F}[z]^m \big/ D\mathbb{F}[z]^m$. However, we gain in concreteness. In view of the fact that, over the years, a computationally efficient algebra for polynomial and rational models has been developed, and furthermore, many important results have been obtained in this way, Theorems 3.1 and 3.2 are typical examples, this decision seems to be fully justified. Moreover, we always keep in mind the operator theoretic analogs of polynomial and rational models, namely the backward shift invariant subspaces and it is in the present formalism that the analogy is brought out in the clearest way.

The polynomial model X_D, and the rational model X^D are isomorphic, the isomorphism given by the multiplication map $f \mapsto D^{-1}f$. It is easily shown that

$$D^{-1} S_D = S^D D^{-1}. \tag{14}$$

It is of great importance to have an easy characterization of elements of polynomial or rational model. Toeplitz operators play an important role throughout the theory of linear systems. We introduce two classes of *Toeplitz operators* in terms of which we can give characterizations of polynomial or rational models. To this end, let $A \in \mathbb{F}\left(\left(z^{-1}\right)\right)^{p\times m}$.

Then the Toeplitz operator $T_A : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^p$ is defined by

$$T_A f = \pi_+ Af, \quad f \in \mathbb{F}[z]^m, \tag{15}$$

whereas the Toeplitz operator $\mathcal{T}_A : z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m \longrightarrow z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^p$ is defined by

$$\mathcal{T}_A h = \pi_- Ah, \quad h \in z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m. \tag{16}$$

For reasons of compatibility with behavioral theory usage, we shall write $\sigma = S_-$ and, for $A \in \mathbb{F}[z]^{p\times m}$

$$A(\sigma)h = \mathcal{T}_A h = \pi_- Ah, \quad h \in \mathbb{F}[z]^m. \tag{17}$$

Both polynomial and rational models have simple representations as kernels of appropriate Toeplitz operators.

PROPOSITION 2.1. *Let $D \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Then*

1. *We have*

$$X_D = \mathrm{Ker}\, T_{D^{-1}}. \tag{18}$$

2. *We have*

$$X^D = \mathrm{Ker}\mathcal{T}_D = \mathrm{Ker}\, D(\sigma). \tag{19}$$

Proof.

1. Assume $f \in X_D$, then $f = \pi_D f = D\pi_- D^{-1} f$ which shows that $D^{-1}f = \pi_- D^{-1} f$ is strictly proper. So $\pi_+ \mathrm{D}^{-1} f = 0$, i.e. $f \in \mathrm{Ker}\, T_{D^{-1}}$.
Conversely, assume $f \in \mathrm{Ker}\, T_{D^{-1}}$, i.e. $\pi_+ D^{-1} f = 0$ which shows that $D^{-1}f$ is strictly proper and hence $D^{-1}f = \pi_- D^{-1} f$ which in turn implies $f = D\pi_- D^{-1} f = \pi_D f$, i.e. $f \in X_D$.
2. Assume $h \in X_D$, i.e. $h = \pi^D h = \pi_- D^{-1}\pi_+ Dh$. This implies $Dh = D\pi_- D^{-1}\pi_+ Dh = \pi_D \pi_+ Dh \in \mathbb{F}[z]^m$ and hence $\pi_- Dh = 0$, i.e. $h \in \mathrm{Ker}\mathcal{T}_D = \mathrm{Ker}\, D(\sigma)$.
Conversely, assume $h \in \mathrm{Ker}\ D(\sigma)$, i.e. $\pi_- Dh = 0$. Equivalently we have $Dh = \pi_+ Dh$. In turn this implies $h = D^{-1}\pi_+ Dh$ and, as $h = \pi_- h$, it follows that $h = \pi_- D^{-1}\pi_+ Dh = \pi^D h \in X^D$. □

An element f of $\mathbb{F}[z]^m$ belongs to X_D if and only if $\pi_+ D^{-1} f = 0$, i.e. if and only if $D^{-1}f$ is a strictly proper rational vector function. Thus we have also the following description of the polynomial model X_D

$$X_D = \left\{ f \in \mathbb{F}[z]^m \,\middle|\, f = Dh,\ h \in z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m \right\} \tag{20}$$

The advantage of this characterization is that it makes sense for an arbitrary $p \times m$ polynomial matrix V. Thus we define following Emre and Hautus [5],

$$X_V = \left\{ f \in \mathbb{F}[z]^p \,\middle|\, f = Vh, \quad h \in z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m \right\} \tag{21}$$

Analogously, $h \in X^D$ if and only if $\pi_- Dh = 0$, i.e. if and only if h is in the kernel of the Toeplitz map $\mathcal{T}_D : z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m \longrightarrow z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ defined by $\mathcal{T}_D h = \pi_- Dh$. Again, given an arbitrary $p \times m$ polynomial matrix V, we define

$$X^V = \left\{ h \in z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m \,\middle|\, Vh \in \mathbb{F}[z]^p \right\} = \mathrm{Ker}\ V(\sigma). \tag{22}$$

Since, for $h \in X^V$, we have $\pi_- Vh = 0$, it follows that

$$\pi_- V(S_- h) = \pi_- V\pi_- zh = \pi_- zVh = \pi_- z\pi_- Vh = 0,$$

it follows that X^V is S_-, or σ invariant. Actually it is a behavior and we will show, in subsection 2.8, that every behavior has such a representation.

2.3. Model Homomorphisms

Given two polynomial models, $X_{\overline{T}}$ and X_T, the homomorphisms between them, i.e. the intertwining linear maps $\phi : X_{\overline{T}} \longrightarrow X_T$ satisfying $\phi S_{\overline{T}} = S_T \phi$ have been characterized in Fuhrmann [6] as follows. A map $\phi : X_{\overline{T}} \longrightarrow X_T$ is a module homomorphism if and only if there exist polynomial matrices $V, \overline{V}$ satisfying

$$V\overline{T} = T\overline{V} \tag{23}$$

in terms of which ϕ is given by

$$\phi f = \pi_T V_f, \quad f \in X_{\overline{T}}. \tag{24}$$

Moreover, ϕ is injective if and only if $\overline{T}, \overline{V}$ are right coprime and surjective if and only if T, V are left coprime. If both coprimeness conditions hold, then ϕ is an isomorphism and its inverse can be obtained from an arbitrary doubly coprime factorization which, due to our coprimeness assumptions, always exists. The details can be found in Fuhrmann [10].

Let

$$\begin{aligned} \begin{pmatrix} -V & T \\ Y & -X \end{pmatrix} \begin{pmatrix} \overline{X} & \overline{T} \\ \overline{Y} & \overline{V} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} \overline{X} & \overline{T} \\ \overline{Y} & \overline{V} \end{pmatrix} \begin{pmatrix} -V & T \\ Y & -X \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \end{aligned} \tag{25}$$

be such a doubly coprime factorization. In particular, we have the equalities

$$\overline{X}T = \overline{T}X; \ \ \overline{Y}V = \overline{V}Y; \ \ \overline{T}Y - \overline{X}V = I. \tag{26}$$

By Proposition 3.6.3 in Fuhrmann [10], see also Bisiacco and Valcher [4] the doubly coprime factorization (25) implies that V is left prime if and only if $\overline{V}$ is. Equalities (26) show that the inverse isomorphism $\psi = \phi^{-1} : X_T \longrightarrow X_{\overline{T}}$ is given by

$$\psi f = \phi^{-1} f = -\pi_{\overline{T}} \overline{X} f, \quad f \in X_T. \tag{27}$$

The isomorphism of the polynomial and rational models given by (14) allows us to characterize the homomorphisms between two rational models. Thus we have.

THEOREM 2.1. *Let $T \in \mathbb{F}[z]^{m \times m}$ and $\overline{T} \in \mathbb{F}[z]^{\overline{m} \times \overline{m}}$ be nonsingular. Then $Z : X^{\overline{T}} \longrightarrow X^T$ is an $\mathbb{F}[z]$-homomorphism if and only if there exist $V, \overline{V} \in \mathbb{F}[z]^{m \times \overline{m}}$ such that*

$$V\overline{T} = T\overline{V} \tag{28}$$

and

$$Zh = \pi_- \overline{V} h = \overline{V}(\sigma) h \quad h \in X^{\overline{T}}. \tag{29}$$

As for the case of polynomial model homomorphisms, Z is injective if and only if $\overline{T}, \overline{V}$ are right coprime and surjective if and only if T, V are left coprime. If both coprimeness conditions hold, then Z is an isomorphism and its inverse can be obtained from an arbitrary doubly coprime factorization (25). In fact, we have

$$\overline{V}(\sigma)^{-1} = -X(\sigma). \tag{30}$$

2.4. Duality

We give a short exposition of duality in the context of polynomial and rational models as developed in Fuhrmann [8, 9].

Given a vector space V over a field $\mathbb{F}$ we denote by V^* the *dual space* of V that is the space of linear functionals on V. Given $v^* \in V^*$ and $v \in V$ we will write

$$\left[v, v^*\right] = v^*(v).$$

In the special case of $V = \mathbb{F}^m$ we can also identify V^* with $\mathbb{F}^m$ and then we write $\left[x, y\right] = \widetilde{y} x$ where $\widetilde{y}$ denotes the transpose of the column vector y. The sole exception will be the complex inner product spaces where $[x, y]$ will be interpreted as the inner product itself. Now given $f, g \in \mathbb{F}((z^{-1}))^m$, we define a pairing

$$[f, g] = \sum_{j=-\infty}^{\infty} [f_j, g_{-j-1}]. \tag{31}$$

It is clear that $[\cdot,\cdot]$ is a bilinear form on $\mathbb{F}((z^{-1}))^m \times \mathbb{F}((z^{-1}))^m$. It is well defined, as in the defining sum at most a finite number of terms are nonzero. Also this form is nondegenerate in the sense that $[f, g] = 0$ for all $g \in \mathbb{F}((z^{-1}))^m$ if and only if $f = 0$. Given a subset $M \subset \mathbb{F}^m$ we define

$$M^{\perp} = \{v^* \in V^* \,|\, [m, v^*] = 0 \quad \forall m \in M\}.$$

Similarly if $M \subset V^*$ we let

$${}^{\perp}M = \{v \in V \,|\, [v, v^*] = 0 \quad \forall v^* \in M\}.$$

It is a simple check of the definitions that $(\mathbb{F}[z]^m)^{\perp} = \mathbb{F}[z]^m$. Moreover, in a natural way, one can identify the dual space of $\mathbb{F}[z]^m$ with $z^{-1}\mathbb{F}[[z^{-1}]]^m$.

As usual, given bilinear forms on $V \times V^*$ and $W \times W^*$ and a map $A\colon V^* \longrightarrow W$ the *dual map* $A^*\colon W^* \longrightarrow V^*$ is defined by the equality

$$[Av, w^*] = [v, A^* w^*].$$

Given $A \in \mathbb{F}((z^{-1}))^{p\times m}$ with $A(z) = \sum_{j=-\infty}^{n} A_j z^j$ we will denote by A^* the element of $\mathbb{F}((z^{-1}))^{m\times p}$ given by

$$A^*(z) = \sum_{j=-\infty}^{n} A_j^* z^j.$$

In the next proposition we summarize, without proofs, the computational rules related to the duality defined by Equation (31).

PROPOSITION 2.2.

1. *Given* $A \in \mathbb{F}((z^{-1}))^{p\times m}$. *Let* $L_A : \mathbb{F}((z^{-1}))^m \longrightarrow \mathbb{F}((z^{-1}))^p$ *be the corresponding Laurent operator defined by*

$$(L_A f)(z) = A(z) f(z) = \sum g_j z^j \tag{32}$$

where $g_j = \sum_{i=-\infty}^{\infty} A_{j-i} f_i$. *Then*

$$(L_A)^* = L_{A^*} \tag{33}$$

2. *The duals of the projections* π_+ *and* π_- *are given by*

$$\pi_+^* = \pi_-, \quad \pi_-^* = \pi_+. \tag{34}$$

3. $\mathbb{F}[z]^m$ *is a submodule, relative to the ring* $\mathbb{F}[z]$, *of* $\mathbb{F}((z^{-1}))^m$ *then* $\mathbb{F}^m[z]$ *is S-invariant thus we can define* S_+ *by*

$$S_+ = S\,|\mathbb{F}[z]^m .$$

We also define $\sigma = S_- : z^{-1}\mathbb{F}[[z^{-1}]]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^m$ *by*

$$\sigma = S_- h = \pi_- z h.$$

4. *The dual of the map* $S_+ : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^m$ *is given by* $S_- = \sigma : z^{-1}\mathbb{F}[[z^{-1}]]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^m$.
5. *Let* $M \subset \mathbb{F}[z]^m$ *be a submodule, then* $M^\perp$ *is a submodule of* $z^{-1}\mathbb{F}[[z^{-1}]]^m$.
6. *Let* $M = D\mathbb{F}[z]^m$ *with* $D \in \mathbb{F}[z]^{m\times m}$ *then* $M^\perp = X^{\widetilde{D}}$.
7. *The adjoint of* π_D *is* $X^{\widetilde{D}}$.

We use the notation σ mainly for compatibility with the behavioral literature.

From our identification of $z^{-1}\mathbb{F}[[z^{-1}]]^m$ as the dual of $\mathbb{F}[z]^m$, it follows that if M is a subset of $\mathbb{F}[z]^m$ then $M^\perp$ is a subset of $z^{-1}\mathbb{F}[[z^{-1}]]^m$.

For a nonsingular $D \in \mathbb{F}[z]^{m\times m}$, we have with the pairing (31),

$$\left(\mathbb{F}[z]^m / D\mathbb{F}[z]^m\right)^* \simeq \left(D\mathbb{F}[z]^m\right)^\perp = X^{\widetilde{D}}, \tag{35}$$

and, using the isomorphism (11), we have $X_D^* \simeq \left(\mathbb{F}[z]^m / D\mathbb{F}[z]^m\right)^*$, it follows that

$$X_D^* \simeq X^{\widetilde{D}}. \tag{36}$$

This corresponds to the pairing $[f, h]$, with $f \in X_D$ and $h \in X^{\widetilde{D}}$ based on the bilinear form (31). Here, as throughout the paper, $\widetilde{A}$ denotes the transpose of A. Of course, since $X^{\widetilde{D}}$ is isomorphic to $X_{\widetilde{D}}$, we could have the identification.

$$X_D^* = X_{\widetilde{D}} \tag{37}$$

with the use of the pairing

$$\langle f, g\rangle = \left[D^{-1} f, g\right], \tag{38}$$

for $f \in X_D$ and $g \in X_{\widetilde{D}}$.

We are now in a position to find the adjoints of the maps ϕ and ψ defined by (24) and (27), respectively. Dualizing equality (23), we obtain

$$\widetilde{\overline{T}}^{-1}\widetilde{\overline{V}} = \widetilde{V}\widetilde{T}^{-1} \tag{39}$$

or equivalently the intertwining relation

$$\widetilde{\overline{V}}\widetilde{T} = \widetilde{\overline{T}}\widetilde{V}. \tag{40}$$

Extending this idea, by transposing the doubly coprime factorization (25), opens the door to the computation of the adjoints of the maps ϕ and ψ.

PROPOSITION 2.3.

1. *Transposing the doubly coprime factorization* (25), *we obtain the dual doubly coprime factorization*

$$\begin{aligned} \begin{pmatrix} \widetilde{\overline{X}} & \widetilde{\overline{Y}} \\ \widetilde{\overline{T}} & \widetilde{\overline{V}} \end{pmatrix} \begin{pmatrix} -\widetilde{V} & \widetilde{Y} \\ \widetilde{T} & -\widetilde{X} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} -\widetilde{V} & \widetilde{Y} \\ \widetilde{T} & -\widetilde{X} \end{pmatrix} \begin{pmatrix} \widetilde{\overline{X}} & \widetilde{\overline{Y}} \\ \widetilde{\overline{T}} & \widetilde{\overline{V}} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \end{aligned} \tag{41}$$

2. *The adjoint of the map* $\phi : X_{\overline{T}} \longrightarrow X_T$ *defined in* (24) *is the map* $\phi^* : X_{\widetilde{T}} \longrightarrow X_{\widetilde{\overline{T}}}$ *given by*

$$\phi^* g = \pi_{\widetilde{\overline{T}}}\widetilde{\overline{V}} g, \quad g \in X_{\widetilde{T}}, \tag{42}$$

3. *The adjoint of the map* $\psi : X_T \longrightarrow X_{\overline{T}}$ *defined in* (27) *is the map* $\psi^* : X_{\widetilde{\overline{T}}} \longrightarrow X_{\widetilde{T}}$ *given by*

$$\psi^* g = -\pi_{\widetilde{T}} \widetilde{X} g, \quad g \in X_{\widetilde{\overline{T}}} \tag{43}$$

With the identification $X_T^* = X_{\widetilde{T}}$, $X_{\overline{T}}^* = X_{\widetilde{\overline{T}}}$ the dual to the map $X_T \xrightarrow{\psi=\phi^{-1}} X_{\overline{T}}$ is $X_{\widetilde{T}} \xleftarrow{\psi^*} X_{\widetilde{\overline{T}}}$. Using the fact that $(\psi^*)^{-1} = (\psi^{-1})^*$, we have $X_{\widetilde{T}} \xrightarrow{\phi^*} X_{\widetilde{\overline{T}}}$.

The isomorphism (11) formed the basis for the identification $X_D^* = X^{\widetilde{D}} = \operatorname{Ker} \widetilde{D}(\sigma)$. This can be extended to arbitrary polynomial quotient modules. Let $\mathcal{M} = M\mathbb{F}[z]^k \subset \mathbb{F}[z]^m$ be a submodule. We can easily check that $\left(\mathbb{F}[z]^m / M\mathbb{F}[z]^k\right)^* \simeq \left(M\mathbb{F}[z]^k\right)^\perp = \operatorname{Ker} \widetilde{M}(\sigma)$.

2.5. The Shift Realization

The next theorem defines the shift realization associated with a particular representation of the transfer function, see [6, 7]. This representation generalizes matrix fraction representations and plays a central role in polynomial matrix descriptions of linear systems, see [29]. The importance of the shift realization cannot be overemphasized. Initially, it provided the link between state space theory, module theory and the theory of polynomial system matrices. Later it became the bridge between input/output based theories of linear systems and the newer behavioral approach. It serves to extend the theory of strict system equivalence to that of various behavior representations and to elucidate connections between behaviors and geometric control. Thus it is the ultimate tool for unifying all existent approaches to linear systems.

