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A note on continuous behavior homomorphisms☆

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Abstract

The purpose of this note is the characterization of the class of behavior homomorphisms that have nice polynomial representations. The key to this is a judicious application of duality theory together with additional continuity requirements.

Keywords: Behaviors; Continuous behavior homomorphisms; Duality

1. Introduction

Behaviors were introduced and extensively studied in a series of seminal papers, Willems [6–8]. In Fuhrmann [4,5], a study of discrete time behaviors was undertaken, of which probably the central contribution to the behavioral literature was the introduction and characterization of behavior homomorphisms. The approach adopted in that paper used duality theory as a core idea. The identification of the space of formal vectorial power series, $z^{-1}F^m[[z^{-1}]]$, as the dual to the space of vector polynomials, $F^m[z]$, a duality introduced in Fuhrmann [3], led essentially to an algebraic analysis. For the characterization of behaviors in terms of kernel representation, a result due to Willems, one had to add a completeness condition on the behavior. This condition, though of an algebraic flavor, is actually a topological condition that is equivalent to a closure condition in a suitable topology. Behaviors turned out to be generalizations of rational models introduced and studied in many papers beginning with Fuhrmann [2], in fact a rational model is just what is referred to in the behavioral literature as an autonomous behavior. However, contrary to the case of polynomial and rational models which are always finite dimensional as linear vector spaces, this is no longer the case for behaviors. Thus, we are in the context of infinite-dimensional spaces, and nonreflexive at that. In fact, a crucial point in Fuhrmann [5] was overlooked and so part of Theorem 4.5, a central result in that paper, is incorrect as stated. This can be remedied. In fact, the process of restating the result correctly sheds some more light on behaviors and behavior homomorphisms and may be of use in the analysis of behaviors in different settings.

We attempt now to point out the difficulty. It was quite easy to extend the principal results about polynomial model homomorphisms to the case of quotient modules of $F^m[z]$. Clearly, the dual of an $F[z]$-homomorphism $Z : F^p[z]/M(z)F^m[z] \to F^{\bar{p}}[z]/\bar{M}(z)F^{\bar{m}}[z]$ is an $F[z]$-homomorphism $Z^* : \operatorname{Ker}\tilde{M}(\sigma) \to \operatorname{Ker}\tilde{\bar{M}}(\sigma)$. However, and this is a gap in the statement of Theorem 4.5 in Fuhrmann [5], not every $F[z]$-homomorphism $\bar{Z} : \operatorname{Ker}\tilde{M}(\sigma) \to$

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$\operatorname{Ker} \tilde{\bar{M}}(\sigma)$ is the dual of an $F[z]$-homomorphism $Z : F^p[z]/M(z)F^m[z] \rightarrow F^{\bar{p}}[z]/\bar{M}(z)F^{\bar{m}}[z]$. The characterization of the class of behavior homomorphisms which are themselves duals is the main result of the present paper. It turns out that we have to add a continuity requirement on the homomorphisms $\bar{Z}$ in order for it to be an adjoint. The continuity is with respect to the weak* topologies in the two behaviors. The end result is that, with the addition of a single word, the problematic theorem remains true.

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2. Preliminaries

Let F denote an arbitrary field. We will denote by F^m the space of all m-vectors with coordinates in F. $F^m[z]$ the space of all polynomials with coefficients in F^m, $z^{-1}F^m[[z^{-1}]]$ the space of formal power series vanishing at infinity and $F^m((z^{-1}))$ the space of truncated Laurent series. Let π_+ and π_- denote the projections of $F^m((z^{-1}))$ on $F^m[z]$ and $z^{-1}F^m[[z^{-1}]]$ respectively. Since

$$F^m((z^{-1})) = F^m[z] \oplus z^{-1}F^m[[z^{-1}]], \tag{1}$$

π_+ and π_- are complementary projections. All three spaces $F^m((z^{-1}))$, $F^m[z]$ and $z^{-1}F^m[[z^{-1}]]$ carry natural $F[z]$-module structures. In the first two, polynomials act by multiplication. In $z^{-1}F^m[[z^{-1}]]$, the module structure is given by

$$p \cdot h = p(\sigma)h = \pi_- ph, \qquad h \in z^{-1}F^m[[z^{-1}]]. \tag{2}$$

The operator σ is called the backward shift, or simply the shift. This can be generalized. For a polynomial matrix $U \in F^{p\times m}[z]$ we define a map $U(\sigma) : z^{-1}F^m[[z^{-1}]] \rightarrow z^{-1}F^p[[z^{-1}]]$ by

