ELSEVIER Linear Algebra and its Applications 351–352 (2002) 303–380
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

A study of behaviors ☆

P.A. Fuhrmann [1]

Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva, Israel

Received 11 February 2001; accepted 5 February 2002

Submitted by D. Hinrichsen

Abstract

The paper presents the results of the study of behavior theory as developed by J.C. Willems from the point of view of polynomial and rational models. Considering behaviors, in the discrete time case, to be generalizations of rational models, a natural focal point becomes the concept of a behavior homomorphism. We give a characterization of behavior homomorphisms and analyze their invertibility properties in terms of embeddings in unimodular polynomial matrices. These results, which are of intrinsic interest, are then applied to the uniform derivation of a large number of results for equivalence in different classes of behavior representations. To a certain extent, these are generalizations of the strict system equivalence concept for the class of polynomial matrix description of systems in the style of Rosenbrock. A study of behavioral controllability is undertaken and gives some new insights into connections with geometric control theory. © 2002 Published by Elsevier Science Inc.

Keywords: Linear systems; Behaviors; Behavior homomorphism; Strict system equivalence; Polynomial models; Rational models

1. Introduction

As the title indicates, the present paper is an individual, and obviously very subjective, account of the author's experience in the study of Jan Willems' behavioral theory, see Willems [1986,1989,1991] and Polderman and Willems [1997]. Since any two individuals are different, they would look at the same object, be it a work of art, a piece of music, a novel or for that matter a scientific theory from a different, highly personal, perspective. Here we use the word study in a broad sense. We do

☆ Partially supported by GIF under Grant No. I-526-034 and ISF Grant 235/01.
E-mail address: paf@CS.bgu.ac.il (P.A. Fuhrmann).
[1] Earl Katz Family Chair in Algebraic System Theory.

PII: S0024-3795(02)00282-3

not equate study with the learning of all the facts available, but rather incorporating the facts in a broader context that represents the person's *Weltanschauung*.

It is with this in mind that the present work was undertaken. Clearly, even if a complete study and representation of behaviors, after more than two decades of research, was possible, it is certainly beyond the ability of the present author. It would be questionable if a record of the author's struggle to understand behaviors justifies publishing a paper of this length. However, in the course of studying the subject, new insights came up which enable a more coherent and compact description of basic results in behavioral theory. Moreover, the approach taken in this paper not only clarifies many underlying connections to classical system theory, but introduces methods that relate to the circle of ideas in operator theory that centers around the commutant lifting theorem. This connection is not new and, in fact, was introduced by the author, see Fuhrmann [1976], for the study of homomorphisms of polynomial models, their invertibility properties and their use in analyzing equivalence of different system representation. The study of strict system equivalence in Fuhrmann [1977] is a case in point. As a matter of fact the introduction and characterization of behavior homomorphism and their application to a unified derivation of equivalence results for different behavior representations are the principal new results described in this paper.

Behavioral theory is an attempt to present a mathematical framework for the descriptions of dynamical systems from a point of view that is not based on the input/output paradigm. Thus, it is a radical change from the standard linear system theory and its emphasis on a state space point of view. In Willems' definition a *dynamical system* Σ as a triple

$$\Sigma = (T, W, \mathscr{B}), \tag{1}$$

where $T \subset \mathbf{R}$ is the *time axis*, W is an abstract set called the *signal alphabet* and $\mathscr{B} \subset W^{\mathrm{T}}$ is called the *behavior*. The elements of $\mathscr{B}$ are called the *trajectories* of the system. The term behavior can be traced back, in the automata theory context, to Eilenberg [1974]. For a detailed discussion of this, see Willems [1989]. Thus, for us, a system has variables taking values in the set W. The specifics of the system are given in terms of its time behavior, namely the set of all permissible time trajectories. The behavior is thus the result of the underlying laws that govern the dynamical evolution of the system. As such, the definition of a dynamical system is very general and little can be said unless more specific assumptions are made. Since we are interested principally in finite dimensional linear systems, we will restrict the class of behaviors significantly. This work was partially motivated by the applicability of behavioral theory to coding theory, see Rosenthal [2000]. For this as well as for technical reasons, we restrict ourselves to the case of linear discrete-time systems over an arbitrary field F. Moreover, we take the time axis to be $T = \mathbf{Z}_+$. By the assumption of linearity, the signal alphabet is a linear vector space which we identify with F^m. The space of all time trajectories, that is W^{T}, we identify with $z^{-1}F^m[[z^{-1}]]$, the space of all formal power series in z^{-1} with vanishing constant term. The space $z^{-1}F^m[[z^{-1}]]$ has a natural $F[z]$-module structure induced by the *left* or *backward shift operator* S_- or σ defined by

$$S_-h = \sigma h = \pi_- zh, \quad h \in z^{-1}F^m[[z^{-1}]], \tag{2}$$

where π_- is the projection of $F^m((z^{-1}))$ onto $z^{-1}F^m[[z^{-1}]]$ corresponding to the direct sum representation

$$F^m((z^{-1})) = F^m[z] \oplus z^{-1}F^m[[z^{-1}]]. \tag{3}$$

The complementary projection is denoted by π_+. In principle, behaviors are linear, shift invariant subspaces, i.e. $F[z]$-submodules, of $z^{-1}F^m[[z^{-1}]]$. However, this class of submodules is too large and we further restrict it by requiring behaviors to be linear, shift invariant and complete subspaces. In our context, completeness turns out to be a purely algebraic constraint and this will be explained in Section 3. It is equivalent to the existence of an autoregressive (AR), or kernel representation, of the behavior. This is what makes the study of these system managable by algebraic techniques.