THEOREM 2.2. *Let a* $p \times m$ *proper rational function have the representation*

$$G = VT^{-1}U + W. \tag{44}$$

Then the system defined, in the state space X_T, *by*

$$\begin{cases} Af = S_T f & f \in X_T \\ B\xi = \pi_T U \xi, & \xi \in \mathbb{F}^m \\ Cf = \left(VT^{-1} f\right)_{-1} & f \in X_T \\ D = G(\infty). \end{cases} \tag{45}$$

is a realization of G. *This realization is observable if and only if* V *and* T *are right coprime and it is reachable if and only if* T *and* U *are left coprime.*

We will denote the shift realization (45) by $\sum(VT^{-1}U + W)$. If we consider the nonsingular polynomial matrix T to be the left denominator in a left matrix fraction $T^{-1}V$, then we will denote by (C_T, A_T) the observable pair defined by

$$\begin{aligned} C_T f &= \left(T^{-1} f\right)_{-1} \qquad f \in X_T \\ A_T f &= S_T f. \end{aligned} \tag{46}$$

Similarly, for a right denominator $\overline{T}$, we will define the reachable pair $\left(A_{\overline{T}}, B_{\overline{T}}\right)$ by

$$\begin{aligned} A_{\overline{T}} f &= S_{\overline{T}} f \qquad f \in X_{\overline{T}} \\ B_{\overline{T}} \xi &= \pi_T \xi, \qquad \xi \in \mathbb{F}^m \end{aligned} \tag{47}$$

Using the isomorphism of $X_{\overline{T}}$ and $X^{\overline{T}}$, the pair $\left(A_{\overline{T}}, B_{\overline{T}}\right)$ is similar to the pair $\left(A^{\overline{T}}, B^{\overline{T}}\right)$ given by

$$\begin{aligned} A^{\overline{T}} f &= S^{\overline{T}} f \qquad f \in X_{\overline{T}} \\ B^{\overline{T}}\xi &= \pi_- \overline{T}^{-1}\xi \quad \xi \in \mathbb{F}^m \end{aligned} \tag{48}$$

It should be noted that every observable pair (C, A) is isomorphic to the pair (C_T, A_T) arising from the coprime factorization $C(zI - A)^{-1} = T^{-1}V$. A similar result holds for reachable pairs.

2.6. Wiener–Hopf Factorizations

Toeplitz operators play an imortant role in the sequel. The analysis of invertibility of Toeplitz operators is closely related to the study of Wiener–Hopf factorizations and the associated factorization indices. The importance of the factorization indices in the multivariable context stems from the fact that the degree of a matrix polynomial is an inaccurate measure of its 'size', whereas the indices capture it much better on top of being associated to the controllability and observability indices of a system, see Fuhrmann and Willems [18]. These indices are introduced next.

DEFINITION 2.1. *Let* $G \in \mathbb{F}\left(\left(z^{-1}\right)\right)^{p\times m}$ *be rational. A* left Wiener–Hopf factorization *at infinity is a factorization of G of the form*

$$G = G_- D G_+ \tag{49}$$

with $G_+ \in \mathbb{F}[z]^{m\times m}$ *unimodular,* $G_- \in \mathbb{F}\left[\left[z^{-1}\right]\right]^{p\times p}$ *biproper and*

$$D(z) = \begin{pmatrix} \Delta(z) & 0 \\ 0 & 0 \end{pmatrix}$$

where $\Delta(z) = \mathrm{diag}(z^{\kappa_1}, ..., z^{\kappa_r})$. *The integers* κ_i, *assumed decreasingly ordered, are called the* left factorization indices *at infinity. A* right factorization *and the* right factorization indices *are analogously defined with the plus and minus signs in* (49) *reversed.*

The Wiener–Hopf factorization indices characterize the invertibility of Toeplitz operators. Consider the Toeplitz operator $T_G : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^p$, defined by (15), with G rational. Clearly, for T_G to be injective, it is necessary that G has full column rank, whereas for surjectivity it is necessary that G has full row rank. Now assume G has full column rank, hence a left Wiener–Hopf factorization of the form $G = G_+ D G_-$ exists with $D(z) = \begin{pmatrix} \Delta(z) \\ 0 \end{pmatrix}$ and $\Delta(z) = \mathrm{diag}(z^{\kappa_1}, ..., z^{\kappa_m})$. Now $T_G f = \pi_+ G_+ D G_- f = \pi_+ G_+ \pi_+ D(G_- f)$.

The map $g \mapsto \pi_+ G_+ g$ is invertible with inverse given by $g \mapsto \pi_+ G_+{}^{-1} g$. The invertibility of the multiplication by G_+ is obviously invetible. Thus $T_G f = 0$ if and only if $T_D(G_+ f) = 0$.

Now $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in \mathrm{Ker}\, T_D$ if and only if $T_{z^{\kappa_i}} f_i = 0$. Injectivity of T_G is therefore equivalent to $\kappa_i \geq 0$ for $i = 1,..., m$. Similarly, if G has full row rank, then T_G is surjective if and only if $\kappa_i \le 0$ for $i = 1,\ldots,$ m. Thus for a Toeplitz operator to be invertible, G has to be square with all left Wiener–Hopf factorization indices equal to zero.

The invertibility of Toeplitz operators is not the only way Wiener–Hopf factorization indices enter the picture. Given a nonsingular polynomial matrix $D(z)$, then the left Wiener–Hopf factorization indices of D are equal to the reachability indices of the pair (A_D, B_D) defined by (47). Similarly, the right Wiener–Hopf factorization indices of D are equal to the observability indices of the pair (C_D, A_D) (46).

Two reachable pairs $\left(A_{D_i}, B_{D_i}\right)$, $i = 1, 2$, defined by (47) are feedback equivalent if and only if D_1 and D_2 have the same left factorization indices. This connection of coprime factorization and feedback is the key to the celebrated pole placement theorem of Rosenbrock [29].

2.7. Elements of Behavior Theory

The object of this subsection is to present the basics of behavior theory in the setting of discrete time systems. We give the definition of dynamical systems and behaviors as used in Willems [32]. In this setting the notion of completeness can be addressed purely from the algebraic point of view. This we do and rederive the kernel representation of behaviors. This is a key result in behavioral theory inasmuch as it allows to reformulate the problems and study behaviors in polynomial terms. Thus essentially the study of behaviors is reducible to the study of rectangular polynomial matrices arising through a kernel, or AR, representation of the behavior. We proceed to the study of subbehaviors and their connection to factorization theory. This is an extension of the fact that in the theory of polynomial and rational models invariant subspaces relate to factorizations. Next we proceed to introduce and study doubly unimodular embeddings. This is an important technical subject that is used throughout the rest of the paper. We conclude with some more resuls on factorizations of polynomial matrices and behaviors.

The behavioral approach differs from the classical approach, dominated by Kalman's ideas, see Chapter 10 in Kalman et al. [22], in changing the emphasis from input/output maps to either full time or future trajectories. In the Kalman approach to linear systems, realization theory is the corner stone. The realization procedure is based on the restricted i/o map, i.e. a Hankel operator, that maps past inputs to future outputs. In fact, under Nerode type equivalence, the past inputs provide a natural abstract state space. In behavior theory to the contrary one looks at the set of future trajectories. In the case of i/o systems we look at the map from state at time zero and future inputs to future outputs. In principle, all the information on the system should be recoverable from this data.

We follow [32] in defining a *dynamical system* Σ as a triple

$$\Sigma = (T, W, \mathcal{B}), \tag{50}$$

where $\mathbf{T} \subset \mathbf{R}$ is the *time axis*, W is an abstract set called the *signal alphabet* and $\mathcal{B} \subset W^T$ is called the *behavior*. The elements of $\mathcal{B}$ are called the *trajectories* of the system.

This definition is very general and is representation-free. In the context of this paper we will identify T with $\mathbf{Z}_+$, the set of positive integers, assume $\mathbb{F}$ is an arbitrary field and take $W = \mathbb{F}^m$. We identify W^T with $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m$. The space $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m$ has a standard $\mathbb{F}[z]$-module structure induced by the *left* or *backward shift operator* S_- or σ defined by (3).

Given a polynomial matrix $P(z) \in \mathbb{F}[z]^{p\times m}$, it defines a map $P(\sigma)$: $z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m \longrightarrow z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^p$ via

$$P(\sigma)h = \pi_- Ph, \quad h \in z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m. \tag{51}$$

Clearly the operators of the form $P(\sigma)$ are a special class of Toeplitz operator and it is their kernels that are of interest to us. In fact we would like to characterize those subspaces of $z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^p$ that are representable in the form Ker $P(\sigma)$ for some polynomial matrix $P(z)$. This *kernel representation*, due to Willems [30], is the key result for the study of behaviors. In what follows we shall describe a purely algebraic approach to this representation result. To this end, let X be a linear vector space over an arbitrary field $\mathbb{F}$ and let X^* be its algebraic dual. Given a subspace $M \subset X$, we denote by $M^\perp$ its *annihilator*, i.e. $M^\perp = \{h \in x^*h | M = 0\}$.

Similarly, given a subspace $V \subset X^*$, we denote by ${}^\perp V$ its *preannihilator*, i.e. ${}^\perp V = \{x \in X | \, h(x) = 0, \forall h \in V\}$.

Since submodules of the space $\mathbb{F}[z]^m$ of vector polynomial are well studied and have a nice representation in terms of polynomial matrices, it leads immediately to a nice representation of those submodules of $z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ that are annihilators of submodules of $\mathbb{F}[z]^m$.

As an $\mathbb{F}[z]$ module, the space $z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ has a multitude of submodules, i.e. linear, shift invariant subspaces. In this class we single out a special, small, subclass which is determined by the extra property of completeness. To introduce completeness, we define in $z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ the projections P_n, $n \in Z_+$ by

$$P_n \sum_{i=1}^{\infty} \frac{h_i}{z^i} = \sum_{i=1}^{n} \frac{h_i}{z^i}. \tag{52}$$

We say that a subset $\mathcal{B} \subset z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ is *complete* if for any $w = \sum_{i=1}^{\infty} w_i z^{-i} \in z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ and for each positive integer N, $P_N w \in P_N(\mathcal{B})$ implies $w \in \mathcal{B}$. A *behavior* in our context is defined as a linear, shift invariant and complete subspace of $z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$. It has been shown in Fuhrmann [10] that a subspace $\mathcal{V} \subset z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ is complete ifand only if

$$\left({}^\perp\mathcal{V}\right)^\perp = \mathcal{V}. \tag{53}$$

The principal characterization of behaviors, due to Willems [30], is now an easy corollary.

THEOREM 2.3. *A subset $\mathcal{B} \subset z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ is a behavior if and only if it admits a* kernel representation*, i.e. there exists a $p \times m$ polynomial matrix $P(z)$ for which*

$$\mathcal{B} = \text{Ker } P(\sigma) = \left\{ h \in z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m | \pi_- Ph = P(\sigma)h = 0 \right\}. \tag{54}$$

An important subclass of behaviors arises when we restrict the polynomial matrix in a kernel representation to be nonsingular. Following Willems, we say that a behavior $\mathcal{B}$ is *autonomous* if it is finite dimensional as a vector space over $\mathbb{F}$. We have the following.

PROPOSITION 2.4. *The following statements are equivalent:*

1. *The behavior $\mathcal{B}$ is autonomous.*
2. $\mathcal{B} = \text{Ker } D(\sigma)$ *for some nonsingular polynomial matrix* $D(s)$.
3. $\mathcal{B}$ *is equal to the rational model* X^D.
4. *There exists an observable pair* (C, A) *for which*
$$\mathcal{B} = \left\{ C(sI - A)^{-1}\xi \,\middle|\, \xi \in \mathbb{F}^n \right\}. \tag{55}$$

We omit the details of the proof.

Since behaviors are generalizations of rational models, we find it convenient to use the notation

$$X^V = \text{Ker } V(\sigma). \tag{56}$$

The identification of autonomous behaviors with rational models is of great importance as, over the years, polynomial and rational models have been studied quite extensively, most importantly the characterization of homomorphisms. This provides much needed intuition into the study of general behaviors. For more on this, see Fuhrmann [10, 12].

2.8. Subbehaviors

Central results in the polynomial approach to the study of linear transformations and linear systems are the representation of submodules of the free module $\mathbb{F}[z]^p$ and the transformation of the analysis of the lattice of submodules to the arithmetic of factorizations of polynomial matrices. Since there is, via duality theory as in Theorem 2.3, a bijective correspondence between behaviors in $z^{-1}\mathbb{F}\left[\left[z^{-1}\right]\right]^m$ and submodules of the free module $\mathbb{F}[z]^p$, we expect to use this correspondence for the study of the lattice of subbehaviors of a given behavior and relate it to factorizations. This we proceed to do, and we begin by defining subbehaviors.

DEFINITION 2.2. *A subset* $\mathcal{B}_0 \subset \mathcal{B}$ *is called a* subbehavior *if it is itself a behavior, i.e. it is linear, shift invariant and closed.*

We wish to point out that not every linear, shift invariant subspace of a behavior is a subbehavior. Closure is necessary.

Given a behavior $\mathcal{B}$ with kernel representation $\mathcal{B} = \text{Ker } R(\sigma)$, then its subbehaviors are related to factorizations of R. In fact, $\mathcal{B}_s \subset \mathcal{B}$ is a subbehavior if and only if there exists a factorization $R = R_1 R_2$ for which $\mathcal{B}_s = \text{Ker } R_2(\sigma)$. In every behavior there is a canonical subbehavior, namely the reachable subbehavior which we proceed to introduce. Since subbehaviors are described via factorizations, it is of interest to see what factorization describes the reachable subbehavior.

DEFINITION 2.3. *Let* $\mathcal{B}$ *be a behavior defined on* $\mathbf{Z}_+$.

1. *A trajectory* $w \in \mathcal{B}$ *is* reachable *if there exists a* $k \in \mathbf{Z}_+$, *a polynomial vector* $\sum_{i=0}^{k-1} f_i z^i \in \mathbb{F}[z]^m$ *such that for all* $T \in \mathbf{Z}_+$, *there exists a* $\overline{w} \in \mathcal{B}$ *for which*
$$\overline{w}_t = \begin{cases} 0 & 1 \le t \le T \\ f_{T+k-t} & T+1 \le t \le T+k \\ w_{t-T-k} & T+k+1 \le t. \end{cases} \tag{57}$$

2. *The behavior $\mathcal{B}$ is* reachable *if every trajectory $w \in \mathcal{B}$ is reachable.*
3. *The set of all reachable trajectories is a linear subspace and will be denoted by $\mathcal{B}_r$.*
4. *Let $\mathcal{B}$ have the AR representation $\mathcal{B} = X^V = \mathrm{Ker}\ V(\sigma)$, with $V(z)$ of full row rank, then we have*

$$\mathcal{B}_r = \left\{h \in \mathcal{B} \,\middle|\, \exists g \in \mathbb{F}^p\ ,\ Vg = Vh\right\}. \tag{58}$$

If $V = E_\rho V_\rho$ is an internal/external factorization, then we have $\mathcal{B}_r = \mathrm{Ker}\ V_\rho(\sigma) = X^{V_\rho}$.

5. *Let $\mathcal{B}$ have the AR representation $\mathcal{B}_r = X^V = \mathrm{Ker}\ V(\sigma)$, with $V(z)$ of full row rank, then $\mathcal{B}$ is a reachable behavior if and only if $V(z)$ is left prime, i.e. has a polynomial right inverse.*

2.9. Doubly Unimodular Embeddings

We proceed to prove a proposition that is the analog, in the behavioral setting of the doubly coprime factorizations that play such an important role in standard system theory. The importance is due to the fact, already apparent in the statement and its proof, that they provide the key to many duality results.

Given a pair of polynomial matrices K_2, L_1 such that $K_2L_1 = 0$, we say that there exists a *doubly unimodular embedding*, if there exist polynomial matrices K_1, L_2 such that

$$\begin{pmatrix} K_1(z) \\ K_2(z) \end{pmatrix}\begin{pmatrix} L_1(z) & L_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{59}$$

Obviously, in such a case, both matrices on the left are unimodular.

LEMMA 2.1. *Given a pair of polynomial matrices K_2, L_1. Then*

1. *There exists a doubly unimodular embedding, if and only if K_2 is left prime, L_1 right prime and*

$$\mathrm{Ker}\ K_2(z) = \mathrm{Im} L_1(z). \tag{60}$$

2. *There exists a doubly unimodular embedding for K_2 and L_1 if and only if there exists a doubly unimodular embedding for $\begin{pmatrix} K_2(z) & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} L_1(z) \\ 0 \end{pmatrix}$.*
3. *Given polynomial matrices satisfying*

$$N_2M_1 = M_2N_1, \tag{61}$$

with M_1, M_2 square and nonsingular. Then a doubly unimodular embedding for

$$\begin{pmatrix} -N_2 & M_2 \end{pmatrix}, \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}$$

exists if and only if M_1, N_1 are right coprime and M_2, N_2 are left coprime.

Doubly unimodular embeddings have a very rigid structure and properties of some entries are reflected in those of other entries. Thus, if

$$\begin{pmatrix} V_1(z) & V_2(z) \\ N_1(z) & N_2(z) \end{pmatrix}\begin{pmatrix} M_1(z) & U_1(z) \\ M_2(z) & U_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{62}$$

is a doubly unimodular embedding of $\begin{pmatrix} N_1(z) & N_2(z) \end{pmatrix}$ and $\begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix}$, then M_1 is a left prime polynomial matrix if and only if N_2 is, N_1 is a right prime polynomial matrix if and only if M_2 is, N_1 has full column rank if and only if M_2 has and N_1 is nonsingular if and only if M_2 is. For the details, see Bisiacco and Valcher [4] and Fuhrmann [10].