$$U(\sigma)h = \pi_- Uh. \tag{3}$$

Given $x^* \in X^*$ and $x \in X$ we will write

$$[x, x^*] = x^*(x).$$

In the special case of $X = F^m$, we can also identify X^* with F^m and then we write $[x, y] = \tilde{y}x$ where $\tilde{y}$ denotes the transpose of the column vector y. The sole exception will be the complex inner-product spaces where $[x, y]$ will be interpreted as the inner product itself. Now given $f \in F^m((z^{-1}))$ and $g \in F^m((z^{-1}))$ we define a pairing

$$[f, g] = \sum_{j=-\infty}^{\infty} [f_j, g_{-j-1}]. \tag{4}$$

It is clear that $[\cdot,\cdot]$ is a bilinear form on $F^m((z^{-1})) \times F^m((z^{-1}))$. It is well defined as in the defining sum at most a finite number of terms are nonzero. Also this form is *nondegenerate* in the sense that $[f, g] = 0$ for all $g \in F^m((z^{-1}))$ if and only if $f = 0$. Given a subset $M \subset X$ we define its *annihilator* $M^\perp$ by

$$M^\perp = \{x^* \in X^* \mid [m, x^*] = 0 \ \forall m \in M\}.$$

Similarly if $M \subset X^*$ we define the *preannihilator* ${}^\perp M$ by

$${}^\perp M = \{x \in X \mid [x, x^*] = 0 \ \forall x^* \in M\}.$$

It is a simple check of the definitions that $F^m[z]^\perp = F^m[z]$ and ${}^\perp(F^m[z]) = F^m[z]$. It is well known, see Fuhrmann [3], that the dual space of $F^m[z]$ can be identified with $z^{-1}F^m[[z^{-1}]]$.

We digress a bit about topology. Let F be an arbitrary field. We make into a topological space by adopting the discrete topology, where every subset is open. This means that even when working with the real or complex number fields which have their own metric topology, these topologies are disregarded. Let X be a linear vector space over the field F and let X^* be its algebraic dual. An element $x \in X$ can be viewed as linear functional $\hat{x}$ on X^* by defining $\hat{x}(x^*) = x^*(x)$ for every $x^* \in X^*$. The X topology of X^*, or more commonly referred to as the weak* topology, is the weakest topology that makes all such functionals continuous. Thus, a local base at $0 \in X^*$ is given by sets of the form $\{x^* \mid x^*(x_i) = 0,\ i = 1, \ldots, n; n \in \mathbf{Z}_+, x_i \in X\}$.

Given a linear space $\mathscr{Y}$ and its dual $\mathscr{Y}^*$, we can make $\mathscr{Y}$ a linear topological space by choosing the weakest topology that makes all linear functionals of the form $\phi(y^*) = [y^*, y]$ continuous. This is referred to as the weak* topology of $\mathscr{Y}^*$. We note that a net $y^*_\alpha \in \mathscr{Y}^*$ converges to $y^* \in \mathscr{Y}^*$ in the weak* topology if for every $y \in \mathscr{Y}$ we have $[y^*_\alpha, y] \rightarrow [y^*, y]$.

3. Behaviors

The following proposition, which is probably standard, was proved in Fuhrmann [5].

Proposition 3.1. *Let X be a linear vector space over a field F and let X^* be its algebraic dual. Then a subspace $\mathscr{V} \subset X^*$ satisfies*

$$({}^{\perp}\mathscr{V})^{\perp} = \mathscr{V} \tag{5}$$

if and only if for any $h_0 \notin \mathscr{V}$ there exists an $x \in {}^{\perp}\mathscr{V}$ such that $h_0(x) \neq 0$.

Definition 3.1. In $z^{-1}F^m[[z^{-1}]]$ we define the projections $P_n, n \in \mathbf{Z}_+$ by

$$P_n \sum_{i=1}^{\infty} \frac{h_i}{z^i} = \sum_{i=1}^{n} \frac{h_i}{z^i}. \tag{6}$$

We say that a subset $\mathscr{B} \subset z^{-1}F^m[[z^{-1}]]$ is *complete* if for any $w = \sum_{i=1}^{\infty} w_i z^{-i} \in z^{-1}F^m[[z^{-1}]]$ and for each positive integer N, $P_N w \in P_N(\mathscr{B})$ implies $w \in \mathscr{B}$.

A *behavior* in our context is defined as a linear, shift invariant and complete subspace of $z^{-1}F^m[[z^{-1}]]$.

The following proposition was proved in Fuhrmann [5].