As soon as this definition of a behavior is adopted, one has to recall the fundamental insight of Kalman, see Chapter 10 in Kalman et al. [1969], of treating a finite dimensional, linear time invariant system as an $F[z]$-module. Of course, the setting in which Kalman worked was that of input/output descriptions. Indeed, it seems that the principal break of behavioral theory from the classical theory lies in changing the emphasis from input/output maps to either full time or future trajectories. In the Kalman approach to linear systems, realization theory is the corner stone. The realization procedure is based on the restricted i/o map, i.e. a Hankel operator, that maps past inputs to future outputs. In fact, under Nerode type equivalence, the past inputs provide a natural abstract state space. In behavioral theory, as presented in this paper, one looks, to the contrary, at the set of future trajectories. In the case of i/o systems we look at the map from state at time zero and future inputs to future outputs. In principle, all the information on the system structure, up to natural equivalences, should be recoverable from this data, i.e. from future trajectories. The history of the use of spaces of trajectories in the analysis of linear systems predates behavior theory. In particular one should note the contributions of Rosenbrock [1970], Pernebo [1977], Hinrichsen and Prätzel-Wolters [1980a,b], Prätzel-Wolters [1981], Callier and Desoer [1982] and Blomberg and Ylinen [1983].

As far as this author is concerned, the principal insight that was needed to gain a better understanding of behaviors is the fact that a behavior is a generalization of a rational model, see Fuhrmann [1976]. It is easily established that rational models are identical to a subclass of behaviors, specifically to the subclass of autonomous behaviors. Since a principal tool in the study of polynomial and rational models was the characterization of the corresponding model homomorphisms and isomorphisms, it is self-evident that a corresponding study has also to be undertaken in the behavioral setting. Thus, given a behavior $\mathscr{B}$, it is natural to consider the map $\sigma^{\mathscr{B}}$ which is defined as the restriction of the (backward) shift σ to the behavior. Given two behaviors $\mathscr{B}_i$, $i = 1, 2$, a behavior homomorphism is defined to be a map

$Z : \mathscr{B}_1 \to \mathscr{B}_2$ satisfying $Z\sigma^{\mathscr{B}_1} = \sigma^{\mathscr{B}_2} Z$. Thus behavior homomorphisms are intertwining maps and their analysis relate to the celebrated commutant lifting theorem of Sarason and Sz.-Nagy–Foias. Thus it is expected that the method presented in this paper will be found to be applicable in other contexts, most notably in the setting of Hardy spaces. Some of the relevant mathematics for this can be found in Fuhrmann [1981b] and Fuhrmann [1994].

We recall that the approach to the study of equivalence in the setting of polynomial matrix descriptions of linear systems taken in Fuhrmann [1977] is based on the characterization of isomorphism of two polynomial models as derived in Fuhrmann [1976]. The derivation of this result is split into the characterization of all module homomorphisms of two polynomial models and, once this has been established, the characterization of invertibility conditions on the homomorphisms in terms of coprimeness conditions. Our aim in this paper is to adopt this philosophy and apply it to the study of behaviors. The principal insight is the fact that a behavior is a generalization of a rational model, see Fuhrmann [1976]. Thus the homomorphisms of rational models can be easily derived from the characterization of the homomorphism of polynomial models. This gives us a clue to the characterization of behavior homomorphisms which we derive in Section 4.

The paper is structured as follows: Section 2 presents some basic material, mostly on the representation of $F[z]$-submodules of $F^m[z]$, polynomial models, duality, model homomorphisms and the shift realization. This material is a prerequisite for all that follows either because it is actually used or, more importantly, as it serves as a guide to the appropriate extensions. Finally, we recall the shift realization developed by the author in Fuhrmann [1976,1977].

In Section 3 we study the concept of completeness of a submodule of $z^{-1} F^m[[z^{-1}]]$ and use it to derive the kernel representation, due to Willems [1986], of a behavior. We proceed to study subbehaviors in terms of factorization theory and extend some results on sums and intersections of rational models to the behavioral setting. The idea is to study geometry in terms of the arithmetic of, in this case rectangular, polynomial matrices. We proceed to study the concept of a doubly unimodular embedding which turns out to be a principal technical tool in our development.

In Section 4 we fill what seems to be a gap in the behavioral literature by introducing the concept of a behavior homomorphism and, more specifically, that of a behavior isomorphism. These seem to be basic objects and, once they are characterized, the study of the equivalence of different behavior representations is simplified. All the results contained in this section could be easily adapted for the study of integral matrices, i.e. matrices over the ring of integers or, even more generally, to matrices over an arbitrary Euclidean domain. For reasons of readability, we leave the setting as it is.