2.10. Behavior Homomorphisms and Isomorphisms

A central tool in behavior theory, introduced in Fuhrmann [10] is that of a behavior homomorphism. Given two behaviors, $\mathcal{B}_1$, $\mathcal{B}_2$, we define for the backward shift operator σ its restriction to the behaviors by $\sigma^{\mathcal{B}_i} = \sigma \mid \mathcal{B}_i$. If the behaviors are given in kernel representations $\mathcal{B}_i = \text{Ker } P_i(\sigma)$, we will write also σ^{P_i} for $\sigma^{\mathcal{B}_i}$. A *behavior homomorphism*, or *B-homomorphism*, is an $\mathbb{F}[z]$-homomorphism with respect to the natural $\mathbb{F}[z]$-module structure in the behaviors, i.e. it satisfies $Z\sigma^{P_1} = \sigma^{P_2} Z$. Our interest is in the characterization of behavior homomorphisms. It turns out that no general characterization of behavior homomorphisms is available. However, adding some continuity constraints makes the problem tractable by duality theory. We will say that a linear map $\overline{Z} : \text{Ker } M(\sigma) \longrightarrow \text{Ker } \overline{M}(\sigma)$ is continuous if it is continuous with respect to the w^* topologies in the two behaviors. In that case, $\overline{Z}$ is the adjoint of a map $Z : \mathbb{F}[z]^{\overline{m}} / \widetilde{\overline{M}} \mathbb{F}[z]^{\overline{p}} \longrightarrow \mathbb{F}[z]^m / \widetilde{M} \mathbb{F}[z]^p$. $\overline{Z}$ is a B-homomorphism if and and only if Z is an $\mathbb{F}[z]$-homomorphism between the quotient modules. We say that two behaviors, $\mathcal{B}_1$, $\mathcal{B}_2$, are *isomorphic* if there exists an invertible continuous B-homomorhism $Z : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$. Thus we can state.

THEOREM 2.4. *Let $M \in \mathbb{F}[z]^{p\times m}$ and $\overline{M} \in \mathbb{F}[z]^{\overline{p}\times\overline{m}}$ be of full row rank. Then* Ker *$M(\sigma)$ is an $\mathbb{F}[z]$-submodule of $z^{-1}\mathbb{F}[[z^{-1}]]^m$ and* Ker *$\overline{M}(\sigma)$ is an $\mathbb{F}[z]$-submodule of $z^{-1}\mathbb{F}[[z^{-1}]]^{\overline{m}}$. Moreover $\overline{Z}$:* Ker *$M(\sigma) \longrightarrow$* Ker *$\overline{M}(\sigma)$ is a continous behavior homomorphism, if and only if there exist $\overline{U} \in \mathbb{F}[z]^{\overline{p}\times p}$ and U in $\mathbb{F}[z]^{\overline{m}\times m}$ such that*

$$\overline{U}(z)M(z) = \overline{M}(z)U(z) \tag{63}$$

and

$$\overline{Z}h = U(\sigma)h \quad h \in \text{Ker } M(\sigma). \tag{64}$$

Next we discuss the invertibility properties of behavior homomorphisms.

THEOREM 2.5. *Given two full row rank polynomial matrices $M \in \mathbb{F}[z]^{p\times m}$, $\overline{M} \in \mathbb{F}[z]^{\overline{p}\times\overline{m}}$ describing the behaviors $\mathcal{B} =$* Ker *$M(\sigma)$ and $\overline{\mathcal{B}} =$* Ker *$\overline{M}(\sigma)$, respectively. Let $\overline{X}$, X be appropriately sized polynomial matrices satisfying*

$$\overline{X}(z)M(z) = \overline{M}(z)X(z), \tag{65}$$

and let Z : Ker *$M(\sigma) \longrightarrow$* Ker *$\overline{M}(\sigma)$ be defined by*

$$Zh = X(\sigma)h = \pi_- Xh \quad h \in \text{Ker } M(\sigma). \tag{66}$$

Then

1. *Z is injective if and only if M, X are right coprime.*
2. *Z is subjective if and only if $\overline{X}$, $\overline{M}$ are left coprime and*

$$\text{Ker}\begin{pmatrix} -\overline{X}(z) & \overline{M}(z) \end{pmatrix} = \text{Im}\begin{pmatrix} M(z) \\ X(z) \end{pmatrix}. \tag{67}$$

3. *Z as defined above is the zero map if and only if, for some appropriately sized polynomial matrix $L(s)$, we have*

$$X(z) = L(z)M(z). \tag{68}$$

4. *Z defined in (66) as invertible if and only if there exists a doubly unimodular embedding*

$$\begin{pmatrix} \overline{X} & \overline{M} \\ \overline{Y} & \overline{V} \end{pmatrix}\begin{pmatrix} -V & M \\ Y & -X \end{pmatrix} = \begin{pmatrix} -V & M \\ Y & -X \end{pmatrix}\begin{pmatrix} \overline{X} & \overline{M} \\ \overline{Y} & \overline{V} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{69}$$

of $\left(-\overline{X}(z)\ \overline{M}(z)\right)$ and $\begin{pmatrix} M(z) \\ X(z) \end{pmatrix}$

5. *If Z is invertible, then in terms of the doubly unimodular embedding (59), its inverse Z^{-1} : Ker $\overline{M}(\sigma) \longrightarrow$ Ker $M(\sigma)$ is given by*

$$Z^{-1} = -\overline{V}(\sigma). \tag{70}$$

The doubly coprime factorization (25) contains much more than the information on isomorphisms between the polynomial models X_T and $X_{\overline{T}}$. In fact, $X^V = \text{Ker } V(\sigma)$ and $X^{\overline{V}} = \text{Ker } \overline{V}(\sigma)$ are behaviors in $z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ and we have, as a special case of Theorem 2.5,

PROPOSITION 2.5. *Given the doubly coprime factorization (25), then*

1. *$Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ is a behavior isomorphism,*
2. *The inverse isomorphism is given by $\overline{T}(\sigma) : X^{\overline{V}} \longrightarrow X^V$.*

Proof.

1. Follows from the characterization of behavior homomorphisms in Fuhrmann [10] and Theorem 3.4 in Fuhrmann [12].
2. Equality (23) together with the embeddability in the doubly coprime factorization (25) shows that $\overline{T}(\sigma) : X^{\overline{V}} \longrightarrow X^V$ is a behavior isomorphism. We use the Bezout equation in Equation (26) to compute, for $h \in X^V$

$$\begin{aligned} \overline{T}(\sigma)\, Y(\sigma)\, h &= \pi_-\overline{T}\pi_-Yh = \pi_-\overline{T}Yh \\ &= \pi_-\left(I + \overline{X}V\right)h = \pi_-h = h. \end{aligned}$$

Similarly, for $h \in X^{\overline{V}}$,

$$Y(\sigma)\overline{T}(\sigma)h = \pi_-Y\overline{T}h = \pi_-\left(I + X\overline{V}\right)h = h.$$

□

The previous proposition shows that behaviors are present even in an input/output based formulation.

It is expected that B-homomorphisms preserve the essential information about behaviors. The next proposition shows that a B-isomorhism provides a bijection between reachable subbehaviors.

PROPOSITION 2.6. *Assume* X^V *and* $X^{\overline{V}}$ *are two isomorphic behaviors with* $V, \overline{V}$ *of full row rank. Let* $V = E_\rho V_\rho$ *and* $\overline{V} = \overline{E}_\rho \overline{V}_\rho$ *be the respective internal/external factorizations. Let* $Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ *be a continuous behavior isomorphism and let* $X^{V_\rho} \subset X^V$ *and* $X^{\overline{V}_\rho} \subset X^{\overline{V}}$ *be the respective reachable subbehaviors. Then*

$$Y(\sigma) X^{V_\rho} = X^{\overline{V}_\rho}. \tag{71}$$

Proof. Since $Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ is a continuous behavior isomorphism, it follows from Fuhrmann [10, 12] that there exists a polynomial matrices $Y, \overline{Y}$ for which

$$\overline{V} Y = \overline{Y} V, \tag{72}$$

and this relation can be embedded in a doubly unimodular factorization

$$\begin{pmatrix} -V & U \\ Y & -X \end{pmatrix} \begin{pmatrix} \overline{X} & \overline{U} \\ \overline{Y} & \overline{V} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} \overline{X} & \overline{U} \\ \overline{Y} & \overline{V} \end{pmatrix} \begin{pmatrix} -V & U \\ Y & -X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{73}$$

□

By the characterization of the reachable subbehavior, see Polderman and Willems [28] or Fuhrmann [10], it follows that there exists an internal/external factorization $V = E_\rho V_\rho$ with E_ρ nonsingular and V_ρ left prime. In the same way, there exists an internal/external factorization of $\overline{V}$ namely

$$\overline{V} = \overline{E}_\rho \overline{V}_\rho. \tag{74}$$

Also, as $Y(\sigma) X^{V_\rho}$ is a subbehavior of $X^{\overline{V}}$, there exists a factorization

$$\overline{V} = \widehat{E}_\rho \widehat{V}_\rho \tag{75}$$

for which

$$Y(\sigma) X^{V_\rho} = X^{\widehat{V}_\rho}. \tag{76}$$

By the same reasoning as before, there exists a polynomial matrix $\hat{Y}$ for which

$$\widehat{V}_\rho Y = \widehat{Y} V_\rho, \tag{77}$$

moreover, this is embedable in a doubly unimodular embedding. In particular, by the rigidity of doubly unimodular embeddings discussed in subsection 2.9, the left primeness of V_ρ implies that of $\widehat{V}_\rho$. From (74) and (75) we have the equality

$$\overline{E}_\rho \overline{V}_\rho = \widehat{E}_\rho \widehat{V}_\rho \tag{78}$$

Now, as linear spaces, the quotient space X^V / X^{V_ρ} is finite dimensional. Thus also $X^{\overline{V}} / X^{\widehat{V}_\rho}$ is finite dimensional. This implies the nonsingularity of $\widehat{E}_\rho$. Equality (78) implies now that $\widehat{E}_\rho, \overline{E}_\rho$ differ at most by a unimodular right factor. Without loss of generality, we can assume therefore that $\widehat{E}_\rho = \overline{E}_\rho$, and $\widehat{V}_\rho, \overline{V}_\rho$. Now (76) implies (71).

3. On Controlled and Conditioned Invariant Subspaces

Invariant subspaces are the cornerstone of the structure of finite dimensional linear systems. It is therefore not very surprising, using perfect hindsight, that relaxing the notion of invariance, when dealing with not one linear transformation but rather with triples or quadruples, turns out to be a basic part of the analysis of multivariable systems. Thus controlled and conditioned invariant subspaces were introduced early in the development of state space oriented linear system theory, see Basile and Marro [2, 3] and Wonham and Morse, see Wonham [34].

Relaxing the notion of invariance is done via the use of feedback and output injection. This we proceed to introduce. Given two reachable pairs, (A_1, B_1), (A_2, B_2) with input spaces $\mathcal{U}_1, \mathcal{U}_2$ and state spaces $\mathcal{X}_1, \mathcal{X}_2$, respectively, we say that they are *feedback equivalent* if there exists an invertible map $Z : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ for which

$$B_2 = ZB_1$$

$$\mathrm{Im}\,(A_2 Z - ZA_1) \subset \mathrm{Im}\, B_2. \tag{79}$$

Suppose next that (79) is satisfied with the map $Z : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ injective. Let $Z^\#$ be any left inverse of Z. Clearly $\mathrm{Im}\, Z \subset \mathcal{X}_2$ is a subspace. Moreover, equations (79) imply the existence of a map K' for which $A_2 Z = ZA_1 + BK'Z^\# Z$. Equivalently, $ZA_1 = (A_2 - BK)Z$, with $K = K'Z^\#$. This implies that $\mathcal{V} = \mathrm{Im}\, Z$ is invariant with respect to $A_2 - BK$ and we have the isomorphism $A_1 \simeq (A_2 - BK)\,|\mathcal{V}$.

Similarly, given two observable pairs, (C_1, A_1), (C_2, A_2) with output spaces $\mathcal{Y}_1, \mathcal{Y}_2$ and state spaces $\mathcal{X}_1, \mathcal{X}_2$, respectively, we say that they are *output injection equivalent* if there exists an invertible map $W : \mathcal{X}_2 \longrightarrow \mathcal{X}_1$ for which

$$C_2 = C_1 W$$

$$\mathrm{Ker}(WA_2 - A_1 W) \supset \mathrm{Ker}\, C_2. \tag{80}$$

Assuming that equations (80) are satisfied with W surjective having a right inverse $W^\#$, we obtain the existence of a map J for which $W(A_2 - JC_2) = A_1 W$. This implies $(A_2 - JC_2)\mathrm{Ker}\, W \subset \mathrm{Ker}\, W$ as well as the isomorphism $A_1 \simeq \overline{(A_2 - JC_2)}|\mathcal{X}_2/\mathrm{Ker} W$. These considerations lead us to the following.

DEFINITION 3.1.

1. *A subspace $\mathcal{V}$ is* controlled invariant *for a pair (A, B), if and only if there exists a map K for which $\mathcal{V}$ is $(A - BK)$-invariant. Such a map K will be called a* friend *of $\mathcal{V}$. The set of all friends of a controlled invariant subspace $\mathcal{V}$ will be denoted by $\mathcal{F}(\mathcal{V})$. A controlled invariant subspace $\mathcal{V}$ will be called an* reachability subspace *if for each monk polynomial q of degree equal to* $\dim \mathcal{V}$*, there exists a friend $K \in \mathcal{F}(\mathcal{V})$ such that q is the characteristic polynomial of $(A - BK)\,|\mathcal{V}$.*
2. *A subspace $\mathcal{V}$ is* conditioned invariant *for a pair (C, A), if and only if there exists a map J for which $\mathcal{V}$ is $(A - JC)$-invariant. Such a map J will be called a* friend *of $\mathcal{V}$. The set of all friends of a conditioned invariant subspace $\mathcal{V}$ will be denoted by $\mathcal{G}(\mathcal{V})$. A conditioned invariant subspace $\mathcal{V}$ will be called an* observability subspace *if for each monk polynomial q of degree equal to* $\mathrm{codim}\, \mathcal{V}$*, there exists a friend $J \in \mathcal{G}(\mathcal{V})$ such that q is the characteristic polynomial of $(A - JC)|_{\mathcal{X}/\mathcal{V}}$, the map induced on the quotient space $\mathcal{X}\big/\mathcal{V}$ by $(A - JC)$.*

Characterization of controlled and conditioned invariant subspaces have been available for a long time in the context of polynomial and rational models. The characterization of controlled invariant subspaces was derived in Fuhrmann and Willems [19]. That of conditioned invariant subspaces in Fuhrmann [8].

To explain the characterization, we observe that all information, about a reachable pair (A, B), up to similarity, can be encoded in a nonsingular polynomial matrix D, uniquely determined up to a right unimodular factor. This is done by taking a right coprime factorization $H(z)D(z)^{-1}$ of $(zI - A)^{-1}B$. Then the pair (A_D, B_D) defined by the shift realization (47) is similar to the original pair. Using the isomorphism of X_D and X^D, we have also the similarity to the pair (A^D, B^D) defined in X^D by (48).

In much the same way, all information, about an observable pair (C, A), up to similarity, can be encoded in a nonsingular polynomial matrix T, uniquely determined up to a left unimodular factor. This is done by taking a left coprime factorization $T(z)^{-1}L(z)$ of $C(zI - A)^{-1}$. Then the pair (C_T, A_T) defined by the shift realization (46) is similar to the original pair (C, A).

In view of this, we may assume without loss of generality that (A, B) is given by the pair (A_D, B_D) defined in (47). Similarly, for an observable pair (C, A), we may assume without loss of generality that it is given by (C_T, A_T), defined in (46), where the nonsingular polynomial matrix T is defined via the coprime factorization $T(z)^{-1}L(z) = C(zI - A)^{-1}$. It has been shown in Hautus and Heymann [20] as well as in Fuhrmann and Willems [19], that the pair $\left(A_{D_1}, B_{D_1}\right)$ is feedback equivalent to the pair (A_D, B_D) if and only if $DD_1{}^{-1}$ is normalized biproper. In that case, the map $Z\colon X_D \longrightarrow X_{D_1}$, satisfying (79) is given by

$$Z = \pi_{D_1}\mathcal{T}_{D_1 D^{-1}}. \tag{81}$$

Similarly, the pair $\left(C_{T_1}, A_{T_1}\right)$ is output injection equivalent to the pair (C_T, A_T) if and only if $T_1{}^{-1}T$ is normalized biproper. In that case, the polynomial models X_T, X_{T_1} are equal as sets and the map $W : X_{T_1} \longrightarrow X_T$ satisfying (80) is given by

$$W = I. \tag{82}$$

A comparison between (81) and (82) indicates that, although conceptually output injection is more difficult to grasp than state feedback, this is compensated by the fact that, in a functional setting, it is much easier to handle technically.

Of course, the representations (81) and (82) are connected by duality. Using the duality pairings introduced in (31) and (38), we compute for $f \in X_D$ and $g \in X_{\widetilde{D}_1}$

$$\begin{aligned}\left\langle \pi_{D_1}\mathcal{T}_{D_1 D^{-1}} f, g\right\rangle &= \left[D_1^{-1} D_1 \pi_- D_1^{-1} \pi_+ D_1 D^{-1} f, g\right]\\ &= \left[D^{-1} f, \widetilde{D}_1 \pi_- \widetilde{D}_1^{-1} g\right] = \langle f, g\rangle.\end{aligned}$$

Here we used the fact that, for $g \in X_{\widetilde{D}_1}$, we have $\overline{D}_1 \pi_- \widetilde{D}_1^{-1} g = g$. Thus we have $\left(\pi_{D_1}\mathcal{T}_{D_1 D^{-1}}\right)^* = I$.

From the previous discussion it follows that a subspace is controlled invariant for a pair (A, B) if and only if it is the image, under an invertible map, of an invariant subspace for a feedback equivalent pair. For polynomial models, invariant subspaces are related to factorizations. The subspace $\mathcal{V}_1 \subset X_{D_1}$, is invariant if and only if it has a representation of the form $\mathcal{V}_1 = E_1 X_{F_1}$ for some factorization $D_1 = E_1 F_1$ with the factors being nonsingular. Thus $\mathcal{V} \subset X_D$ is controlled invariant if and only if

$$\mathcal{V} = \pi_D T_{DD_1^{-1}}\left(E_1 X_{F_1}\right) = D\pi_- D^{-1}\pi_+ DD_1^{-1} E_1 X_{F_1} = D\pi^D X^{F_1}.$$

Using the isomorphism of X_D and X^D, we have that $\pi^D X^{F_1}$ is a representation of a controlled invariant subspace for (A^D, B^D).