Proposition 3.2. *Let F be a field and let $\mathscr{V} \subset z^{-1}F^m[[z^{-1}]]$ be a subspace. Then $\mathscr{V}$ is complete if and only if*

$$({}^{\perp}\mathscr{V})^{\perp} = \mathscr{V}. \tag{7}$$

Completeness can be shown to be equivalent to closure with respect to an appropriate topology. This topology we proceed to introduce in terms of convergence of nets. We say a net $f^{(\alpha)}(z) = \sum_{j=1}^{\infty}(f_j^{(\alpha)}/z^j) \in z^{-1}F^m[[z^{-1}]]$ converges to $f(z) = \sum_{j=1}^{\infty}(f_j/z^j) \in z^{-1}F^m[[z^{-1}]]$ if for any $n \in \mathbf{Z}_+$, there exists a β such that for all $\alpha > \beta$ we have $f_j^{(\alpha)} = f_j$, $j = 1,\ldots,n$. It is clear that this topology is just the weak* topology of $z^{-1}F^m[[z^{-1}]]$.

Theorem 3.1. *A subspace $\mathscr{V} \subset z^{-1}F^m[[z^{-1}]]$ is closed in the above topology if and only if*

$$({}^{\perp}\mathscr{V})^{\perp} = \mathscr{V}. \tag{8}$$

Proof. Assume (8) holds. Each element $f \in {}^{\perp}\mathscr{V}$ induces a linear functional ϕ_f on $\mathscr{V}$ by $\phi_f(h) = [f,h]$. This functional is continuous in the weak* topology of $z^{-1}F^m[[z^{-1}]]$ as this is by definition the weakest topology that makes all functionals ϕ_f continuous. By continuity $\operatorname{Ker}\phi_f$ is a closed subspace. Any intersection of closed sets is closed, so $\mathscr{V} = ({}^{\perp}\mathscr{V})^{\perp} = \bigcap_{f \in {}^{\perp}\mathscr{V}} \operatorname{Ker}\phi_f$ is closed.

To prove the converse, assume without loss of generality that $\mathscr{V}$ is a proper subspace of $z^{-1}F^m[[z^{-1}]]$. Applying Proposition 3.1, it suffices to show that for every $h \in z^{-1}F^m[[z^{-1}]] - \mathscr{V}$ there exists an $f \in {}^{\perp}\mathscr{V}$ such that $[f,h] \neq 0$. Define for each integer $n \in \mathbf{Z}_+$ the projections P_n by (6). Clearly, for all $n \in \mathbf{Z}_+$, $P_n(\mathscr{V})$ is a finite-dimensional vector space, in fact $\dim P_n(\mathscr{V}) \leqslant mn$. Since $h \notin \mathscr{V}$ and $\mathscr{V}$ is closed, it follows that for some n_0, $P_{n_0}h \notin P_{n_0}(\mathscr{V})$. By finite dimensionality, there exists a polynomial vector $f \in F^m[z]$ of degree $< n_0$ such that $[f, P_{n_0}(\mathscr{V})] = 0$ and $[f, P_{n_0}h] \neq 0$. It is obvious that $[f,\mathscr{V}] = 0$, i.e. $f \in {}^{\perp}\mathscr{V}$, and $[f,h] \neq 0$. □

Corollary 3.1. *A subspace $\mathscr{V} \subset z^{-1}F^m[[z^{-1}]]$ is complete if and only if it is closed.*

Proof. Both conditions are equivalent to $({}^{\perp}\mathscr{V})^{\perp} = \mathscr{V}$. □

For the principal result we will need the following theorem. In the context of Banach spaces, it was proved by Banach. In Yosida [9], it is proved in the context of locally convex linear topological vector spaces. Since we are working over an arbitrary field, we adapt Yosida's proof to that context.

Theorem 3.2. *Let $\mathscr{X}$ be a be a vector space over the field F and let X^* be its dual. A linear functional f on $\mathscr{X}^*$ is of the form*

$$f(x^*) = [x^*, x] = \hat{x}(x^*) \tag{9}$$

for some $x \in \mathscr{X}$ if and only if f is continuous in the weak* topology of $\mathscr{X}^*$.

Proof. Clearly, every functional of the form $f = \hat{x}$ is continuous by the definition of the weak* topology.