To study behavior homomorphisms, we first extend the analysis and characterization of the linear maps intertwining two polynomial or rational models and their invertibility properties. We do this by looking at factor modules of the free module of polynomial vectors, that is at factor modules of the form $F^p[z]/MF^m[z]$, with $M(z)$ a $p \times m$ polynomial matrix which, without loss of generality, can be taken to be of full column rank. We characterize the torsion submodule of such a factor module in terms of factorization theory. The $F[z]$-homomorphisms between such factor spaces are described and their invertibility properties characterized in terms of coprimeness conditions and the existence of doubly unimodular embeddings. As a byproduct, we obtain the classical characterization of finitely generated modules over the polynomial ring $F[z]$. By using appropriately duality theory, we extend these results to behavior homomorphisms. As indicated above, all of this is related to lifting theorems, which are the algebraic analogs of the commutant lifting theorem. While we discuss these results for the case of the ring $F[z]$, the results could easily proved for other Euclidean rings or even more generally principal ideal domains. Of particular potential interest is the derivation of these over Z, i.e. a theory of integral matrices.

Section 5 is devoted to the description of various representations of behaviors. We pay special attention to a representation we call a normalized ARMA (NARMA) representation which turns out to be of exceptional importance for the analysis of various classes of representations. This has been studied before in Schumacher [1989] where it is called an AR/MA representation. Its importance lies in the fact that every behavior representation is reducible to a NARMA one. The transformation to first order representations, essentially realization theory, is described via the use of the shift realization. This section is very close in spirit to Schumacher [1989] but differs somewhat in results and techniques.

Section 6 is devoted to the study of the behavioral controllability concept of Willems. We introduce the notion of reachability in the behavioral setting. It turns out to be equivalent to controllability but easier to apply. Moreover, we show the relation of behavioral controllability to the classical controllability characterization of the shift realization as well as to the concept of controllability in geometric control theory. We also discuss briefly the question of stability in the behavioral setting.

In Section 7 we present the principal application to behavioral theory and that is the unified derivation of equivalence results for different behavior representations. The question of equivalence is to find characterization of two system representations which give rise to the same behavior. These problems are not new. The Kalman state space isomorphism, see Kalman et al. [1968], result is of this type. So is Rosenbrock's [1970] notion of strict system equivalence and its modification known as Fuhrmann [1977] system equivalence, see also Kailath [1980] and Özgüler [1994]. In the context of behaviors of particular importance is Hinrichsen and Prätzel-Wolters [1980a,b], the work of Kuijper [1992,1994] and of Schumacher [1989]. In fact, part of the insight for the present work is due to several, highly suggestive, formulas in Kuijper's thesis. More recently, in the context of multidimensional systems, Zerz [2000] as well as Hou et al. [1997] contain similar ideas. The paper by Valcher [2000] may also be relevant. This section is concluded with a study of minimality of several behavior representations. The results are familiar and appear in Kuijper [1992,1994], however the derivations are different inasmuch as they use elementary operations on polynomial matrices rather than state space based iterative algorithms.

Limitations of space and time have excluded several topics from this paper. We have already alluded to the possible extensions of the behavioral theory to the Hardy space context. There is already some work in this direction, see Weiland and Stoorvogel [1997]. However, we feel that the approach taken in the present paper has a great potential usefulness in that context. Moreover, since it is functional oriented, it might be extended to some infinite dimensional situations. Another topic that was not treated sufficiently, but only hinted at, is the study of the connections between behaviors and geometric control theory in the style of Basile and Marro [1973] or Wonham [1979]. We find it surprising that this connection has not been sufficiently addressed in the numerous publications on behaviors. While behaviors can and are introduced in the setting of time trajectories, it is easy to reformulate the problems and study behaviors in polynomial terms. Thus essentially the study of behaviors is reducible to the study of rectangular polynomial matrices arising through AR or ARMA representations of the behavior. Rectangular polynomial matrices appear most naturally in the study of finite dimensional, linear, time invariant systems and they represent numerator matrices in matrix fraction representations or polynomial system matrices in the Rosenbrock [1970] formulation of linear system theory. Incidentally, polynomial system matrices have their own interpretation as representing the zero structure, see Hautus and Fuhrmann [1980] for the details. Thus the study of rectangular polynomial matrices is intimately related to the study of zeros of rational matrices. This in turn is the focal point of geometric control. Another approach to the analysis of zeros of rational matrices is the abstract module theoretic approach initiated by Wyman and Sain [1981] and Wyman et al. [1989]. The link between geometric control theory to polynomial theory has its origin in Emre and Hautus [1980] and Antoulas [1980], with later developments by Fuhrmann and Willems [1979,1980], Fuhrmann [1981], Khargonekar and Emre [1982] as well as the work of Özgüler [1986]. All this body of work is based on the theory of polynomial models introduced in Fuhrmann [1976,1977]. Thus it is to be expected that deep links exist between behavioral theory and geometric control. This will be part of a continuation of the present research. Another topic that has not been addressed in this paper is the study of symmetries in the behavioral setting.