Similarly, in view of the simplicity of the map W in (82), one expects that the derivation of a representation of conditioned invariant subspaces would be even simpler. Indeed, this is the case. A subspace $\mathcal{V} \subset X_D$ is conditioned invariant for the pair (C_D, A_D) if and only if it is invariant for an output injection equivalent pair which without loss of generality can be assumed to be an invariant subspace of $\left(C_{D_1}, A_{D_1}\right)$ with $D_1{}^{-1}D$ normalized biproper. Since $X_{D_1} = X_D$, such a subspace has the representation $E_1 X_{F_1}$, for some factorization $D_1 = E_1 F_1$ with the factors being nonsingular. It is simple to check that in this case $\mathcal{V} = X_D \cap E_1\mathbb{F}[z]^m$.

From the previous discussion follows the characterization of controlled and conditioned invariant subspaces.

PROPOSITION 3.1.

1. *Let $D \in \mathbb{F}[z]^{p\times p}$ be nonsingular. Then a subspace $\mathcal{V} \subset X_D$ is conditioned invariant if and only if it has a representation of the form*

$$\mathcal{V} = X_D \cap E_1\mathbb{F}[z]^p \tag{83}$$

for some nonsingular polynomial matrix E_1 for which all left Wiener–Hopf factorization indices of $D^{-1}E_1$ are nonpositive.

2. *Let $T \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Then a subspace $\mathcal{V} \subset X_T$ is controlled invariant if and only if it has a representation of the form $\mathcal{V} = T\pi^T X^{F_1}$ for some nonsingular polynomial matrix F_1 for which all right Wiener–Hopf factorization indices of $F_1 T^{-1}$ are nonpositive.*

One should note that, in both cases, the representation of the subspaces are given in terms of nonsingular polynomial matrices. However, these polynomial matrices are, in general, not uniquely determined by the corresponding subspaces. In order to obtain unique representations, we need to go through another stage where the representations are even less unique than in the previous proposition.

We note that X^{F_1} is an autonomous subbehavior. It is only natural to check whether $\pi^D\mathcal{B}$ would be a controlled invariant subspace for an arbitrary behavior. This is indeed the case, as is stated in Theorem 3.1. Similarly, we expect that the submodule $E_1\mathbb{F}[z]^m$ can be replaced by an arbitrary submodule $\mathcal{M} \subset \mathbb{F}[z]^m$, which is easy to verify. Combining the two characterizations we can state the following. The characterizations of controlled and conditioned invariant subspaces can be summed up in the following theorem. The first characterization is taken from Fuhrmann and Willems [19], with the minor change of restricting ourselves to behaviors rather than to arbitrary backward shift invariant subspaces. The difference between the two is the completeness property of behaviors. The second characterization is from Fuhrmann [8].

THEOREM 3.1.

1. *Let $T \in \mathbb{F}[z]^{p\times p}$ be nonsingular. Then, with respect to the shift realization in the state space X_T, a subspace $\mathcal{V} \subset X_T$ is controlled invariant if and only if there exists a submodule $\mathcal{M} \subset \mathbb{F}[z]^p$ for which*

$$\mathcal{V} = X_T \cap \mathcal{M}. \tag{84}$$

2. *Let $\overline{T} \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Then, with respect to the shift realization in the state space $X^{\overline{T}}$, given by (48), a subspace $\overline{\mathcal{V}} \subset X^{\overline{T}}$ is controlled invariant if and only if there exists a behavior $\mathcal{B} \subset z^{-1}\mathbb{F}\left[\!\left[z^{-1}\right]\!\right]^m$ for which*

$$\overline{\mathcal{V}} = \pi^{\overline{T}}\mathcal{B}. \tag{85}$$

The characterizations given above are module theoretic in nature and very concise, one might even venture to say elegant. Their main drawback is that such representations are, in general, highly nonunique. As a simple example, consider the case of a scalar, manic polynomial d. A submodule $\mathcal{M}$ of $\mathbb{F}[z]$ is an ideal and hence has a representation $\mathcal{M} = h\mathbb{F}[z]$ for an essentially unique polynomial h. In particular, for the zero subspace (0) we have the representation $\{0\} = X_d \cap h\mathbb{F}[z]$ whenever deg $h \geq$ deg d. Moreover, the first representation uses D as a right denominator while the second representation uses T as a left denominator. Our aim is to derive the two characterizations in the same setting. Now one encouraging aspect of Theorem 3.1 is the fact that, applying the duality introduced in subsection 2.4, there is a bijective correspondence between polynomial submodules and behaviors, see Oberst [26] and Fuhrmann [8, 10]. This shows that, as expected, there is a duality relation in the functional approach to geometric control, but the technicalities are nontrivial.

In order to overcome the nonuniqueness issue, inherent in the representations (84) and (85), we shall look for a representative submodule $\mathcal{M}_{\mathcal{V}}$ of $\mathbb{F}[z]^m$ that is uniquely determined by $\mathcal{V}$ and a uniquely determined behavior $\mathcal{B}_{\overline{\mathcal{V}}}$. Indeed, this can be done. For the case of conditioned invariant subspaces, we follow Hinrichsen, Münzner and Prätzel-Wolters [21], see also the discussion in Fuhrmann and Helmke [16], from which the first part of the following proposition is quoted. The second part follows by duality considerations.

PROPOSITION 3.2.

1. *Let $\mathcal{V} \subset X_T$ be a conditioned invariant subspaces with respect to the pair (C_T, A_T) defined by te shift realization.*
   - (a) *Let $\langle\mathcal{V}\rangle$ be the submodule of $\mathbb{F}[z]^p$ generated by $\mathcal{V}$, that is the smallest submodule of $\mathbb{F}[z]^p$ that contains $\mathcal{V}$. Then*
   $$\mathcal{V} = X_T \cap \langle\mathcal{V}\rangle. \tag{86}$$
   - (b) *If $\mathcal{E} \subset X_T$ is a subspace, then $X_T \cap \langle\mathcal{E}\rangle$ is the smallest conditioned invariant subspace of X_T that contains $\mathcal{E}$.*
   - (c) *A subspace $\mathcal{V} \subset X_T$ is a conditioned invariant subspace if and only if it has a representation of the form*
   $$\mathcal{V} = X_T \cap H(z)\mathbb{F}[z]^k, \tag{87}$$
   where $H(x)$ is a full column rank $p \times k$ polynomial matrix whose columns are in $\mathcal{V}$. $H(z)$ is uniquely determined up to a tight $k \times k$ unimodular factor.
2. *Let $\overline{\mathcal{V}} \subset X^{\overline{T}}$ be a controlled invariant subspaces with respect to the pair $\left(A^{\overline{T}}, B^{\overline{T}}\right)$ defined by the shift realization and given by (48). Then*
   - (a) *There exists a unique maximal behavior $\mathcal{B}_{\overline{\mathcal{V}}}$, given by*
   $$\mathcal{B}_{\overline{\mathcal{V}}} = \left\langle{}^{\perp}\overline{\mathcal{V}}\right\rangle^{\perp}. \tag{88}$$
   for which
   $$\overline{\mathcal{V}} = \pi^{\overline{T}}\mathcal{B}_{\overline{\mathcal{V}}}. \tag{89}$$
   - (b) *The subspace ${}^{\perp}\overline{\mathcal{V}} \subset X_{\tilde{\overline{T}}}$ is conditioned invariant with respect to the pair $\left(C_{\tilde{\overline{T}}}, A_{\tilde{\overline{T}}}\right)$ given by the shift realization.*

(c) *If* $\langle {}^{\perp}\overline{\mathcal{V}} \rangle$ *has the representation* $\langle {}^{\perp}\overline{\mathcal{V}} \rangle = \widetilde{R}\mathbb{F}[z]^k$, *then* $\mathcal{B}_{\overline{\mathcal{V}}}$ *has the autoregressive representation* $\mathcal{B}_{\overline{\mathcal{V}}} = \text{Ker } R(\sigma)$.

Proof.

1. (a) Let $\mathcal{V} = X_T \cap \mathcal{M}$. This implies $\mathcal{M} \supset \mathcal{V}$ and hence, $\mathcal{M}$ being a submodule, we have $M \supset \langle \mathcal{V} \rangle$. As a result, we have

$$\mathcal{V} = X_T \cap \mathcal{M} \supset X_T \cap \langle \mathcal{V} \rangle \supset \mathcal{V}.$$

Hence we have the equality (86).
(b) Obvious.
(c) Follows by choosing a basis for $\langle \mathcal{V} \rangle$ and using minimality. See Furmann and Helmke [16].

2. (a) First we show that $\pi^{\overline{T}}\mathcal{B}_{\overline{\mathcal{V}}} = \overline{\mathcal{V}}$. Since $\overline{\mathcal{V}} \subset \mathcal{B}_{\overline{\mathcal{V}}}$, we have

$$\overline{\mathcal{V}} = \pi^{\overline{T}}\overline{\mathcal{V}} \subset \pi^{\overline{T}}\mathcal{B}_{\overline{\mathcal{V}}} \subset \overline{\mathcal{V}},$$

and hence the equality.
Let now $\mathcal{B}$ be any behavior satisfying $\overline{\mathcal{V}} = \pi^{\overline{T}}\mathcal{B}$. This shows that ${}^{\perp}\overline{\mathcal{V}} \subset {}^{\perp}\mathcal{B}$, and hence $\mathcal{B}_{\overline{\mathcal{V}}} = \langle {}^{\perp}\overline{\mathcal{V}} \rangle^{\perp} \supset \left({}^{\perp}\mathcal{B} \right)^{\perp} = \mathcal{B}$. This shows the maximality of $\mathcal{B}_{\overline{\mathcal{V}}}$.
(b) Follows by duality as $X^{\overline{T}} = \left(X_{\widetilde{\overline{T}}} \right)^*$.
(c) Follows from the identity $\left(M\mathbb{F}[z]^k \right)^{\perp} = \text{Ker } \tilde{M}(\sigma)$. □

We note two things related to representation (87). First, the representation (87) of conditioned invariant subspaces is closely linked to Toeplitz operators. In fact, it can be easily verified, see Fuhrmann and Helmke [16], that

$$X_T \cap H(z)\mathbb{F}[z]^k = H\text{Ker } \mathcal{T}_{T^{-1}H}. \tag{90}$$

The second point to note is that the representative matrix H is rectangular. This indicates that we are faced with a polynomial matrix completion problem, in order to pass on to a representation of the form (83).

Closely related, and more general, characterization of conditioned invariant subspaces, based on polynomial system matrices, see Rosenbrock [29], or equivalently on representations of proper rational functions in the form $G = VT^{-1}U + W$, are given in Özgüler [27].

Proposition 3.2 is the key to the parametrization of all conditioned invariant subspaces of a given observable pair (C, A), that can be taken, without loss of generality, to be in dual Brunovsky form. The basic results are those of Hinrichsen et al. [21] with extensions given in Fuhrmann and Helmke [16]. As a result of the above, all information, up to similarity, on the conditioned invariant subspace, given in (87), is, in principle, derivable from the polynomial matrices $D(Z)$ and $H(z)$. In particular, because of our interest in observers, we will emphasize the characterization of observability subspaces.

It is well known, and easy to see, that the set of controlled invariant subspaces is closed under sums and the set of conditioned invariant subspaces is closed under intersections. Thus, given a subspace $\mathcal{E}$ of the state space of a linear system, there exists a unique maximal controlled invariant subspace contained in $\mathcal{E}$, which we denote by $\mathcal{V}^*(\mathcal{E})$ and a unique minimal conditioned invariant subspace containing

$\mathcal{E}$ which we denote by $\mathcal{V}_*(\mathcal{E})$. Similarly, we denote by $\mathcal{R}^*(\mathcal{E})$ the maximal reachability subspace contained in $\mathcal{E}$ and by $\mathcal{O}_*(\mathcal{E})$ the minimal observability subspace containing $\mathcal{E}$. Clearly, we have the inclusions $\mathcal{O}_*(\mathcal{E}) \supset \mathcal{V}_*(\mathcal{E}) \supset \mathcal{E} \supset \mathcal{V}^*(\mathcal{E}) \supset \mathcal{R}^*(\mathcal{E})$.

Let us assume now that the triple (C, A, B) is a realization of a strictly proper rational function. Clearly, Im B and Ker C are subspaces of the state space. In case the system is assumed, we denote by $\mathcal{V}^* = \mathcal{V}^*(\text{Ker } C)$ the maximal output nulling controlled invariant subspace and by $\mathcal{R}^* = \mathcal{R}^*(\text{Ker } C)$ the maximal reachability output nulling subspace. Similarly, $\mathcal{V}_* = \mathcal{V}_*(\text{Im } B)$ and $\mathcal{O}_* = \mathcal{O}_*(\text{Im } B)$ are the minimal input containing conditioned invariant subspace and the minimal input containing observability subspace, respectively. These subspaces are the most important objects in geometric control and there exist state space algorithms to compute them. Our interest is, given a matrix fraction representation $G = T^{-1}V$ of a (strictly) proper rational function, to give explicit formulas for these subspaces with respect to the shift realization in the state space X_T. The initial result in this direction was the characterization of $\mathcal{V}^*$ given in Emre and Hautus [5], see also Fuhrmann and Willems [19]. The following theorem generalizes these results as well those of Fuhrmann [8]. Equalities (92) are due to Morse [25]. For a more detailed, state space analysis, see Aling and Schumacher [1].

THEOREM 3.2. *Let a strictly proper rational function have the left matrix fraction representation $G = T^{-1}V$, with $T \in \mathbb{F}[z]^{p\times p}$ nonsingular, and let (C, A, B) be the associated shift realization, given by* (45), *in the state space X_T. Then we have the following characterizations, namely*

$$\begin{aligned} \mathcal{O}_* &= X_V + X_T \cap V\mathbb{F}[z]^k \\ \mathcal{V}^* &= X_V \\ \mathcal{V}_* &= X_T \cap V\mathbb{F}[z]^k \\ \mathcal{R}^* &= X_V \cap V\mathbb{F}[z]^k. \end{aligned} \tag{91}$$

Moreover, we have the Morse relations, *see Morse* [24],

$$\begin{aligned} \mathcal{R}^* &= \mathcal{V}^* \cap \mathcal{V}_* \\ \mathcal{O}_* &= \mathcal{V}^* + \mathcal{V}_* \end{aligned} \tag{92}$$

as well as the following isomorphism

$$\mathcal{O}_*/\mathcal{V}_* \simeq \mathcal{V}^*/\mathcal{R}^*. \tag{93}$$

The inclusions are summarized by the following diagram

$$\begin{tikzcd} & \mathcal{O}_* = \mathcal{V}^* + \mathcal{V}_* \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{V}^* \arrow[dr, dash] & & \mathcal{V}_* \arrow[dl, dash] \\ & \mathcal{R}^* = \mathcal{V}^* \cap \mathcal{V}_* & \end{tikzcd} \tag{94}$$

Diagram 3.1.

Proof. That $\mathcal{V}^* = X_V$ was proved in Emre and Hautus [5] and also in Fuhrmann and Willems [19].

That $V_* = X_T \cap V\mathbb{F}[z]^k$ can be proved, as we shall see in Section 4, from the Emre–Hautus result by rather intricate duality considerations. However, a shockingly short, direct proof is available. Since $\mathcal{V}_*$ is in particular a conditioned invariant subspace of X_T, it has, by Theorem 3.1, a representation of the form $\mathcal{V}_* = X_T \cap \mathcal{M}$ for some submodule $\mathcal{M} \cap \mathbb{F}[z]^p$. Since $\mathcal{V}_*$ is input containing, we must have $\{\pi_T V(z)\xi \mid \xi \in \mathbb{F}^m\} \subset \mathcal{M}$. However, by the assumed strict properness of $T^{-1}V$, we have $\pi_T V\xi = V\xi$, so $\{V(z)\xi \mid \xi \in \mathbb{F}^m\} \subset \mathcal{M}$. Since $\mathcal{M}$ is a submodule, we have $V\mathbb{F}[z]^m \subset \mathcal{M}$. By minimality, we must have the equality $V\mathbb{F}[z]^m \subset \mathcal{M}$.

The other two equalities follow from the Morse relations (92). □

Note that the proof of the characterization of $\mathcal{V}_*$ does not require any coprimeness conditions, the same holds for the characterization of $\mathcal{V}*$. Thus, as the Morse relations hold in general, the statement of the theorem is true in general. For proofs of the characterization of $\mathcal{R}^*$ and $\mathcal{O}_*$ that are not dependent on the Morse relations, see Fuhrmann [8] and Fuhrmann and Trumpf [17], respectively.

4. Duality in Geometric Control

Given a state space system $\Sigma = (A, B, C)$ in the state space X, then the dual system is defined to be $\Sigma = (A^*, C^*, B^*)$ in the state space X^*.

Given two systems $\Sigma = (A, B, C)$ and $\overline{\Sigma} = \left(\overline{A}, \overline{B}, \overline{C}\right)$ in the state spaces X and $\overline{X}$, respectively, an invertible linear map $\psi : X \longrightarrow \overline{X}$ is said to *intertwine* Σ and $\overline{\Sigma}$, which we denote by $\Sigma \overset{\psi}{\rightsquigarrow} \overline{\Sigma}$, if the relations

$$\begin{aligned} \psi A &= \overline{A}\psi \\ \psi B &= \overline{B} \\ C &= \psi\overline{C} \end{aligned} \tag{95}$$

are satisfied. We say two systems, Σ and $\overline{\Sigma}$, are *isomorphic* if there exists an isomorphism $\psi : X \longrightarrow \overline{X}$ such that $\Sigma \overset{\psi}{\rightsquigarrow} \overline{\Sigma}$. Here isomorphism was introduced for state space descriptions. Starting from behaviors and their various representations, a more general equivalence theory can be established, see Fuhrmann [10].

Intertwining isomorphisms preserve various system chracteristics. In the same way, passing from a system Σ to its dual Σ^*, the basic geometric control objects are dualized via annihilators. We summarize the basic properties in the following proposition, omitting the simple details.

PROPOSITION 4.1.