Conversely, let $f : \mathscr{X}^* \to F$ be a linear functional continuous in the weak* topology. Since $f^{-1}(0)$ is an open neighborhood of $0 \in \mathscr{X}^*$, it contains a set of the form $\{x^*|x^*(x_i)=0, i=1,\dots,n; n\in\mathbf{Z}_+, x_i\in X\}$. Thus, $x^*x_i=0, i=1,\dots,n$ implies $f(x^*)=0$. Define a linear transformation $L:\mathscr{X}^*\to F^n$ by $L(x^*)=(x^*(x_1),\dots,x^*(x_n))$. Clearly $L(x_1^*)=L(x_2^*)$ if and only if $x_1^*(x_i)=x_2^*(x_i)$ for $i=1,\dots,n$. Thus, the functional $\Phi:L(\mathscr{X}^*)\to F$ given by $\Phi(x^*(x_1),\dots,x^*(x_n))=f(x^*)$ is well defined. We extend this functional to F^n which we still denote by Φ. Clearly, there exists $\alpha_i\in F$ such that $\Phi(\xi_1,\dots,\xi_n)=\sum_{i=1}^n \alpha_i\xi_i$. We compute

$$f(x^*)=\Phi(x^*(x_1),\dots,x^*(x_n))=\sum_{i=1}^n \alpha_i x^*(x_i)$$

$$=\left[x^*,\sum_{i=1}^n \alpha_i x_i\right].$$

Thus $f=\hat{x}$ with $x=\sum_{i=1}^n \alpha_i x_i$. □

A linear transformation $\bar{T}:\mathscr{Y}^*\to\mathscr{X}^*$ is continuous with respect to the weak* topologies in $\mathscr{Y}^*$ and $\mathscr{X}^*$, or simply continuous, if for every weak* convergent net $y_\alpha^*\to y^*$, we have the weak* convergence $\bar{T}y_\alpha^*\to\bar{T}y^*$. This is equivalent to $[\bar{T}y_\alpha^*,x]\to[\bar{T}y^*,x]$ for every $x\in\mathscr{X}$.

The following theorem is given as an exercise in Dunford and Schwartz [1] where the context is that of Banach spaces. Using the characterization of weak* continuous functionals given in Yosida [9], it is easily extended to arbitrary, locally convex linear topological vector spaces. We give the proof for the case of an arbitrary field.

Theorem 3.3. *Let $\mathscr{X},\mathscr{Y}$ be vector spaces over an arbitrary field F. Let the dual spaces, $\mathscr{X}^*,\mathscr{Y}^*$, be endowed with the respective w^* topologies. Let $\bar{T}:\mathscr{Y}^*\to\mathscr{X}^*$ be a linear transformation that is continuous with respect to these topologies. Then there exists a linear transformation $T:\mathscr{X}\to\mathscr{Y}$ such that $\bar{T}=T^*$.*

Proof. First we show that $T^*:\mathscr{Y}^*\to\mathscr{X}^*$ is continuous. Let y_α^* be a net converging to y^* in the weak* topology of $\mathscr{Y}^*$. We compute

$$[T^*y_\alpha^*,x]=[y_\alpha^*,Tx]\to[y^*,Tx]=[T^*y^*,x].$$

To prove the converse, let $\bar{T}:\mathscr{Y}^*\to\mathscr{X}^*$ be a continuous linear transformation. Thus, given any $x\in\mathscr{X}$, the map $\phi_x:\mathscr{Y}^*\to F$ defined by

$$\phi_x(y^*)=[\bar{T}y^*,x] \tag{10}$$

is a continuous linear functional in the weak* topology of $\mathscr{Y}^*$. By Theorem 3.2, there exists a vector $\bar{y}\in\mathscr{Y}$ such that $[\bar{T}y^*,x]=[y^*,\bar{y}]$. Clearly, $\bar{y}$ depends linearly on x and so we can write $\bar{y}=Tx$ for some linear transformation $T:\mathscr{X}\to\mathscr{Y}$. It follows that

$$[\bar{T}y^*,x]=[y^*,Tx]$$

for all $y^*\in\mathscr{Y}^*$ and $x\in\mathscr{X}$. This implies $\bar{T}=T^*$. □

A central tool in behavior theory, introduced in Fuhrmann [5] is that of a behavior homomorphism. Given two behaviors $\mathscr{B}_1,\mathscr{B}_2$, we define for the backward shift operator σ its restriction to the behaviors by $\sigma^{\mathscr{B}_i}=\sigma|\mathscr{B}_i$. If the behaviors are given in kernel representations $\mathscr{B}_i=\operatorname{Ker}P_i(\sigma)$, we will write also σ^{P_i} for $\sigma^{\mathscr{B}_i}$. A behavior homomorphism is an $F[z]$-homomorphism with respect to the natural $F[z]$-module structure in the behaviors, i.e. it satisfies $Z\sigma^{P_1}=\sigma^{P_2}Z$. Our interest is in the characterization of behavior homomorphisms. It turns out that no general characterization of behavior homomorphisms is available. However, adding some continuity constraints makes the problem tractable by duality theory. We will say that a linear map $Z:\operatorname{Ker}M(\sigma)\to\operatorname{Ker}\bar{M}(\sigma)$ is continuous if it is continuous with respect to the w^* topologies in the two behaviors. Thus we can state:

Theorem 3.4. *Let $M\in F^{p\times m}[z]$ and $\bar{M}\in F^{\bar{p}\times\bar{m}}[z]$ be of full row rank. Then $\operatorname{Ker}M(\sigma)$ is an $F[z]$-submodule of $z^{-1}F^m[[z^{-1}]]$ and $\operatorname{Ker}\bar{M}(\sigma)$ is an $F[z]$-submodule of $z^{-1}F^{\bar{m}}[[z^{-1}]]$. Moreover $\bar{Z}:\operatorname{Ker}M(\sigma)\to\operatorname{Ker}\bar{M}(\sigma)$ is a continuous behavior homomorphism, if and only if there exist $\bar{U}\in F^{\bar{p}\times p}[z]$ and U in $F^{\bar{m}\times m}[z]$ such that*

$$\bar{U}(z)M(z)=\bar{M}(z)U(z) \tag{11}$$

and

$$\bar{Z}h=U(\sigma)h,\quad h\in\operatorname{Ker}M(\sigma). \tag{12}$$

Proof. Assume $\bar{Z}:\operatorname{Ker}M(\sigma)\to\operatorname{Ker}\bar{M}(\sigma)$ is a continuous $F[z]$-homomorphism. For a linear space X and a subspace $\mathscr{V}\subset X$, we have the isomorphism $(X/\mathscr{V})^*\simeq\mathscr{V}^\perp$. We note that

$$(\tilde{M}(z)F^p[z])^\perp=\operatorname{Ker}M(\sigma), \tag{13}$$

and this leads to

$$(F^m[z]/\tilde{M}F^p[z])^* = \operatorname{Ker} M(\sigma), \tag{14}$$

with the duality pairing

$$[h, [f]_{\tilde{M}}] = [h, f]. \tag{15}$$

It is easily checked, using (13), that this is independent of the choice of equivalence class representative. By Theorem 3.3, there exists a map $Z : F^{\bar{m}}[z]/\tilde{\bar{M}}F^{\bar{p}}[z] \to F^m[z]/\tilde{M}F^p[z]$ such that $\bar{Z} = Z^*$. The identity $\bar{Z}S^M = S^{\bar{M}}\bar{Z}$, i.e. $\bar{Z}S^M = S^{\bar{M}}\bar{Z}$, leads to $ZS_{\tilde{\bar{M}}} = S_{\tilde{M}}Z$, that is Z is an $F[z]$-module homomorphism. By Theorem 4.2 in Fuhrmann [5], there exist polynomial matrices $\bar{U} \in F^{\bar{p}\times p}$ and $U \in F^{\bar{m}\times m}$, satisfying $\tilde{U}\tilde{\bar{M}} = \tilde{M}\tilde{\bar{U}}$, which is equivalent to (11), and for which

$$Z[f]_{\tilde{\bar{M}}} = [\tilde{\bar{U}}f]_{\tilde{M}}.$$

We can easily check now that necessarily $\bar{Z}$: $\operatorname{Ker} M(\sigma) \to \operatorname{Ker} \bar{M}(\sigma)$ is given by (12).

Conversely, let $h \in \operatorname{Ker} M(\sigma)$. Then $M(\sigma)(\sigma h) = \sigma(M(\sigma)h) = 0$, i.e. $\sigma h \in \operatorname{Ker} M(\sigma)$ which shows that it is a submodule. Similarly for $\operatorname{Ker} \bar{M}(\sigma)$. Let $\bar{Z}$ be defined by (12), with (11) holding. Then, for $h \in \operatorname{Ker} M(\sigma)$, $\bar{M}(\sigma)\bar{Z}h = \bar{M}(\sigma)(U(\sigma)h) = \bar{U}(\sigma)(M(\sigma)h) = 0$, i.e. $\bar{Z}h \in \operatorname{Ker} \bar{M}(\sigma)$. Moreover, we compute

$$\bar{Z}S^M h = U(\sigma)\sigma h = \sigma U(\sigma)h = S^{\bar{M}}\bar{Z}h,$$

that is $\bar{Z}$ is an $F[z]$-homomorphism. The continuity of $\bar{Z}$ in the weak* topology follows from the equality $[U(\sigma)h, f] = [h, \tilde{U}f]$, holding for all $h \in \operatorname{Ker} M(\sigma)$ and $f \in F^m[z]$. □

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