The author would like to thank an anonymous reviewer who pointed out the reference to the seminal paper, Oberst [1990], which resulted in significant simplifications and streamlining of the original exposition.

2. Preliminaries

2.1. Polynomial and rational models

Let F denote an arbitrary field. We will denote by F^m the space of all m-vectors with coordinates in F. $F^m[z]$ the space of all polynomials with coefficients in F^m,

$z^{-1}F^m[[z^{-1}]]$ the space of formal power series vanishing at infinity and $F^m((z^{-1}))$ the space of truncated Laurent series. Let π_+ and π_- denote the projections of $F^m((z^{-1}))$ on $F^m[z]$ and $z^{-1}F^m[[z^{-1}]]$ respectively. Since

$$F^m((z^{-1})) = F^m[z] \oplus z^{-1}F^m[[z^{-1}]], \tag{4}$$

π_+ and π_- are complementary projections. We recall that $U \in F^{m\times m}[z]$ is *unimodular* if it has a polynomial inverse. An element $P \in F^{p\times m}[z]$ is *left prime* if it has a polynomial right inverse. Two polynomial matrices Q, P are *left coprime* if the polynomial matrix $(Q \;\; P)$ is left prime. Right primeness and right coprimeness are analogously defined. We proceed to introduce polynomial and rational models. Given a nonsingular polynomial matrix D in $F^{m\times m}[z]$ we define two projections $\pi_D : F^m[z] \to F^m[z]$ and $\pi^D : z^{-1}F^m[[z^{-1}]] \to z^{-1}F^m[[z^{-1}]]$ by

$$\pi_D f = D\pi_- D^{-1} f \quad \text{for } f \in F^m[z], \tag{5}$$

$$\pi^D h = \pi_- D^{-1}\pi_+ Dh \quad \text{for } h \in z^{-1}F^m[[z^{-1}]], \tag{6}$$

and define two linear subspaces of $F^m[z]$ and $z^{-1}F^m[[z^{-1}]]$ by

$$X_D = \operatorname{Im}\pi_D \tag{7}$$

and

$$X^D = \operatorname{Im}\pi^D. \tag{8}$$

An element f of $F^m[z]$ belongs to X_D if and only if $\pi_+ D^{-1} f = 0$, i.e. if and only if $D^{-1}f$ is a strictly proper rational vector function. Thus we have also the following description of the polynomial model X_D:

$$X_D = \left\{ f \in F^m[z] \mid f = Dh,\; h \in z^{-1}F^m[[z^{-1}]] \right\}. \tag{9}$$

The advantage of this characterization is that it makes sense for an arbitrary $p \times m$ polynomial matrix V. Thus we define the following Emre and Hautus [1980]:

$$X_V = \left\{ f \in F^p[z] \mid f = Vh,\; h \in z^{-1}F^m[[z^{-1}]] \right\}. \tag{10}$$

Analogously, $h \in X^D$ if and only if $\pi_- Dh = 0$, i.e. if and only if h is in the kernel of the Toeplitz map $\mathscr{T}_D : z^{-1}F^m[[z^{-1}]] \to z^{-1}F^m[[z^{-1}]]$ defined by $\mathscr{T}_D h = \pi_- Dh$. We shall also, for reasons of compatibility with behavioral theory usage, write $\sigma = S_-$ and $D(\sigma) = \mathscr{T}_D$.

We shall refer to X_D as a *polynomial model* whereas to X^D as a *rational model*. We turn X_D into an $F[z]$-module by defining

$$p \cdot f = \pi_D pf \quad \text{for } p \in F[z], \quad f \in X_D. \tag{11}$$

Note that $\operatorname{Ker}\pi_D = DF^m[z]$ and that $\pi_D : F^m[z] \to X_D$ is a surjective module homomorphism. Thus we have the important isomorphism

$$X_D \simeq F^m[z]/DF^m[z]. \tag{12}$$

The representation of quotient modules is more general inasmuch as it makes sense for arbitrary submodules. We shall return to this in Section 3. We recall, see Fuhrmann [1976], that a submodule $M \subset F^m[z]$ is a *full submodule* if $F^m[z]/M$ is a torsion module; also, M is full if and only if M has a representation $M = DF^m[z]$ with D nonsingular. One might therefore mimic the construction of polynomial and rational models for the case of nonfull submodules or, equivalently, for the case of rectangular polynomial matrices. Indeed, this can be done and has important implications for behaviors. We will pursue this subject in Section 3. It is easy to check that the set of all rational models coincides with the set of all finitely generated torsion submodules of $z^{-1}F^m[[z^{-1}]]$. For more on this, see Proposition 3.4.