1. *Let $\psi : X \longrightarrow \overline{X}$ be an isomorphism such that $\Sigma \overset{\psi}{\rightsquigarrow} \overline{\Sigma}$. Then*

$$\begin{aligned} \psi\left(\mathcal{V}^*(\Sigma)\right) &= \mathcal{V}^*\left(\overline{\Sigma}\right) \\ \psi\left(\mathcal{V}_*(\Sigma)\right) &= \mathcal{V}_*\left(\overline{\Sigma}\right) \end{aligned} \tag{96}$$

In the same way, we have

$$\psi\left(\mathcal{R}^*(\Sigma)\right) = \mathcal{R}^*\left(\overline{\Sigma}\right)$$
$$\psi\left(\mathcal{O}_*(\Sigma)\right) = \mathcal{O}_*\left(\overline{\Sigma}\right), \tag{97}$$

2. *We have*

$$\left(\mathcal{V}^*(\Sigma)\right)^{\perp} = \mathcal{V}_*\left(\Sigma^*\right)$$
$$\left(\mathcal{V}_*(\Sigma)\right)^{\perp} = \mathcal{V}^*\left(\Sigma^*\right) \tag{98}$$

and

$$\left(\mathcal{R}^*\left(\overline{\Sigma}\right)\right)^{\perp} = \mathcal{O}_*\left(\overline{\Sigma}^*\right)$$
$$\left(\mathcal{O}_*(\Sigma)\right)^{\perp} = \mathcal{R}^*\left(\Sigma^*\right). \tag{99}$$

Our aim now is to go back to the characterizations given in Theorem 3.1 and 3.2 and to clarify the duality relations associated with them.

We note that in the characterization of controlled invariant subspaces given in Theorem 3.1, we considered $\overline{T}$ as a right denominator, i.e. corresponding to a matrix fraction of the form $\overline{V}\overline{T}^{-1}$. On the other hand, in the characterization of conditioned invariant subspaces given in the same theorem, we consider T to be a left denominator, i.e. corresponding to a matrix fraction of the form $T^{-1}V$.

On the other hand, all characterizations given in Theorem 3.2 were given in terms of one left matrix representation, $G = T^{-1}V$. The proof of Theorem 3.2 is complete if we allow ourselves the use of the Morse relations (92). However, if we want an independent, functional oriented, proof then we should get all characterizations in (91) without recourse to the Morse relations.

To circumvent this difficulty, we employ a two step process. With the left matrix fraction $T^{-1}V$, which momentarily we assume to be left coprime, we associate a right coprime factorization $\overline{V}\overline{T}^{-1}$. Using a previously introduced notation, with the two coprime factorizations, we have two associated shift realizations which we denote by

$$\Sigma = \Sigma\left(T^{-1}V\right), \quad \overline{\Sigma} = \Sigma\left(\overline{V}\overline{T}^{-1}\right) \tag{100}$$

The dual systems corresponding to the coprime factorizations (39) and are given by

$$\Sigma^* = \Sigma\left(\widetilde{V}\widetilde{T}^{-1}\right), \quad \overline{\Sigma}^* = \Sigma\left(\widetilde{\overline{T}}^{-1}\widetilde{\overline{V}}\right) \tag{101}$$

Note that formally we have

$$\overline{\widetilde{T}}^{-1}\overline{\widetilde{V}} = \widetilde{\overline{T}}^{-1}\widetilde{\overline{V}}. \tag{102}$$

The map $\psi : X_T \longrightarrow X_{\overline{T}}$, defined in (27), intertwines the realizations Σ and $\overline{\Sigma}$, and this implies that $\Sigma^* \overset{\phi^*}{\rightsquigarrow} \overline{\Sigma}^*$. Again, formally, we have

$$\overline{\Sigma^*} = \overline{\Sigma}^*. \tag{103}$$

Using (96) and (98), it is clear that given a characterization of one of the subspaces $\mathcal{V}^*(\Sigma)$, $\mathcal{V}^*\left(\overline{\Sigma}\right)$, $\mathcal{V}_*(\Sigma^*)$, $\mathcal{V}_*\left(\overline{\Sigma}^*\right)$, we can, in principle, derive all other characterizations using isomorphisms and annihilators.

Our starting point is the characterization $V^*(\Sigma) = X_V$, given in Theorem 3.2. This characterization is due to Emre and Hautus [5], see also Fuhrmann and Willems [19]. The standard duality in the context of polynomial models, developed in Fuhrmann [8], identifies the dual space of X_T with either the polynomial model $X_{\widetilde{T}}$ or with the rational model $X^{\widetilde{T}}$, depending on the duality pairing used. Here $\widetilde{T}$ denotes the transpose of T. However, this forces us to pass from a left matrix fraction $G = T^{-1}V$ to a right matrix fraction $\overline{V}\,\overline{T}^{-1}$. This is not of use, as our aim is to obtain all the characterizations in terms of a single representation of the system.

LEMMA 4.1. *Let $G = T^{-1}V = \overline{V}\,\overline{T}^{-1}$ be coprime factorizations of a $p \times m$, strictly proper rational function, and let (25) be an associated doubly coprime factorization. With ψ, defined by (27), the following is a commutative diagram.*

$$\begin{tikzcd}
X_T & & X^V \arrow[ll, "M_V"'] \arrow[d, "Y(\sigma)"] \\
X_{\overline{T}} \arrow[from=u, "\psi"'] & & X^{\overline{V}} \arrow[ll, "-\overline{T}\pi^{\overline{T}}"']
\end{tikzcd}$$

Diagram 4.1.

Proof. We compute, for $h \in X^V$ and using (26),

$$\begin{aligned}
\psi M_V h &= -\pi_{\overline{T}}\overline{X}Vh = \overline{T}\pi_-\overline{T}^{-1}\pi_+\big(I - \overline{T}Y\big)h \\
&= -\overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{T}(\pi_+ + \pi_-)Yh = \overline{T}\pi_-\overline{T}^{-1}\overline{T}\pi_+Yh - \overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{T}\pi_-Yh \\
&= -\overline{T}\pi^{\overline{T}}Y(\sigma)h.
\end{aligned}$$

□

We can state now the following theorem which gives the characterizations of these subspaces. The thing to note is that $\mathcal{V}^*(\Sigma)$ and $\mathcal{V}_*\left(\overline{\Sigma}^*\right)$ are given in terms of left coprime factorizations whereas those of $\mathcal{V}^*\left(\overline{\Sigma}\right)$ and $\mathcal{V}_*(\Sigma^*)$ in terms of right coprime factorizations. Statement 3 was proved in Theorem 3.2. It is included here in order to clarify the duality relations. Thus, in principle, one can use Theorem 3.2 in order to prove the Emre–Hautus characterization given by (104). Statement 4 of Theorem 4.1 does not provide new insights and is included mostly for the sake of completeness. Representation (107) can be proved by isomorphism from (106) or by duality from (104).

THEOREM 4.1. *Let $G = T^{-1}V = \overline{V}\,\overline{T}^{-1}$ be coprime factorizations of a $p \times m$, strictly proper rational function. We have*

1.

$$\mathcal{V}^*(\Sigma) = X_V. \tag{104}$$

2. *We have*

$$\mathcal{V}^*(\overline{\Sigma}) = \psi(\mathcal{V}^*(\Sigma)) = -\pi_{\overline{T}}\overline{X}VX^V = \overline{T}\pi^{\overline{T}}X^{\overline{V}}. \tag{105}$$

3.

$$\mathcal{V}_*\left(\overline{\Sigma}^*\right) = X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}\mathbb{F}[z]^p. \tag{106}$$

4.

$$\mathcal{V}_*(\Sigma^*) = (X_V)^\perp = \pi_{\widetilde{T}}\mathrm{Ker}\ \mathcal{T}_{\widetilde{V}\widetilde{T}^{-1}}. \tag{107}$$

If V has full row rank, then $\pi_{\widetilde{T}}\,|\mathrm{Ker}\mathcal{T}_{\widetilde{V}\widetilde{T}^{-1}}$ is injective.

Proof.

1. This was quoted in Theorem 3.2. The result is due to Emre and Hautus [5], see also Fuhrmann and Willems [19].
2. Note that from the doubly coprime factorization (25), we have in particular the relation $\overline{Y}V = \overline{V}Y$. Using Propositions 2–5, $Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ is a behavior isomorphism. In particular $Y(\sigma)X^V \longrightarrow X^{\overline{V}}$. The result follows now from Lemma 4.1.
3. Using (105), we will compute $\mathcal{V}_*\left(\overline{\Sigma}^*\right) = \left(\mathcal{V}^*(\overline{\Sigma})\right)^\perp = \left(-\pi_{\overline{T}}\overline{X}VX^V\right)^\perp$ and show that $\left(-\pi_{\overline{T}}\overline{X}VX^V\right)^\perp = X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}\mathbb{F}[z]^p$.
Assume $f \in \mathcal{V}^*(\overline{\Sigma}) = \overline{T}\pi^{\overline{T}}X^{\overline{V}}$, then $f = \overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{T}h$ with $h \in X^{\overline{V}}$. Assume $g \in X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}\mathbb{F}[z]^p$, i.e. $g = \widetilde{\overline{V}}\widehat{g}$. Then

$$\begin{aligned}\langle f, g\rangle &= \left[\overline{T}^{-1}\overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{T}h, g\right]\\ &= \left[\pi_-\overline{T}^{-1}\pi_+\overline{T}h, g\right] = \left[h, \widetilde{\overline{T}}\pi_-\widetilde{\overline{T}}^{-1}g\right]\\ &= \left[h, g\right] = \left[h, \widetilde{\overline{V}}\widehat{g}\right] = \left[\overline{V}h, g\right] = \left[\pi_-\overline{V}h, g\right] = 0.\end{aligned}$$

as $h \in \mathrm{Ker}\ \overline{V}(\sigma) = X^{\overline{V}}$. This shows the inclusion $X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}\mathbb{F}[z]^p \subset \left(\mathcal{V}^*(\overline{\Sigma})\right)^\perp$.
To prove the converse inclusion, assume $g \in \left(\mathcal{V}^*(\overline{\Sigma})\right)^\perp \subset X_{\widetilde{\overline{T}}}$. Let f be an arbitrary element of $\overline{T}\pi^{\overline{T}}X^{\overline{V}}$, i.e. $f = \overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{T}h$ with $h \in X^{\overline{V}}$. We compute

$$\begin{aligned}0 = \langle f, g\rangle &= \left[\overline{T}^{-1}\overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{T}h, g\right]\\ &= \left[\pi_-\overline{T}^{-1}\pi_+\overline{T}h, g\right] = \left[h, \widetilde{\overline{T}}\pi_-\widetilde{\overline{T}}^{-1}g\right]\\ &= \left[h, \pi_{\widetilde{\overline{T}}}g\right] = \left[h, g\right].\end{aligned}$$

Since this holds for an arbitrary element $h \in X^{\overline{V}}$, it follows, see Fuhrmann [10, 12], that $g = \widetilde{\overline{V}}\widehat{g}$ for some polynomial vector $\hat{g}$. Thus $g \in X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}\mathbb{F}[z]^p$ and hence $\left(\mathcal{V}^*(\overline{\Sigma})\right)^\perp \subset X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}\mathbb{F}[z]^p$. The equality (106) follows from the two inclusions. □

We expect that duality relations similar to those for the maximal output nulling and minimal input containing subspaces that were studied in Theorem 4.1, will have a counterpart for the maximal reachability output nulling and minimal observability input containing subspaces. This we proceed to study. As was the case before, statement 4 is included for the sake of completeness.

THEOREM 4.2. *Let $G = T^{-1}V = \overline{V}\overline{T}^{-1}$ be coprime factorizations of a $p \times m$, strictly proper rational function, with $V, \overline{V}$ of full row rank.*

$$\begin{aligned} V &= E_\rho V_\rho \\ \overline{V} &= \overline{E}_\rho \overline{V}_\rho \end{aligned} \tag{108}$$

be internal/external factorizations. Let the maps ϕ and ψ be defined by (24) and (27), respectively. Then we have

1.
$$\mathcal{R}^*(\Sigma) = E_\rho X_{V_\rho} = VX^{V_\rho} = X_V \cap V\mathbb{F}[z]^m, \tag{109}$$
i.e. $\mathcal{R}^*(\Sigma)$ is the image, under the multiplication by V map, of the reachable subbehavior X^{V_ρ} X^V.
2. *We have*
$$\mathcal{R}^*\left(\overline{\Sigma}\right) = -\pi_{\overline{T}}\overline{X}VX^{V_\rho} = \overline{T}\pi^{\overline{T}}X^{\overline{V}_\rho} \tag{110}$$
3.
$$\mathcal{O}_*\left(\overline{\Sigma}^*\right) = X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}_\rho\mathbb{F}[z]^p. \tag{111}$$
4.
$$\mathcal{O}_*\left(\Sigma^*\right) = \pi_{\widetilde{T}}\widetilde{X}\widetilde{\overline{V}}_\rho \mathrm{Ker}\, \mathcal{T}_{\widetilde{\overline{T}}^{-1}\widetilde{\overline{V}}_\rho}. \tag{112}$$

Proof.

1. This was proved, under somewhat restrictive conditions, in Fuhrmann [8].
2. The map $\psi : X_T \longrightarrow X_{\overline{T}}$ intertwines Σ and $\overline{\Sigma}$, hence $\psi(\mathcal{R}^*(\Sigma)) = \mathcal{R}^*\left(\overline{\Sigma}\right)$. We compute, using (97) and the isomorphism ψ defined by (27),
$$\begin{aligned} \mathcal{R}^*\left(\overline{\Sigma}\right) &= \psi(\mathcal{R}^*(\Sigma)) = -\pi_{\overline{T}}\overline{X}E_\rho X_{V_\rho} = -\pi_{\overline{T}}\overline{X}V\, X^{V_\rho} \\ &= \pi_{\overline{T}}\pi_+\left(-\overline{X}V\right)X^{V_\rho} = \pi_{\overline{T}}\pi_+\left(I - \overline{T}Y\right)X^{V_\rho} \\ &= \pi_{\overline{T}}\pi_+\overline{T}YX^{V_\rho} = \pi_{\overline{T}}\pi_+\overline{T}(\pi_+ + \pi_-)YX^{V_\rho} \\ &= \pi_{\overline{T}}\pi_+\overline{T}Y(\sigma)X^{V_\rho} = \overline{T}\left(\pi_-\overline{T}^{-1}\pi_+\overline{T}\right)Y(\sigma)X^{V_\rho} \\ &= T\pi^{\overline{T}}Y(\sigma)X^{V_\rho} = \overline{T}\pi^{\overline{T}}X^{\overline{V}_\rho}, \end{aligned}$$
as we saw, in Proposition 2.6, that $Y(\sigma)X^{V_\rho} = X^{\overline{V}_\rho}$.
3. Since $\mathcal{R}^*\left(\overline{\Sigma}\right)^\perp = \mathcal{O}_*\left(\overline{\Sigma}^*\right)$, it suffices to show that $\left(\pi_{\overline{T}}\overline{X}VX^{V_\rho}\right)^\perp = X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}_\rho\mathbb{F}[z]^p$. The proof follows the same line as the proof of Part 3 of Theorem 4.1.
4. Since $\psi^* : X_{\widetilde{\overline{T}}} \longrightarrow X_{\widetilde{T}}$ is an isomorphism, we have $\mathcal{O}_*(\Sigma^*) = \psi^*\left(\mathcal{O}_*\left(\overline{\Sigma}^*\right)\right)$, and we use Equation (151) and the fact that, see Fuhrmann and Helmke [16], $X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{V}}_\rho\mathbb{F}[z]^p = \widetilde{\overline{V}}_\rho \mathrm{Ker}\, \mathcal{T}_{\widetilde{\overline{T}}^{-1}\widetilde{\overline{V}}_\rho}$. □

Theorems 4.1 and 4.2 dealt with the maximal output nulling, controlled invariant (reachability) subspaces and the minimal input containing, conditioned invariant (observability) subspaces. The next proposition deals with arbitrary controlled invariant (reachability) and conditioned invariant (observability) subspaces.

PROPOSITION 4.2.

1. (a) *Given an observable pair* (C, A) *in the state space* $\mathcal{X}$ *and a conditioned invariant subspace* $\mathcal{V} \subset \mathcal{X}$*. Then there exists a full column rank matrix* B *for which* $\mathcal{V} = \mathcal{V}_*(\Sigma)$*, with* $\Sigma = (A, B, C)$.
   (b) *Given an observable pair* (C, A) *in the state space* $\mathcal{X}$ *and an observability subspace* $\mathcal{O} \subset \mathcal{X}$*. Then there exists a full column rank matrix* B *for which* $\mathcal{O} = \mathcal{O}_*(\Sigma)$*, with* $\Sigma = (A, B, C)$.
2. (a) *Given a reachable pair* (A, B) *in the state space* $\mathcal{X}$ *and a controlled invariant subspace* $\mathcal{V} \subset \mathcal{X}$*. Then there exists a full row ranlc matrix* C *for which* $\mathcal{V} = \mathcal{V}^*(\Sigma)$*, with*$\Sigma = (A, B, C)$.
   (b) *Given a reachable pair* (A, B) *in the state space* $\mathcal{X}$ *and a reachability subspace* $\mathcal{R} \subset \mathcal{X}$*. Then there exists a full row rank matrix* C *for which* $\mathcal{R} = \mathcal{R}^*(\Sigma)$*, with* $\Sigma = (A, B, C)$.

Proof.

1. (a) With (C, A) assumed to be observable, let $T(z)^{-1}L(z)$ be an arbitrary left coprime factorization of $C(zI - A)^{-1}$. It is well known that the columns of L provide a basis for the polynomial model X_T. Moreover, without loss of generality, we can assume that X_T is the state space and (C_T, A_T) are given by (46). Let now $\mathcal{V} \subset X_T$ be conditioned invariant. Bt Theorem 3.1 and Proposition 3.2, we have $\mathcal{V} = X_T \cap H(z)\mathbb{F}[z]^k$ with H of full column rank and $T^{-1}H$ strictly proper. As the columns of H are in X_T, there exists a constant matrix B_0 for which $H(z) = L(z)B_0$. Since H has full column rank, so has B_0. Defining the input map B, for $\xi \in \mathbb{F}^k$, by $B\xi = H\xi$, we have $\mathcal{V} \supset \operatorname{Im} B$. In fact, any conditioned invariant subspace that contains $\operatorname{Im} B$ contains $\mathcal{V} = X_T \cap H(z)\mathbb{F}[z]^k$. So $\mathcal{V}$ is the smallest such subspace, i.e. $\mathcal{V} = \mathcal{V}_*(\Sigma)$ with $\Sigma = (A, B, C)$.
   (b) We use the previous part. Since $\mathcal{O}$ is an observability subspace, it is in particular conditioned invariant. In the representation $\mathcal{O} = X_T \cap H(z)\mathbb{F}[z]^k$, H is, by Theorem 3.4 in Fuhrmann and Trumpf [17], right prime. By the previous part, it is the smallest conditioned invariant subspace containing $\operatorname{Im} B$, hence it is also the smallest observability subspace that contains $\operatorname{Im} B$, i.e. $\mathcal{O}_*(\Sigma)$.
2. Follows from Part 1 by duality. □

5. Spectral Assignment

We proceed to describe the basic spectral assignability problems of geometric control. We use the term spectral assignability rather than pole placement as we would like to explore also the feasibility of changing the fine structure, i.e. the corresponding invariant factors. This is in the spirit of Rosenbrock's theorem.