Similarly, we introduce in X^D a module structure by

$$p \cdot h = \pi_- ph \quad \text{for } p \in F[z], \;\; h \in X^D. \tag{13}$$

In X_D we will focus on a special map S_D, a generalization of the classical companion matrix, which corresponds to the action of the identity polynomial z, i.e.

$$S_D f = \pi_D z f \quad \text{for } f \in X_D. \tag{14}$$

It is easily checked that

$$S_D f = zf(z) - D(z)\xi_f, \tag{15}$$

where the constant vector ξ_f depends linearly on f. In fact we have $\xi_f = \pi_+ zD(z)^{-1} f$. It follows from (14) that the module structure in X_D is identical to the module structure induced by S_D through $p \cdot f = p(S_D) f$. With this definition the study of S_D is identical to the study of the module structure of X_D. In particular the invariant subspaces of S_D are just the submodules of X_D which are characterized next. They are related to factorization of polynomial matrices.

Similarly, we introduce in X^D a module structure, given by

$$S_D h = \pi_- z h, \quad h \in X^D, \tag{16}$$

i.e. S^D is the restriction of the backward shift operator to the backward shift invariant subspace X^D.

Polynomial and rational models are closely related. Thus we have the following result obtainable via a trivial computation.

Proposition 2.1. *The polynomial model X_D and the rational model X^D are isomorphic, with the isomorphism $\rho_D : X^D \to X_D$ given by $f \mapsto D^{-1} f$. Moreover we have*

$$S_D \rho_D = \rho_D S^D. \tag{17}$$

The set of all full submodules is a lattice and the set operations are given by the arithmetic of nonsingular polynomial matrices.

Theorem 2.1.
1. *Given nonsingular polynomial matrices $D_1, D_2 \in F^{m\times m}[z]$, then $D_1 F^m[z] \subset D_2 F^m[z]$ if and only if $D_1 = D_2 E$ for some nonsingular polynomial matrix E.*
2. *Given nonsingular polynomial matrices $D_i \in F^{m\times m}[z]$, $i = 1, \ldots, k$, then*

$$\sum_{i=1}^{k} D_i F^m[z] = DF^m[z], \tag{18}$$

where D is a greatest common left divisor (g.c.l.d.) of the D_i.

3. *Given nonsingular polynomial matrices $D_i \in F^{m\times m}[z]$, $i = 1, \dots, k$, then*

$$\bigcap_{i=1}^{k} D_i F^m[z] = EF^m[z], \tag{19}$$

where E is a least common right multiple (l.c.r.m.) of the D_i.

The following theorem, a consequence of the previous one, is of great importance as it connects factorization theory to the geometry of invariant subspaces.

Theorem 2.2.
1. *Let $D \in F^{m\times m}[z]$ be a nonsingular polynomial matrix. A subset M of X_D is a submodule, or equivalently an S_D invariant subspace, if and only if $M = D_1 X_{D_2}$ for some factorization $D = D_1 D_2$ with D_i $m \times m$, necessarily nonsingular, polynomial matrices.*
2. *A subset M of X^D is a submodule, or equivalently an S^D invariant subspace, if and only if $M = X^{D_2}$ for some factorization $D = D_1 D_2$ with D_i $m \times m$, necessarily nonsingular, polynomial matrices.*
3. *A subset M of X^D is a submodule, or equivalently an S^D invariant subspace, if and only if $M = X^{D_1}$ for some factorization $D = D_1 D_2$ with $D_i \in F^{m\times m}[z]$.*

We summarize now the all important connection between the geometry of invariant subspaces and the arithmetic of polynomial matrices. This allows us to make factorization theory one of the cornerstones of algebraic system theory.

Theorem 2.3. *Let M_i, $i = 1, \dots, s$, be submodules of X_D, having the representations $M_i = E_i X_{F_i}$, that correspond to the factorizations*

$$D = E_i F_i.$$

Then the following statements are true:
1. *$M_1 \subset M_2$ if and only if $E_1 = E_2 R$, i.e. if and only if E_2 is a left factor of E_1.*
2. *$\bigcap_{i=1}^{s} M_i$ has the representation $E_\nu X_{F_\nu}$ with E_ν the l.c.r.m. of the E_i and F_ν the g.c.r.d. of the F_i.*
3. *$M_1 + \cdots + M_s$ has the representation $E_\mu X_{F_\mu}$ with E_μ the g.c.l.d. of the E_i and F_μ the l.c.l.m. of all the F_i.*

Corollary 2.1. *Let $D = E_i F_i$ for $i = 1, \dots, s$. Then*
1. *We have*

$$X_D = E_1 X_{F_1} + \cdots + E_s X_{F_s}$$

if and only if the E_i are left coprime.

2. *We have* $\bigcap_{i=1}^{s} E_i X_{F_i} = 0$ *if and only if the* F_i *are right coprime*.
3. *The decomposition*

$$X_D = E_1 X_{F_1} \oplus \cdots \oplus E_s X_{F_s}$$

is a direct sum if and only if $D = E_i F_i$ *for all* i, *the* E_i *are left coprime and the* F_i *are right coprime*.

The following theorem, proved in Fuhrmann [1976], is the algebraic version of the celebrated commutant lifting theorem proved, in the context of operator theory in Hilbert spaces, by Sarason in the scalar case and by Sz.-Nagy and Foias in the general case.

Theorem 2.4. *Let* $D \in F^{p\times m}[z]$ *and* $\overline{D} \in F^{\overline{p}\times\overline{m}}[z]$ *be nonsingular. Then* $Z : X_D \to X_{\overline{D}}$ *is an* $F[z]$*-homomorphism if and only if there exist* $\overline{N} \in F^{\overline{p}\times p}$ *and* $N \in F^{\overline{m}\times m}[z]$ *such that*

$$\overline{N} D = \overline{D} N \tag{20}$$

and

$$Zf = \pi_{\overline{D}} \overline{N} f. \tag{21}$$

The following theorem, proved in Fuhrmann [1976], characterizes the invertibility properties of $F[z]$-module homomorphisms between polynomial models. In turn, it is based on operator theoretic results, see Fuhrmann [1981b] and the further references therein.

Theorem 2.5. *Let* $Z : X_D \to X_{D_1}$ *be the module homomorphism defined by*

$$Zf = \pi_{\overline{D}} \overline{N} f \tag{22}$$

with

$$\overline{N} D = \overline{D} N \tag{23}$$

holding. *Then*:
1. Ker $Z = EX_G$, *where* $D = EG$ *and* G *is a g.c.r.d. of* D *and* N.
2. Im $Z = E_1 X_{G_1}$, *where* $\overline{D} = E_1 G_1$ *and* E_1 *is a g.c.l.d. of* $\overline{D}$ *and* $\overline{N}$.
3. Z *is invertible if and only if* D *and* N *are right coprime and* $\overline{D}$ *and* $\overline{N}$ *are left coprime*.

Both Theorems 2.4 and 2.5 will be generalized in Section 3. The isomorphism between polynomial and rational models proved in Proposition 2.1 allows us to translate the content of Theorems 2.4 and 2.5 to the rational model context. Thus we have:

Theorem 2.6. *Let $D \in F^{m\times m}[z]$ and $\overline{D} \in F^{\overline{m}\times\overline{m}}[z]$ be nonsingular. Then $Z : X^D \to X^{\overline{D}}$ is an $F[z]$-homomorphism if and only if there exist $\overline{N} \in F^{\overline{p}\times p}$ and $N \in F^{\overline{m}\times m}[z]$ such that*

$$\overline{N}D = \overline{D}N \tag{24}$$

and

$$Zh = \pi_- Nh, \quad h \in X^D. \tag{25}$$

Theorem 2.7. *Let $Z : X^D \to X^{\overline{D}}$ be the module homomorphism defined by* (25), *with condition* (24) *satisfied. Then*:

1. Ker $Z = X^G$, *where $D = EG$ and G is a g.c.r.d. of D and M.*
2. Im $Z = X^{\overline{G}}$, *where $\overline{D} = E\overline{G}$ and E is a g.c.l.d. of $\overline{D}$ and N.*
3. *Z is invertible if and only if D and N are right coprime and $\overline{D}$ and $\overline{N}$ are left coprime.*

2.2. Duality

In this section we review basic duality results as developed in Fuhrmann [1981]. These are crucial for the study of behaviors. Given a vector space V over a field F we denote by V^* the *dual space* of V that is the space of linear functionals on V. Given $v^* \in V^*$ and $v \in V$ we will write

$$[v, v^*] = v^*(v).$$

In the special case of $V = F^m$ we can also identify V^* with F^m and then we write $[x, y] = \tilde{y}x$ where $\tilde{y}$ denotes the transpose of the column vector y. The sole exception will be the complex inner product spaces where $[x, y]$ will be interpreted as the inner product itself. Now given $f \in F^m((z^{-1}))$ and $g \in F^m((z^{-1}))$ we define a pairing

$$[f, g] = \sum_{j=-\infty}^{\infty} [f_j, g_{-j-1}]. \tag{26}$$

It is clear that $[\cdot,\cdot]$ is a bilinear form on $F^m((z^{-1})) \times F^m((z^{-1}))$. It is well defined as in the defining sum at most a finite number of terms are nonzero. Also this form is *nondegenerate* in the sense that $[f, g] = 0$ for all $g \in F^m((z^{-1}))$ if and only if $f = 0$. Given a subset $M \subset V$ we define its *annihilator* $M^\perp$ by

$$M^\perp = \left\{v^* \in V^* \mid [m, v^*] = 0 \ \forall m \in M\right\}.$$

Similarly if $M \subset V^*$ we define the *preannihilator* $^\perp M$ by

$$^\perp M = \left\{v \in V \mid [v, v^*] = 0 \ \forall v^* \in M\right\}.$$

It is a simple check of the definitions that $F^m[z]^\perp = F^m[z]$ and $^\perp(F^m[z]) = F^m[z]$.