Originally, the questions of pole assignment by state feedback, motivated by stabilization, and its dual, the problem of pole assignment by output injection, motivated by the construction of state observers, were a great driving force in the development of linear system theory. The problem was eventually completely solved in Rosenbrock [29]. Much more delicate are the problems of pole placement by feedback in controlled invariant subspaces and pole placement by output injection in quotient spaces modulo coditioned invariant subspaces. In turn this led to the analysis of reachability and observability subspaces.

We note that if (A, B) is a reachable pair and $\mathcal{V}$ a controlled invariant subspace, then for any $K \in \mathcal{F}(\mathcal{V})$, the induced pair in the quotient space, namely $\big((A - BK)|_{\mathcal{X}/\mathcal{V}}, P_{\mathcal{X}/\mathcal{V}}B\big)$, is reachable and hence, applying Rosenbrock's theorem, $(A - BK)|_{\mathcal{X}/\mathcal{V}}$ is fully spectral assignable with the only constraints given by the reachability indices of the induced pair. This leaves open the question on what is the extent of control we have over the fine structure of $(A - BK)|\,\mathcal{V}$. We will refer to this as *the internal spectral assignability problem*.

Analogously, if (C, A) is an observable pair and $\mathcal{V}$ a conditioned invariant subspace, then for every $J \in \mathcal{G}(\mathcal{V})$, the reduced pair $(C|\,\mathcal{V}, (A - JC)|\,\mathcal{V})$ is observable and hence, again applying Rosenbrock's theorem, $(A - JC)|\,\mathcal{V}$ is fully spectral assignable, the only constraints given by the observability indices of the reduced pair. This leaves open the question on what is the extent of control we have over the fine structure of $(A - JC)|_{\mathcal{X}/\mathcal{V}}$. In line with previous terminology, we will refer to this as *the external spectral assignability problem*. Actually, the question of outer spectral assignability is not only more delicate but also more important as it is the cornerstone of observer design for partial states.

Recently, both problems have been given a solution in a functional setting. The first one in Fuhrmann [14], whereas the second in Fuhrmann and Trumpf [17]. Both solutions are using polynomial matrix extensions. Since, in a very natural way, from the state space point of view, the two problems are dual, it might be expected that a functional based duality theory should provide a relation between the two solutions. This turns out to be the case but the analysis of duality turns out to be rather intricate.

Exactly as the notions of controlled and conditioned invariant subspaces are dual concepts, the same holds for output nulling and input containing subspaces. Moreover, given a linear system Σ defined by the triple (A, B, C), with the dual system Σ^* given by (A^*, C^*, B^*), then it is easy to check that $\mathcal{V}_*(\Sigma^*) = (\mathcal{V}^*(\Sigma))^{\perp}$. One expects therefore that we can apply duality theory to connect polynomial characterizations of $\mathcal{V}^*$ and $\mathcal{V}_*$. Indeed, this can be done, but not in a straightforward way. The reason for this is that starting from a left matrix fraction $G = T^{-1}V$ of a proper rational function, the polynomial characterization of the maximal output nulling subspace is given by $\mathcal{V}^* = X_V \subset X_T$, where the polynomial model space X_T is the natural state space for the shift realization, which we denote by Σ.

For conditioned invariant subspaces of a polynomial model realization, associated with $D^{-1}N$, we have two representations. One, given in Proposition 3.2, is $\mathcal{V} = X_D \cap \langle \mathcal{V} \rangle = X_D \cap H(z)\mathbb{F}[z]^k$, where $H(z)$ is a basis matrix for the free submodule $\langle \mathcal{V} \rangle \subset \mathbb{F}[z]^p$ whose columns are in $\mathcal{V}$. The other, see Proposition 3.1, is a representation of the form $\mathcal{V} = X_D \cap E\mathbb{F}[z]^p$ for some nonsingular polynomial matrix for which all right Wiener–Hopf factorization indices are nonnegative. This means that there exists a $D_1 = EF$, with the factors E, F nonsingualar, for which ${D_1}^{-1}D$ is biproper. In this case $\mathcal{V} = EX_F$. Thus, comparing the two characterizations, it seems reasonable to expect that the nonsingular polynomial matrix E can be obtained from H by an extension process, i.e. by letting $E = \begin{pmatrix} H & \widehat{H} \end{pmatrix}$ or an appropriate polynomial matrix $\widehat{H}$. Naturally, we don't expect such an extension to be unique. A full analysis of this issue and its relation to kernel representations of conditioned invariant subspaces can be found in Fuhrmann and Helmke [16]. The analysis of this extension procedure is central to the understanding of the error dynamics of observers, the analysis of the amount of freedom we have in the choice of observer dynamics and in particular to the construction procedures for such observers. To begin with, we assume the polynomial matrix H to be right prime. Thus there exist unimodular extensions $\begin{pmatrix} H & H' \end{pmatrix}$, with inverse $\begin{pmatrix} K \\ K' \end{pmatrix}$, and we fix one. Then it can be shown that, see Fuhrmann and Trumpf [17], that up to a right unimodular factor, we have $E = \begin{pmatrix} H & \widehat{H} \end{pmatrix} = \begin{pmatrix} H & H's \end{pmatrix}$ for some nonsingular polynomial matrix S. Now, for such an extension, we want S to be 'large' enough so that we have the equality

$$\mathcal{V} = X_D \cap H\mathbb{F}[z]^k = X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p, \tag{113}$$

but at the same time 'small' enough so that there exists a module structure on X_D for which we have the isomorphism $X_D \big/ X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p \simeq X_S$. It turns out that the right measure of small and large is in terms of Wiener–Hopf factorization indices. Indeed, in Theorem 3.5 of Fuhrmann and Trumpf [17], it is shown that equality (113) holds if and only if all right Wiener–Hopf factorization indices of $S^{-1}K'D$ are nonpositive while the isomorphism $X_D \big/ \big(X_D \cap \begin{pmatrix} H & H'S \end{pmatrix}\mathbb{F}[z]^p\big) \simeq H'X_S$ holds if and only if all right Wiener–Hopf factorization indices of $S^{-1}K'D$ are nonnegative.

A similar situation occurs for controlled invariant subspaces. Assuming $T^{-1}V$ is a left coprime factorization, then a controlled invariant subspace $\mathcal{V} \subset X_T$ has a representation $\mathcal{V} = X_V$, for some $p \times m$ polynomial matrix V of full row rank. We assume now that V is left prime. Clearly, we have $X_V = VX^V$. Of course, if V is not square, then X^V is infinite dimensional and the multiplication map $M_V : X^V \longrightarrow X_V$ cannot be injective. Now X^V is a behavior, and as V is assumed to be left prime, a reachable behavior at that, whereas X_V is finite dimensional. So, we would like to study autonomous subbehaviors of X^V and see how M_V reduces to these subbehaviors. As V is assumed to be left prime, any autonomous subbehavior of X^V has a representation of the form $X^{\begin{pmatrix} V \\ QV' \end{pmatrix}}$, where $\begin{pmatrix} V \\ V' \end{pmatrix}$, is a fixed unimodular extension and Q nonsingular. We would like to choose Q 'small' enough so that the map $M_V : X^{\begin{pmatrix} V \\ QV' \end{pmatrix}} \longrightarrow X_V$ is injective and, at the same time, 'large' enough so that the same map is surjective. Not surprisingly, see Fuhrmann [14] for the details, both conditions can be expressed in terms of Wiener–Hopf factorization indices. Injectivity is equivalent to all left Wiener–Hopf factorization indices of $W'Q^{-1}$ being nonnegative, while surjectivity is equivalent to all right Wiener–Hopf factorization indices of $W'Q^{-1}$ being nonpositive.

In the next lemma we give some auxiliary results needed for the study of the internal spectral assignability problem in the special case that in the left coprime factorization $T^{-1}V$, with T and V being $p \times p$ and $p \times m$ polynomial matrices, respectively, and V is assumed to be left prime. In fact, for a full row rank matrix $V(z)$, we have $V\mathbb{F}[z]^m = \mathbb{F}[z]^p$ if and only if $V(z)$ is left prime and applying Theorem 3.2, this is equivalent to the equality $\mathcal{V}^* = \mathcal{R}^*$, i.e. this is the case where there are no transmission zeros.

LEMMA 5.1. *Given the coprime factorizations $T^{-1}V = \overline{V}\,\overline{T}^{-1}$ of a $p \times m$, strictly proper rational function, Assume V is left prime. Let $\begin{pmatrix} V \\ V' \end{pmatrix}$ be a fixed unimodular extension.*

Let a $\begin{pmatrix} W & W' \end{pmatrix} = \begin{pmatrix} V \\ V' \end{pmatrix}^{-1}$

Let $Q \in \mathbb{F}[z]^{(m-p)\times(m-p)}$ be nonsingular and let $E = \begin{pmatrix} V \\ QV' \end{pmatrix}$. Using the doubly coprime factorizations (25), let $Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ be the B-isomorphism defined in Proposition 2.5. Then

1. *$Y(\sigma)X^E$ is an autonomous subbehavior of $X^{\overline{V}}$ and there exists a polynomial matrix $\bar{E}$, unique up to a left unimodular factor, for which*
$$X^{\overline{E}} = Y(\sigma)X^E. \tag{114}$$
$\bar{E}$ has a representation of the form
$$\overline{E} = \begin{pmatrix} \overline{V} \\ \overline{QV}' \end{pmatrix} \tag{115}$$
and furthermore, there exists a polynomial matrix $\overline{Y}_a$ uniquely defined up to a left unimodular factor, for which
$$\overline{Y}_a E = \overline{E}Y. \tag{116}$$
Without loss of generality, we can assume that
$$\overline{Y}_a = \begin{pmatrix} \overline{Y} & 0 \\ \overline{R} & I \end{pmatrix} \tag{117}$$
2. *The intertwining relation (116) is embeddable in the factorization*
$$\begin{aligned} \left(\begin{array}{cc|c} \overline{X} & 0 & \overline{T} \\ \hline \overline{Y} & 0 & \overline{V} \\ \overline{R} & I & \overline{QV'} \end{array}\right) \left(\begin{array}{c|cc} -V & T & 0 \\ -QV' & S & I \\ \hline Y & -X & 0 \end{array}\right) &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ \left(\begin{array}{c|cc} -V & T & 0 \\ -QV' & S & I \\ \hline Y & -X & 0 \end{array}\right) \left(\begin{array}{cc|c} \overline{X} & 0 & \overline{T} \\ \hline \overline{Y} & 0 & \overline{V} \\ \overline{R} & I & \overline{QV}' \end{array}\right) &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \end{aligned} \tag{118}$$
with $S = \overline{QV}'X - \overline{R}T$.
3. *The nonsingular polynomial matrices E and $\bar{E}$ have the same invariant factors. In particular, we have*
$$\deg\det Q = \deg\det E = \deg\det \overline{E} = \deg\det \overline{Q}. \tag{119}$$
4. *With ψ defined by (27), the following is a commutative diagram.*

$$\begin{tikzcd} X_T \arrow[d, "\psi"'] & X^E \arrow[l, "M_V"'] \arrow[d, "Y(\sigma)"] \\ X_{\overline{T}} & X^{\overline{E}} \arrow[l, "\overline{T}\pi^{\overline{T}}"] \end{tikzcd}$$

Diagram 5.1.
and the maps $M_V : X^E \longrightarrow X_T$ *and* $\overline{T}\pi^{\overline{T}} : X^{\overline{E}} \longrightarrow X_{\overline{T}}$ *are equivalent.*

5. *Let* $X_{\tilde{\overline{T}}} = X^*_{\overline{T}}$ *with respect to the pairing*

$$\langle f, g\rangle = \left[\overline{T}^{-1} f, g\right], \tag{120}$$

for $f \in X_{\overline{T}}, g \in X_{\tilde{\overline{T}}}$ *and the bilinear form* $[\cdot,\cdot]$ *defined by* (31). *Identify* $X_{\tilde{\overline{E}}} = X^{\overline{E}}$ *with respect to the pairing* $[f, h]$, $f \in X_{\tilde{\overline{E}}}$ *and* $h \in X^{\overline{E}}$. *Then the adjoint of the map* $\overline{T}\pi^{\overline{T}} : X^{\overline{E}} \longrightarrow X_{\overline{T}}$ *is given by the projection map* $\pi_{\tilde{\overline{E}}} : X_{\tilde{\overline{T}}} \longrightarrow X_{\tilde{\overline{E}}}$

6. *Using the identifications* $X^*_T = X_{\tilde{T}}$ *and* $\left(X^E\right)^* = X_{\tilde{E}}$, *then the adjoint of the map* $M_V : X^E \longrightarrow X_T$ *is the map* $M^*_V : X_{\tilde{T}} \longrightarrow X_{\tilde{E}}$ *given by* $M^*_V g = \pi_{\tilde{E}} T_{\tilde{V}\tilde{T}^{-1}} g$.
7. $M_V : X^E \longrightarrow X_T$ *is injective if and only if* $\pi_{\tilde{\overline{E}}} : X_{\tilde{\overline{T}}} \longrightarrow X_{\tilde{\overline{E}}}$ *is surjective.*
8. $M_V : X^E \longrightarrow X_V$ *is surjective if and only if* $\pi_{\tilde{\overline{E}}} : X_{\tilde{\overline{T}}} \longrightarrow X_{\tilde{\overline{E}}}$ *is injective.*
9. $M_V : X^E \longrightarrow X_V$ *is injective/surjective/bijective if and only if* Q *a nonsingular polynomial matrix for which all left Wiener–Hopf factorization indices of* $W'Q^{-1}$ *are nonpositive/nonnegative/zero.*

Proof.

1. Clearly, a behavior X^E is autonomous if and only if there exists a nonzero polynomial $e \in \mathbb{F}[z]$ for which $e(\sigma)X^E = 0$. Thus, with $X^{\overline{E}} = Y(\sigma)X^E$, we have $e(\sigma)X^{\overline{E}} = e(\sigma)Y(\sigma)X^E = Y(\sigma)e(\sigma)X^E = 0$, i.e. $X^{\overline{E}}$ is autonomous and, without loss of generality, we can assume $\bar{E}$ to be nonsingular. Equality (116) follows from the characterization given Without loss of generality, we can write $\overline{E} = \begin{pmatrix} \overline{V} \\ \overline{\overline{V}} \end{pmatrix}$. Writing $\overline{Y}_a$, as a block matrix, we conclude that $\overline{Y}_a E = \overline{E} Y$ can be rewritten as

$$\begin{pmatrix} \overline{Y}_{11} & \overline{Y}_{12} \\ \overline{Y}_{21} & \overline{Y}_{22} \end{pmatrix} \begin{pmatrix} V \\ QV' \end{pmatrix} = \begin{pmatrix} \overline{V} \\ \overline{\overline{V}} \end{pmatrix} Y. \tag{121}$$

From this, using the equality $\overline{V}Y = \overline{Y}V$, we get

$$\overline{Y}_{11}V + \overline{Y}_{12}QV' = \overline{V}Y = \overline{Y}V,$$

which leads to $(\overline{Y} - \overline{Y}_{11})V - \overline{Y}_{12}QV' = 0$. Since $\begin{pmatrix} V \\ QV' \end{pmatrix}$ is nonsingular, we conclude that $\overline{Y}_{11} = \overline{Y}$ and $\overline{Y}_{12} = 0$. The equality (121) together with the embeddability property implies the isomorphism of the two behaviors $X^{\begin{pmatrix} \overline{Y} & 0 \\ \overline{Y}_{21} & \overline{Y}_{22} \end{pmatrix}}$ and X^Y. However, from the doubly coprime factorization (25), it follows that $\overline{Y}$ and Y are equivalent, and this forces $\overline{Y}_{22}$ to be unimodular. By redefining $\overline{Y}_{22}$ and $\overline{\overline{V}}$, we may assume without loss of generality that $\overline{Y}_{22} = I$. Multiplying on the left by an appropriate unimodular polynomial matrix of the form $\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$ and redefining

$\overline{Y}_{21}$, we can assume without loss of generality that $\begin{pmatrix} \overline{V} \\ \overline{\overline{V}} \end{pmatrix} = \begin{pmatrix} \overline{V} \\ \overline{QV'} \end{pmatrix}$ for some unimodular extension $\begin{pmatrix} \overline{V} \\ \overline{V}' \end{pmatrix}$ which proves the representation (115).

Setting $R = \overline{Y}_{21}$, equality (121) can be rewritten as

$$\begin{pmatrix} \overline{Y} & 0 \\ \overline{R} & I \end{pmatrix} \begin{pmatrix} V \\ QV' \end{pmatrix} = \begin{pmatrix} \overline{V} \\ \overline{QV}' \end{pmatrix} Y. \tag{122}$$

2. From the doubly coprime factorization (25) we immediately obtain the extended factorization

$$\begin{pmatrix} \overline{X} & 0 & \overline{T} \\ \overline{Y} & 0 & \overline{V} \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} -V & T & 0 \\ 0 & 0 & I \\ Y & -X & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{123}$$

From this, using elementary operations, we obtain

$$\left(\begin{array}{cc|c} \overline{X} & 0 & \overline{T} \\ \hline \overline{Y} & 0 & \overline{V} \\ \overline{R} & I & \overline{QV}' \end{array}\right) \left(\begin{array}{c|cc} -V & T & 0 \\ \overline{R}V - \overline{QV}'Y & S & I \\ \hline Y & -X & 0 \end{array}\right) = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \tag{124}$$

with $S = \overline{QV}'X - \overline{R}T$. From (122), we have the equality $\overline{R}V + QV' = \overline{QV}'Y$. This clearly shows that (116) has been embedded in the doubly coprime factorization (118).