Theorem 2.8. *The dual space of $F^m[z]$ is $z^{-1}F^m[[z^{-1}]]$.*

Proof. Clearly every element $h \in z^{-1}F^m[[z^{-1}]]$ defines, by way of the previous pairing (26), a linear functional on $F^m[z]$. Conversely let Φ be a linear functional on $F^m[z]$. It induces linear functionals ϕ_i on F^m by defining

$$\phi_i(x) = [z^i x, \phi] = \Phi(z^i x)$$

and an element $h \in z^{-1}F^m[[z^{-1}]]$ is defined by letting $h(z) = \sum_{j=0}^{\infty} \phi_j z^{-j-1}$. It follows that $\Phi(f) = [f, h]$. □

Throughout the paper, given a matrix A, we will denote by $\tilde{A}$ its transpose. The same holds for polynomial matrices or, more generally, to elements of $F^{p\times m}((z^{-1}))$. Thus given $A \in F^{p\times m}((z^{-1}))$ with $A(z) = \sum_{j=-\infty}^{n} A_j z^j$ we will denote by $\tilde{A}$ the element of $F^{m\times p}((z^{-1}))$ given by

$$\tilde{A}(z) = \sum_{j=-\infty}^{n} \tilde{A}_j z^j.$$

As usual, given bilinear forms on $V \times V^*$ and $W \times W^*$ and a map $A : V \to W$ the *dual map* $A^* : W^* \to V^*$ is defined by the equality

$$[Av, w^*] = [v, A^* w^*].$$

In the following proposition we summarize, without proofs, the computational rules related to the duality defined by (26). For the full details, see Fuhrmann [1981a].

Proposition 2.2.

1. *Given $A \in F^{p\times m}((z^{-1}))$. Let $L_A : F^m((z^{-1})) \to F^p((z^{-1})$ be the corresponding Laurent operator defined by*

$$(L_A f)(z) = A(z) f(z) = \sum g_j z^j, \tag{27}$$

where $g_j = \sum_{i=-\infty}^{\infty} A_{j-i} f_i$. Then

$$(L_A)^* = L_{\tilde{A}}. \tag{28}$$

2. *The duals of the projections π_+ and π_- are given by*

$$\pi_+^* = \pi_-, \quad \pi_-^* = \pi_+. \tag{29}$$

3. *$F^m[z]$ is a submodule, relative to the ring $F[z]$, of $F^m((z^{-1}))$ then $F^m[z]$ is S-invariant thus we can define S_+ by*

$$S_+ = S|F^m[z].$$

We also define $S_- : z^{-1}F^m[[z^{-1}]] \to z^{-1}F^m[[z^{-1}]]$ by

$$S_- h = \pi_- z h.$$

4. *The dual of the map $S_+ : F^m[z] \to F^m[z]$ is given by $S_+^* = S_- : z^{-1}F^m[[z^{-1}]] \to z^{-1}F^m[[z^{-1}]]$.*

5. *Let $M \subset F^m[z]$ be a submodule. Then $M^\perp$ is a submodule of $z^{-1}F^m[[z^{-1}]]$.*
6. *Let $M = DF^m[z]$ with $D \in F^{m\times m}[z]$ nonsingular. Then $M^\perp = X^{\tilde{D}}$.*
7. *The disjoint of π_D is $\pi^{\tilde{D}}$.*

By theorem 2.8, $(F^m[z])^*$ is $z^{-1}F^m[[z^{-1}]]$. However, $F^m[z]$ is not a reflexive space. Thus Proposition 2.2.5 is true in one direction only. Purely on mathematical grounds, it is of great interest to characterize those submodels of $z^{-1}F^m[[z^{-1}]]$ that are annihilators of submodels of $F^m[z]$. This will be done in Section 3. It is a pleasant surprise that these submodels coincide, in our context, with behaviors in the sense of Willems. Thus, they have important system theoretic significance.

In due course, this allows us to analyze set operations on behaviors by reduction to the corresponding analysis of set operations on submodules of $F^m[z]$ and these in turn reduce to the arithmetic of polynomial matrices. These duality results can be traced to Fuhrmann [1981]. A detailed analysis of a related duality theory, in the context of multidimensional systems, is available in the major paper Oberst [1990].

In our characterization of torsion quotient modules of $F^m[z]$ and of finitely generated torsion submodules of $z^{-1}F^m[[z^{-1}]]$ there were many similarities. This is not coincidental but is related to a further study of duality in the functional model setting. With our identification of $z^{-1}F^m[[z^{-1}]]$ as the dual of $F^m[z]$ it follows that if M is a subset of $F^m[z]$ then $M^\perp$ is a subset of $z^{-1}F^m[[z^{-1}]]$.

2.3. The shift realization

We recall now the shift realization introduced in Fuhrmann [1976,1977]. Assume a proper rational function G is given by

$$G(z) = V(z)T(z)^{-1}U(z) + W(z), \tag{30}$$

where V, T, U, W are appropriately sized polynomial matrices and T is nonsingular. This representation is the cornerstone of Rosenbrock's theory. To this representation we associate, following Rosenbrock [1970], the polynomial system matrix

$$\mathscr{P} = \begin{pmatrix} T & -U \\ V & W \end{pmatrix}. \tag{31}$$

To the polynomial system matrix (31), we associate the state space X_T and the maps

$$\begin{aligned} Af &= S_T f = \pi_T z f, \quad f \in X_T, \\ B\eta &= \pi_D U\eta, \quad \eta \in F^m, \\ Cf &= (VT^{-1}f)_{-1}, \\ D &= \pi_+ G. \end{aligned} \tag{32}$$

It was established in Fuhrmann [1977] that this is a realization of G, called the *shift realization*. This realization is controllable (reachable if we consider discrete time) if and only if T, U are left coprime and it is observable if and only if V, T are right coprime.