3. From the doubly coprime factorization embedding (118), we have the equivalence of $\begin{pmatrix} V \\ QV' \end{pmatrix}$ and $\begin{pmatrix} \overline{V} \\ \overline{QV}' \end{pmatrix}$. This implies the following equalities

$$\begin{aligned} \deg\det Q = \deg\det E = \deg\det \begin{pmatrix} V \\ QV' \end{pmatrix} \\ = \deg\det \overline{E} = \deg\det \begin{pmatrix} \overline{V} \\ \overline{QV}' \end{pmatrix} = \deg\det \overline{Q}. \end{aligned}$$

4. The proof is analogous to that of Lemma 4.1. We note that $Y(\sigma) : X^E \longrightarrow X^{\overline{E}}$ is invertible in view of (116), the doubly unimodular embedding (118) and Theorem 2.5.

5. Let $f \in X_{\widetilde{\overline{T}}}$ and $h \in X^{\overline{E}}$. We compute

$$\begin{aligned} \left\langle \overline{T}\pi^{\overline{T}}h, f \right\rangle &= \left[\overline{T}^{-1}\overline{T}\pi^{\overline{T}}h, f \right] = \left[\pi^{\overline{T}}h, f \right] = \left[h, \pi_{\widetilde{\overline{T}}}f \right] \\ &= \left[h, f \right] = \left[\pi^{\overline{E}}h, f \right] = \left[h, \pi_{\widetilde{\overline{E}}}f \right]. \end{aligned} \tag{125}$$

6. Let $h \in X^E$ and $g \in X_{\widetilde{T}}$. Then

$$\begin{aligned} \langle M_V h, g \rangle &= \left[T^{-1}Vh, g \right] = \left[h, \widetilde{V}\widetilde{T}^{-1}g \right] \\ &= \left[h, \pi_+\widetilde{V}\widetilde{T}^{-1}g \right] = \left[\pi^E h, \mathcal{T}_{\widetilde{V}\widetilde{T}^{-1}}g \right] \\ &= \left[h, \pi_{\widetilde{E}}\mathcal{T}_{\widetilde{V}\widetilde{T}^{-1}}g \right]. \end{aligned}$$

7. Since both ψ and $Y(\sigma)$ are invertible maps, it follows from the commutativity of Diagram 5.1 that the injectivity of M_V: $X^E \longrightarrow X_T$ implies the injectivity of $\overline{T}\pi^{\overline{T}} : X^{\overline{E}} \longrightarrow X_{\overline{T}}$. Applying Part 5 the result folows.

8. As above, using the invertibility of ψ and $Y(\sigma)$, it follows from the commutativity of Diagram 5.1 that the surjectivity of M_V: $X^E \longrightarrow X_T$ is equivalent to the surjectivity of $\overline{T}\pi^{\overline{T}} : X^{\overline{E}} \longrightarrow X_{\overline{T}}$. Again, the result follows by applying Part 5.
9. This was proved in Fuhrmann [14].

We saw that, given E defined by (126) with all left Wiener–Hopf factorization indices of $W'Q^{-1}$ equal to zero, the multiplication map M_V: $X^E \longrightarrow X_V$ is bijective. Now X^E has a natural $\mathbb{F}[z]$-module structure, while X_V does not. Thus we can use the isomorphism M_V to induce, for any such extension E, an $\mathbb{F}[z]$-module structure on X_V. We refer to this, following Khargonekar et al. [23], as a *shift module structure*. Note that W' is determined uniquely, up to a right unimodular factor, by V. Thus the condition on the Wiener–Hopf indices in the next theorem, quoted without proof from Fuhrmann [14], is independent of the completion E. Moreover, the following theorem shows that this module structure has a feedback interpretation. This shows that the friends of X_V can be parametrized via equivalence classes of nonsingular polynomial matrix completions of V.

THEOREM 5.1. *Given a strictly proper, $p \times m$ rational function $T^{-1}V$, with V left prime. Let $\begin{pmatrix} V \\ V' \end{pmatrix}$ be any extension to a unimodular polynomial matrix and let $\begin{pmatrix} W & W' \end{pmatrix} = \begin{pmatrix} V \\ V' \end{pmatrix}^{-1}$. Let E be defined by*

$$E = \begin{pmatrix} V \\ QV' \end{pmatrix}, \tag{126}$$

with Q a nonsingular polynomial matrix for which all left Wiener–Hopf factorization indices of $W'Q^{-1}$ are zero. Then

1. *X^E is an autonomous subbehavior of X^V.*
2. *Defining S_E: $X_V \longrightarrow X_V$ via the commutativity of the following diagram*

$$\begin{tikzcd} X_V \arrow[d, "S_E"'] & X^E \arrow[l, "M_V"'] \arrow[d, "S^E"] \\ X_V & X^E \arrow[l, "M_V"] \end{tikzcd}$$

Diagram 5.2.
we have the isomorphism of S_E and S^E and hence an S^E induced module structure on X_V. This is given, for $f = Vh$ with $h \in X^E$ and $\eta = h_{-1}$, by

$$S_E f = VS^E h = V(zh - \eta) = zf - V(z)\eta. \tag{127}$$

3. *The invariant factors of S_E are equal to the invariant factors of Q*

Note that, with respect to the shift realization $\Sigma(T^{-1}V)$, for $K \in \mathcal{F}(\mathcal{V})$, there exists a shift module structure such that $(A - BK)| X_V \simeq S_E$. Thus the spectral characteristics of $(A-BK)X_V$, i.e. its invariant factors, are equal to those of E. Now E is defined by (126) and as $E = \begin{pmatrix} V \\ QV' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} V \\ V' \end{pmatrix}$, with $\begin{pmatrix} V \\ V' \end{pmatrix}$ unimodular, the nontrivial invariant factors of E are those of Q. In turn, Q is constrained only by the requirement that the left Wiener–Hopf factorization indices of $W'Q^{-1}$ are zero. This means that the column indices of Q are equal to those of W'. The column indices of Q, see subsection 2.6, are the reachability indices of S_Q and hence of S_E. Assume now that the column indices of W' are $\kappa_1,\ldots,\kappa_{m-p}$, then by results of Rosenbrock [29], we have

$$\sum_{i=1}^{k} \deg q_i \begin{cases} \geq \sum_{i=1}^{k} \kappa_i & 1 \leq k < m-p \\ = \sum_{i=1}^{k} \kappa_i & k = m-p \end{cases} \tag{128}$$

Thus the invariant factors q_i can be chosen arbitrarily, subject only to the constraints (128) and the divisibility conditions $q_i | q_{i-1}$. Thus, Theorem 5.1 is essentially an extension of Rosenbrock's celebrated generalized pole placement theorem, see [29], to the case of reachability subspaces. The moral of Theorem 5.1 is that, in the case of an output nulling reachability subspace, internal spectral assignability is achieved via a simple polynomial matrix extension. Thus it is only natural to expect that $\mathcal{V}^*(\overline{\Sigma})$, $\mathcal{V}_*\left(\overline{\Sigma}^*\right)$, $\mathcal{V}_*(\Sigma^*)$ can also be characterized in terms of matrix completions. This is achieved by the use of isomorphism and duality. The following theorem describes the results for the special case of controllability and observability subspaces. The general result for controlled and conditioned invariant subspaces will be taken up in Theorem 5.3. The theorem translates the results of Theorem 4.1 in terms of extensions of the numerator polynomial matrix V.

THEOREM 5.2. *Let $G = T^{-1}V = \overline{V}\,\overline{T}^{-1}$ be coprime factorizations of a strictly proper rational function. V is assumed to be left prime. Let $Q \in \mathbb{F}[z]^{(m-p)\times(m-p)}$ be nonsingular and let E be defined by* (126), *where $\begin{pmatrix} V \\ V' \end{pmatrix}$ is a jixed unimodular extension of V with inverse given by $\begin{pmatrix} W & W' \end{pmatrix}$ and where we assume that all left Wiener–Hopf factorization indices of $W'Q^{-1}$ are zero. Let $\bar{E}$ be defined, uniquely up to a left unimodular factor, by $X^{\overline{E}} = Y(\sigma)X^E$. Then, for the shift realizations given by* (100) *and* (101), *we have*

1.

$$\mathcal{V}^*(\Sigma) = X_V = VX^E \tag{129}$$

and

$$\dim X_V = \deg\det E = \deg\det Q. \tag{130}$$

2. *With respect to the shift realization $\overline{\Sigma}$, we have*

$$\mathcal{V}^*\left(\overline{\Sigma}\right) = -\pi_{\overline{T}}\overline{X}VX^E = \overline{T}\pi^{\overline{T}}X^{\overline{E}}. \tag{131}$$

3. *With respect to the shift realization $\overline{\Sigma}^*$, we have*

$$\mathcal{V}_*\left(\overline{\Sigma}^*\right) = X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{E}}\mathbb{F}[z]^p. \tag{132}$$

4. *We have the codimension formula*

$$\operatorname{codim} \mathcal{V}_*\left(\overline{\Sigma}^*\right) = \deg\det \widetilde{\overline{E}} = \deg\det E = \operatorname{codim} \mathcal{V}_*\left(\Sigma^*\right). \tag{133}$$

Proof.

1. We already know that $\mathcal{V}^*(\Sigma) = X_V$. By Theorem 3.1 in Fuhrmann [14], the condition on the Wiener–Hopf factorization indices is equivalent to $M_V : X^E \longrightarrow X_V$ being invertible. In particular, we have (129) and (130).
2. For ψ defined in (27) we have $\Sigma \overset{\psi}{\rightsquigarrow} \overline{\Sigma}$ and hence $\mathcal{V}^*\left(\overline{\Sigma}\right) = \psi(\mathcal{V}^*(\Sigma))$. We compute, using (114),

$$\begin{aligned}
\mathcal{V}^*\left(\overline{\Sigma}\right) &= \psi\left(\mathcal{V}^*\left(\Sigma\right)\right) = -\pi_{\overline{T}}\overline{X}VX^E \\
&= -\overline{T}\pi_-\overline{T}^{-1}\overline{X}VX^E = -\overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{X}VX^E \\
&= \overline{T}\pi_-\overline{T}^{-1}\pi_+\left(I - \overline{T}Y\right)X^E = \overline{T}\pi_-\overline{T}^{-1}\pi_+\overline{T}\left(\pi_+ + \pi_-\right)X^E \\
&= \overline{T}\pi^{\overline{T}}\pi_- YX^E = \overline{T}\pi^{\overline{T}}Y(\sigma)\,X^E \\
&= \overline{T}\pi^{\overline{T}}X^{\overline{E}}.
\end{aligned} \tag{134}$$

3. We have $\mathcal{V}_*\left(\overline{\Sigma}^*\right) = \mathcal{V}^*\left(\overline{\Sigma}\right)^\perp = \overline{T}\pi^{\overline{T}}X^{\overline{E}}$. Using the fact that the adjoint of the map $\overline{T}\pi^{\overline{T}} : X^{\overline{E}} \longrightarrow X_{\overline{T}}$ is, by Lemma 5.1, given by the projection map $\pi_{\widetilde{\overline{E}}} : X_{\widetilde{\overline{T}}} \longrightarrow X_{\widetilde{\overline{E}}}$, it follows that $\left(\operatorname{Im} \overline{T}\pi^{\overline{T}}\middle| X^{\overline{E}}\right)^\perp = \operatorname{Ker} \pi_{\widetilde{\overline{E}}}\,|X_{\widetilde{\overline{T}}} = X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{E}}\mathbb{F}[z]^p$.
4. We compute

$$\begin{aligned}
\dim \mathcal{V}_*\left(\Sigma^*\right) &= \dim\ \operatorname{Ker}\mathcal{T}_{\widetilde{\overline{E}}\widetilde{\overline{T}}^{-1}} \\
&= \dim X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{E}}\mathbb{F}[z]^p \\
&= \deg\det \widetilde{\overline{T}} - \deg\det \widetilde{\overline{E}} = \deg\det T - \deg\det E.
\end{aligned}$$

□

Since the invariant factors of E, $\overline{E}$ and $\widetilde{\overline{E}}$ are equal, this theorem provides a complete solution to the outer spectral assignability problem. For the details, as well as the output injection interpretation, we refer to Fuhrmann and Trumpf [17].

We proceed now to drop the assumption of left primeness of V. In this case, X_V is no longer a reachability subspace for $\Sigma\left(T^{-1}V\right)$ and as a result and there are additional constraints on the invariant factors of the closed loop system reduced to X_V arising from the presence of transmission zeros. The next theorem gives a precise answer to what may be the elementary divisors of $(A - BK)|X_V$, and traces this characterization via isomorphisms and adjoints.

THEOREM 5.3. *Let $G = T^{-1}V = \overline{V}\,\overline{T}^{-1}$ be coprime factorizations of a strictly proper, $p \times m$ rational function and let $V = E_\rho V_\rho$ and $\overline{V} = \overline{E}_\rho \overline{V}_\rho$ be internal/external factorizations. Let $\begin{pmatrix} V_\rho \\ V'_\rho \end{pmatrix}$ be any extension to a unimodular polynomial matrix and* *let* $\begin{pmatrix} w_\rho & w'_\rho \end{pmatrix} = \begin{pmatrix} V_\rho \\ V_{\prime}^{\rho} \end{pmatrix}^{-1}$. *Similarly, let* $\begin{pmatrix} \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}$ *be any extension to a unimodular polynomial matrix and let* $\begin{pmatrix} \overline{w}_\rho & \overline{w}'_\rho \end{pmatrix} = \begin{pmatrix} \overline{V}_\rho \\ \overline{V}_{\prime}^{\rho} \end{pmatrix}^{-1}$, *Let* Q *be any* $(m-p) \times (m-p)$ *nonsingular polynomial matrix for which all right Wiener–Hopf factorization indices of* $W'_\rho Q^{-1}$ *are zero. Let* E *be defined*

$$E = \begin{pmatrix} E_\rho V_\rho \\ QV'_\rho \end{pmatrix} \tag{135}$$

and E_{re} *by*

$$E_{re} = \begin{pmatrix} V_P \\ Q'_\rho \end{pmatrix}. \tag{136}$$

Then

1. *We have*

$$X^E = X^{\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}} \oplus X^{\begin{pmatrix} V_\rho \\ QV'_\rho \end{pmatrix}}. \tag{137}$$

2. *The multiplication map* $M_V : X^E \longrightarrow X_V$*is bijective with*

$$M_V\left(X^{\begin{pmatrix} V_\rho \\ QV'_\rho \end{pmatrix}} \right) = E_\rho X_{V_\rho}$$

$$M_V\left(X^{\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}} \right) = \mathcal{C}_{E_\rho} = \begin{pmatrix} I & 0 \end{pmatrix} X_{\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}} \tag{138}$$

3. *Defining* $S_E : XV \longrightarrow X_V$ *via the commutative diagram 5.2, the induced module structure is given by* (127).
4. $\mathcal{C}_{E_\rho}$ *is a tight controlled invariant subspace of* X_V *and we have the direct sum decomposition*

$$X_V = \mathcal{C}_{E_\rho} \oplus E_\rho X_{V_\rho}. \tag{139}$$

   This is a direct sum of modules with respect to the shift module structure of X_V *induced by* X^E.
5. *Let* $\bar{E}$ *be defined, up to a left unimodular factor, by*

$$X^{\overline{E}} = Y(\sigma) X^E, \tag{140}$$

   then there exists a polynomial matrix $\overline{Y}_a$ *for which*

$$\overline{Y}_a E = \overline{E} Y. \tag{141}$$

6. *We have the representation*

$$\overline{E} = \begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{Q}\overline{V}'_\rho \end{pmatrix}, \tag{142}$$

where $\begin{pmatrix} \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}$ *is some unimodular extension of* $\overline{V}_\rho$. *We define*

$$\overline{E}_{re} = \begin{pmatrix} \overline{V}_\rho \\ \overline{Q V'_\rho} \end{pmatrix} \tag{143}$$

We have the direct sum decomposition

$$X^{\overline{E}} = X^{\begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}} \oplus X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{Q V'_\rho} \end{pmatrix}}. \tag{144}$$

Without loss of generality, we can assume Y_a *has the representation*

$$\overline{Y}_a = \begin{pmatrix} \overline{Y} & 0 \\ \overline{R}_\rho & I \end{pmatrix}, \tag{145}$$

for some polynomial matrix $\overline{R}_\rho$.

7. *With respect to the shift realization* Σ *we have*

$$\begin{aligned} \mathcal{V}^*(\Sigma) &= M_V X^E \\ \mathcal{R}^*(\Sigma) &= M_V X^{\begin{pmatrix} V_\rho \\ Q V'_\rho \end{pmatrix}}. \end{aligned} \tag{146}$$

8. *With respect to the shift realization* $\overline{\Sigma}$, *we have*

$$\begin{aligned} \mathcal{V}^*\left(\overline{\Sigma}\right) &= \overline{T} \pi^{\overline{T}} X^{\overline{E}} \\ \mathcal{R}^*\left(\overline{\Sigma}\right) &= \overline{T} \pi^{\overline{T}} X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{Q V'_\rho} \end{pmatrix}}. \end{aligned} \tag{147}$$

9. *With respect to the shift realization* $\overline{\Sigma}^*$, *we have*

$$\begin{aligned} \mathcal{V}_*\left(\overline{\Sigma}^*\right) &= X_{\tilde{\tilde{T}}} \cap \left(\tilde{\tilde{V}}_\rho \tilde{\tilde{E}}_\rho \; \tilde{\tilde{V}}'_\rho \tilde{\tilde{Q}} \right) \mathbb{F}[z]^p \\ \mathcal{O}_*\left(\overline{\Sigma}^*\right) &= X_{\tilde{\tilde{T}}} \cap \left(\tilde{\tilde{V}}_\rho \; \tilde{\tilde{V}}'_\rho \tilde{\tilde{Q}} \right) \mathbb{F}[z]^p. \end{aligned} \tag{148}$$

10. *With* $\mathcal{T}$ *defined by* $\mathcal{T} = X_{\tilde{\tilde{T}}} \cap \left(\tilde{\tilde{V}}_\rho \tilde{\tilde{E}}_\rho \; \tilde{\tilde{V}}'_\rho \right) \mathbb{F}[z]^p$, we have the transversal intersection representation

$$\mathcal{V}_*\left(\overline{\Sigma}^*\right) = \mathcal{O}_*\left(\overline{\Sigma}^*\right) \cap \mathcal{T}, \tag{149}$$

and the direct sum representation

$$X_{\tilde{\tilde{T}}} \Big/ \mathcal{V}_*\left(\overline{\Sigma}^*\right) = \mathcal{O}_*\left(\overline{\Sigma}^*\right) \Big/ \mathcal{V}_*\left(\overline{\Sigma}^*\right) \oplus \mathcal{T} \Big/ \mathcal{V}_*\left(\overline{\Sigma}^*\right). \tag{150}$$

Proof.

1. The polynomial matrices $\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}$ and $\begin{pmatrix} V_\rho \\ QV'_\rho \end{pmatrix}$, are right coprime as, clearly, $\begin{pmatrix} V_\rho \\ QV'_\rho \\ E_\rho V_\rho \\ V'_\rho \end{pmatrix}$ is right prime. Moreover, E is their 1.c.l.m. This implies the direct sum (137).
2. This was proved in Fuhrmann [14].
3. Follows from the fact that $M_V : X^E \longrightarrow X^V$ is bijective.
4. This was proved in Fuhrmann [14].
5. The doubly coprime factorization (25), and in particular the relation $\overline{Y}V = \overline{V}Y$, imply that $Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ is a B-isomorphism. Also, $X^E \subset X_V$ as an autonomous subbehavior, therefore $Y(\sigma)X^E$ is an autonomous subbehavior of $X^{\overline{V}}$, hence has a representation of the form $X^{\overline{E}}$ for some nonsinular polynomial matrix $\bar{E}$. By Theorem 2.5, there exists a polynomial matrix $\overline{Y}_a$ for which
$$\overline{Y}_a E = \overline{E}Y. \tag{151}$$
6. Let $\overline{V} = \overline{E}_\rho \overline{V}_\rho$ be an internal/external factorization. Since, by Proposition 2.6, $Y(\sigma)$ maps the reachable subbehavior $X^{V_\rho} \subset X^V$ onto the reachable subbehavior $X^{\overline{V}_\rho} \subset X^{\overline{V}}$, there exists a polynomial matrix $\overline{Y}_\rho$, for which
$$\overline{Y}_\rho V_\rho = \overline{V}_\rho Y \tag{152}$$
We compute
$$\left(\overline{E}_\rho \overline{Y}_\rho E_\rho^{-1}\right)V = \left(\overline{E}_\rho \overline{Y}_\rho E_\rho^{-1}\right)E_\rho V_\rho = \overline{E}_\rho \overline{Y}_\rho V_\rho = \left(\overline{E}_\rho \overline{Y}_\rho\right)Y = \overline{V}Y = \overline{Y}V.$$
Since V has full row rank, we conclude that $\overline{E}_\rho \overline{Y}_\rho E_\rho^{-1} = \overline{Y}$ or, equivalently, that
$$\overline{E}_\rho \overline{Y}_\rho = \overline{Y}E_\rho. \tag{153}$$
This can of course be rewritten also as
$$\overline{Y}_\rho = \overline{E}_\rho^{-1}\overline{Y}E_\rho. \tag{154}$$
Next, we consider the autonomous subbehavior of X^V given by $X^{\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}}$. Since $Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ is, a B-isomorphism, then $Y(\sigma)X^{\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}}$ is an autonomous subbehavior of $X^{\overline{V}}$. As such, using Proposition 3.6 in Fuhrmann [14], it has a representation $Y(\sigma)X^{\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}} = X^{\begin{pmatrix} \overline{E}_\beta \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}}$, with $\overline{E}_\beta$ the right factor in the factorization $\overline{E}_\rho = \overline{E}_\alpha \overline{E}_\beta$. Necessarily $\begin{pmatrix} \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}$ is nonsingular. Thus, there

exists a polynomial matrix $\begin{pmatrix} \overline{Y}_{11} & \overline{Y}_{12} \\ \overline{Y}_{21} & \overline{Y}_{22} \end{pmatrix}$ for which

$$\begin{pmatrix} \overline{Y}_{11} & \overline{Y}_{12} \\ \overline{Y}_{21} & \overline{Y}_{22} \end{pmatrix} \begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix} = \begin{pmatrix} \overline{E}_\beta \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix} Y. \tag{155}$$

In particular, this implies the equality $\overline{Y}_{11} V + \overline{Y}_{12} V'_\rho = \overline{E}_\beta \overline{V}_\rho Y$. Multiplying by $\overline{E}_\alpha$ on the left, we have

$$\overline{E}_\alpha \overline{Y}_{11} V + \overline{E}_\alpha \overline{Y}_{12} V'_\rho = \overline{E}_\alpha \overline{E}_\beta \overline{V}_\rho Y = \overline{E}_\rho \overline{V}_\rho Y = \overline{V} Y = \overline{Y} V.$$

Since $\begin{pmatrix} \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}$ is nonsingular, this implies $\overline{Y}_{12} = 0$ and $\overline{E}_\alpha \overline{Y}_{11} V = \overline{V} Y$, or

$$\overline{Y}_{11} V = \overline{E}_\beta \overline{V}_\rho Y. \tag{156}$$

This equation shows that $Y(\sigma) X^V \subset X^{\overline{E}_\beta \overline{V}_\rho} \subset X^{\overline{V}}$. However, we know that $Y(\sigma) X^V = X^{\overline{V}}$, hence we have the equality $X^{\overline{E}_\beta \overline{V}_\rho} = X^{\overline{V}}$ or $\overline{E}_\beta \overline{V}_\rho$ and $\overline{V}$ differ by at most a left unimodular factor. Necessarily, $\overline{E}_\alpha$ is unimodular and, without loss of generality, we may assume $\overline{E}_\alpha = I$ and $\overline{E}_\beta = \overline{E}_\rho$. We conclude from (156) that $Y_{11} = Y$. Setting $\overline{Y}_{21} = \overline{L}_\rho$, we have

$$\begin{pmatrix} \overline{Y} & 0 \\ \overline{L}_\rho & I \end{pmatrix} \begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix} = \begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix} Y, \tag{157}$$

and

$$X^{\begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}} = Y(\sigma) X^{\begin{pmatrix} E_\rho V_\rho \\ \overline{V}'_\rho \end{pmatrix}}. \tag{158}$$

As $X^{\begin{pmatrix} V_\rho \\ Q V'_\rho \end{pmatrix}}$ is an autonomous subbehavior of X^V and $Y(\sigma) : X^V \longrightarrow X^{\overline{V}}$ is a B-isomorphism, then $Y(\sigma) X^{\begin{pmatrix} V_\rho \\ Q V'_\rho \end{pmatrix}}$ is an autonomous subbehavior of $X^{\overline{V}}$, and as such, applying Lemma 5.1, it has a representation of the form

$$X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{Q V}'_\rho \end{pmatrix}} = Y(\sigma) X^{\begin{pmatrix} V_\rho \\ Q V'_\rho \end{pmatrix}} \tag{159}$$

Moreover, there exists a polynomial matrix $\overline{Y}_q$ which has the representation

$$\overline{Y}_q = \begin{pmatrix} \overline{Y}_\rho & 0 \\ \overline{R}_\rho & I \end{pmatrix} \tag{160}$$

for which

$$\overline{Y}_q \begin{pmatrix} V_\rho \\ \overline{Q V}'_\rho \end{pmatrix} = \begin{pmatrix} \overline{V}_\rho \\ \overline{Q V}'_\rho \end{pmatrix} Y \tag{161}$$

holds and for which exists a doubly unimodular embedding.

The B-isomorphism $Y(\sigma)$ maps direct sums into direct sums. Using (158) and (159), we have

$$\begin{aligned} X^{\overline{E}} &= Y(\sigma)\, X^{E} = Y(\sigma)\, X^{\begin{pmatrix} E_\rho V_\rho \\ V'_\rho \end{pmatrix}} \oplus Y(\sigma)\, X^{\begin{pmatrix} V_\rho \\ QV'_\rho \end{pmatrix}} \\ &= X^{\begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}} \oplus X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}} = X^{\begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}}. \end{aligned} \tag{162}$$

Here we also used the fact that the polynomial matrices $\begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}$ and $\begin{pmatrix} \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}$ are right coprime and that $\begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}$ is their least common left multiple. It is clear that (162) implies (142).

The fact that a doubly unimodular embedding for (160) exists implies that Q and $\overline{Q}$ are equivalent in the sense that they have the same invariant factors. Both have the same size, so it is a consequence of the invariant factor algorithm that they are unimodularly equivalent, i.e. there exist unimodular polynomial matrices U_1,U_2 for which $\overline{Q} = U_1 Q U_2$. Modifying trivially the definitions of V'_ρ, $\overline{V}'_\rho$ and $\overline{R}_\rho$, we may assume without loss of generality that $\overline{Q} = Q$.

Multiplying Equation (161) on the left $\begin{pmatrix} \overline{E}_\rho & 0 \\ 0 & I \end{pmatrix}$, we have

$$\begin{aligned} \begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix} Y &= \begin{pmatrix} \overline{E}_\rho \overline{V}_\rho & 0 \\ \overline{R}_\rho & I \end{pmatrix} \begin{pmatrix} V_\rho \\ QV'_\rho \end{pmatrix} \\ &= \begin{pmatrix} \overline{E}_\rho \overline{Y}_\rho E_\rho^{-1} & 0 \\ \overline{R}_\rho E_\rho^{-1} & I \end{pmatrix} \begin{pmatrix} E_\rho V_\rho \\ QV'_\rho \end{pmatrix} \end{aligned}$$

and using Equation (153), we obtain

$$\begin{pmatrix} \overline{Y} & 0 \\ \overline{R}_\rho E_\rho^{-1} & I \end{pmatrix} \begin{pmatrix} E_\rho V_\rho \\ QV'_\rho \end{pmatrix} = \begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix} Y. \tag{163}$$

In turn, this implies $\overline{Y}_a = \begin{pmatrix} \overline{Y} & 0 \\ \overline{R}_\rho E_\rho^{-1} & I \end{pmatrix} = \begin{pmatrix} \overline{Y} & 0 \\ \overline{P}_\rho & I \end{pmatrix}$ with $\overline{P}_\rho$ a polynomial matrix.

Thus we have

$$\overline{Y}_a = \begin{pmatrix} \overline{Y} & 0 \\ \overline{P}_\rho & I \end{pmatrix} \tag{164}$$

and

$$\overline{Y}_q = \begin{pmatrix} \overline{Y}_\rho & 0 \\ \overline{P}_\rho E_\rho & I \end{pmatrix} \tag{165}$$

7. Under our assumptions, we have $M_V X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}} = E_\rho X_{V_\rho} = \mathcal{R}^*(\Sigma)$.
8. Using the commutativity of Diagram 5.1, equations (146) imply (147).

9. Computing annihilators, as in Theorem 4.1, we have

$$\mathcal{V}_*\left(\overline{\Sigma}^*\right) = \left(\overline{T}\pi^{\overline{T}} X^{\begin{pmatrix} E_\rho \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}}\right)^\perp = X_{\widetilde{\overline{T}}} \cap \left(\widetilde{\overline{V}}_\rho \widetilde{\overline{E}}_\rho\ \widetilde{\overline{V}}'_\rho \widetilde{\overline{Q}}\right)\mathbb{F}[z]^p.$$

Similarly,

$$\mathcal{O}_*\left(\overline{\Sigma}^*\right) = \left(\overline{T}\pi^{\overline{T}} X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}}\right)^\perp = X_{\widetilde{\overline{T}}} \cap \left(\widetilde{\overline{V}}_\rho\ \widetilde{\overline{V}}'_\rho \widetilde{\overline{Q}}\right)\mathbb{F}[z]^p.$$

10. $Y(\sigma) : X^E \longrightarrow X^{\overline{E}}$ is invertible and maps the direct sum (137) onto the direct sum (144). Now $\left(\overline{T}\pi^{\overline{T}} X^{\overline{E}}\right)^\perp = X_{\widetilde{\overline{T}}} \cap \widetilde{\overline{E}}\,\mathbb{F}[z]^p$ leads to

$$\begin{aligned}
&\left(\overline{T}\pi^{\overline{T}} X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}} \oplus \overline{T}\pi^{\overline{T}} X^{\begin{pmatrix} \overline{E}_\rho \overline{V}_\rho \\ \overline{V}'_\rho \end{pmatrix}}\right)^\perp \\
&= \left(\overline{T}\pi^{\overline{T}} X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}}\right)^\perp \cap \left(\overline{T}\pi^{\overline{T}} X^{\begin{pmatrix} \overline{V}_\rho \\ \overline{QV}'_\rho \end{pmatrix}}\right)^\perp \\
&= \left(X_{\widetilde{\overline{T}}} \cap \left(\widetilde{\overline{V}}_\rho\ \widetilde{\overline{V}}'_\rho \widetilde{\overline{Q}}\right)\mathbb{F}[z]^p\right) \cap \left(X_{\widetilde{\overline{T}}} \cap \left(\widetilde{\overline{V}}_\rho \widetilde{\overline{E}}_\rho\ \widetilde{\overline{V}}'_\rho\right)\mathbb{F}[z]^p\right) \\
&= \mathcal{O}_*\left(\overline{\Sigma}^*\right) \cap \mathcal{T}
\end{aligned}$$

That the last intersection is transversal follows from the fact that it is the annihilator of a direct sum. □

References

1. Aling, H., Schumacher, J.M.: A nine-fold decomposition for linear systems. Int. J. Control **39**, 779–805 (1984)
2. Basile, G., Marro, G.: Controlled and conditioned invariant subspaces in linear system theory. J. Optim. Theory Appl. **3**, 306–315 (1969)
3. Basile, G., Marro, G.: Controlled and conditioned invariant subspaces in linear system theory. J. Optim. Theory Appl. **3**, 306–316 (1973)
4. Bisiacco, M., Valcher, M.-E.: Behavior decompositions and two-sided diophantine equations. Automatica **37**, 1387–1395 (2001)
5. Emre, E., Hautus, M.L.J.: A polynomial characterization of (A, B)-invariant and reachability subspaces. SIAM J. Control Optim. **18**, 420–436 (1980)
6. Fuhrmann, P.A.: Algebraic system theory: An analyst's point of view. J. Franklin Inst. **301**, 521–540 (1976)
7. Fuhrmann, P.A.: On strict system equivalence and similarity. Int. J. Control **25**, 5–10 (1977)
8. Fuhrmann, P.A.: Duality in polynomial models with some applications to geometric control theory. IEEE Trans. Automat. Contr. **AC-26**, 284–295 (1981)
9. Fuhrmann, P.A.: On symmetric rational matrix functions. Lin. Alg. Appl., Special issue on Linear Control Theory **50**, 167–250 (1983)
10. Fuhrmann, P.A.: A study of behaviors. Lin. Alg. Appl. **351–352**, 303–380 (2002)

11. Fuhrmann, P.A.: On observers and behaviors. *Proceedings of MTNS2002* (2003a)
12. Fuhrmann, P.A.: A note on continuous behavior homomorphisms. Syst. Control. Lett. **49**, 359–363 (2003b)
13. Fuhrmann, P.A.: Output nulling subspaces and system interconnections. Mittag Leffler Institute preprint series, Stockholm (2003c)
14. Fuhrmann, P.A.: Autonomous subbehaviors and output nulling subspaces. Int. J. Control **78**, 1378–1411 (2005)
15. Fuhrmann, P.A., Gombani, A.: On a Hardy space approach to the analysis of spectral factors. Int. J. Control **71**, 277–357 (1998)
16. Fuhrmann, P.A., Helmke, U.: On the parametrization of conditioned invariant subspaces and observer theory. Lin. Alg. Appl. **332–334**, 265–353 (2001)
17. Fuhrmann, P.A., Trumpf, J.: On observability subspaces. To appear Int. J. Control (2006)
18. Fuhrmann, P.A., Willems, J.C.: Factorization indices at infinity for rational matrix functions. Integr. Equ. Oper. Theory **2**, 287–301 (1979)
19. Fuhrmann, P.A., Willems, J.C.: A study of (A, B)-invariant subspaces via polynomial models. Int. J. Control **31**, 467–494 (1980)
20. Hautus, M.L.J., Heymann, M.: Linear feedback – An algebraic approach. SIAM J. Control **16**, 83–105 (1978)
21. Hinrichsen, D., Münzner, H.F., Prätzel-Wolters, D.: Parametrization of (C, A)-invariant subspaces. Syst. Control. Lett. **1**, 192–199 (1981)
22. Kalman, R.E., Falb, P., Arbib, M.: Topics in Mathematical System Theory. McGraw Hill (1969)
23. Khargonekar, P.P., Georgiou, T.T., Özgüller, A.B.: Skew-prime polynomial matrices: The polynomial model approach. Lin. Alg. Appl. **50**, 403–435 (1983)
24. Morse, A.S.: Output controllability and system synthesis. SIAM J. Control **9**, 64–67 (1971)
25. Morse, A.S.: Structural invariants for linear multivariable systems. SIAM J. Control **11**, 446–465 (1973)
26. Oberst, U.: Multidimensional constant linear systems. Acta Appl. Math. **20**, 1–175 (1990)
27. Özgüller, A.B.: Polynomial characterizations of (H, F)-invariant subspaces with applications. Lin. Alg. Appl. **73**, 1–31 (1986)
28. Poldermann, J.W., Willems, J.C.: Introduction to Mathematical System Theory. Springer, Berlin Heidelberg New York (1997)
29. Rosenbrock, H.H.: State-space and Multivariable Theory. Wiley, New York (1970)
30. Willems, J.C.: From time series to linear systems. Part I. Finite-dimensional linear time invariant systems. Automatica **22**, 561–580 (1986)
31. Willems, J.C.: Models for dynamics. In: Kirchgraber, U., Walther, H.O. (eds.) Dynamics Reported, vol. 2, pp. 171–269. Wiley-Teubner (1989)
32. Willems, J.C.: On interconnections, control and feedback. IEEE Trans. Automat. Contr. **AC-42**, 326–337 (1991)
33. Willems, J.C.: On interconnections, control and feedback. IEEE Trans. Automat. Contr. **AC-42**, 326–337 (1997)
34. Wonham, W.M.: Linear Multivariable Control, 2nd ed. Springer, Berlin Heidelberg New York (1979)
35. Yekutieli, A.: On the structure of behaviors. Lin. Alg. Appl. (2004)