From any minimal realization we can easily construct a basis for the model space associated with a right coprime factorization. This extremely useful construction goes back to Hautus and Heymann [1978], see also Wimmer [1979]. We omit the proof.

Theorem 2.9. *Let G be a proper rational function of McMillan degree n and let*

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{33}$$

be a minimal realization. Then $C(zI - A)^{-1} = T(z)^{-1}H(z)$ for some polynomial matrices T and H. Defining $U(z) = H(z)B$ we have

$$G = D + T^{-1}U = T^{-1}(TD + U). \tag{34}$$

A basis for X_T is given by the columns H_i of H and a basis for X^T is given by the columns of $C(zI - A)^{-1}$. In particular, we have

$$X^T = \left\{C(zI - A)^{-1}\xi \mid \xi \in R^n\right\}. \tag{35}$$

As a result we conclude that, given any polynomial matrix N, then $T^{-1}N$ is strictly proper if and only if there exists a constant matrix K for which $N(z) = H(z)K$.

2.4. Bits of geometric control

As noted in Section 1, there are deeper connections between behavior theory and geometric control. This topic is beyond the scope of this paper. However, we will indicate in the sequel some of these connections. To this end we will need some results concerning the polynomial model approach to geometric control. This line of research originated in Emre and Hautus [1980] and continued in Fuhrmann and Willems [1980], Fuhrmann [1981] and Khargonekar and Emre [1982]. Very relevant to this topic is also Özgüler [1986].

We recall that, given a state space system with transfer function

$$G(z) = \left(\begin{array}{c|c} A & B \\ \hline c & D \end{array}\right),$$

then a subspace $\mathscr{V}$ of the state space $\mathscr{X}$ is called *controlled invariant* if, for some feedback map K, we have $(A + BK)\mathscr{V} \subset \mathscr{V}$. It is an *output nulling controlled invariant subspace* if, for some feedback map K, we have $(A + BK)\mathscr{V} \subset \mathscr{V} \subset \operatorname{Ker}(C + DK)$. Finally, a subspace $\mathscr{R} \subset \mathscr{X}$ is an *output nulling reachability subspace* if it is an output nulling controlled invariant subspace and satisfies, for some feedback map K, $\mathscr{R} = \langle A + BK | \mathscr{B} \cap \mathscr{R}\rangle$. It is established in geometric control theory that, given a state space system, there exist unique maximal output nulling controlled invariant and maximal output nulling reachability subspaces which are denoted by $\mathscr{V}^*$ and $\mathscr{R}^*$ respectively. The following theorem presents the relevant characterizations.

Theorem 2.10. *Let $G = Q^{-1}P$ be a $p \times m$ proper rational function. Then, with respect to the associated shift realization* (32), *we have*

$$\begin{aligned} \mathscr{V}^* &= X_P, \\ \mathscr{R}^* &= X_P \cap PF^m[z]. \end{aligned} \tag{36}$$

If G, and hence P, has full row rank then $\mathscr{V}^* = \mathscr{R}^*$ if and only if P is left prime.

For the case of a strictly proper rational function G the characterization of $\mathscr{V}^*$ is due to Emre and Hautus [1980] and Fuhrmann and Willems [1980]. The characterization of $\mathscr{R}^*$, in a slightly different formulation, is due to Fuhrmann [1981] and Khargonekar and Emre [1982]. The full proof of these, and related results, will be published elsewhere.

3. Elements of behavior theory

The object of this section is to present the basics of behavior theory in the setting of discrete time systems. We give the definition of dynamical systems and behaviors as used in Willems [1991]. In this setting the notion of completeness can be addressed purely from the algebraic point of view. This we do and rederive the kernel representation of behaviors. This is a key result in behavioral theory inasmuch as it allows to reformulate the problems and study behaviors in polynomial terms. Thus essentially the study of behaviors is reducible to the study of rectangular polynomial matrices arising through a kernel, or AR, representation of the behavior. We proceed to the study of subbehaviors and their connection to factorization theory. This is an extension of the fact that in the theory of polynomial and rational models invariant subspaces relate to factorizations. Next we proceed to introduce and study doubly unimodular embeddings. This is an important technical subject that is used throughout the rest of the paper for studying the invertibility properties of behavior homomorphisms. We conclude with some more results on factorizations of polynomial matrices and behaviors.

The behavioral approach differs from the classical approach, dominated by Kalman's ideas, see Chapter 10 in Kalman et al. [1969], in changing the emphasis from input/output maps to either full time or future trajectories. In the Kalman approach to linear systems, realization theory is the corner stone. The realization procedure is based on the restricted i/o map, i.e. a Hankel operator, that maps past inputs to future outputs. In fact, under Nerode type equivalence, the past inputs provide a natural abstract state space. In behavior theory to the contrary one looks at the set of future trajectories. In the case of i/o systems we look at the map from state at time zero and future inputs to future outputs. In principle, all the information on the system should be recoverable from this data. The history of the use of spaces of trajectories in the analysis of linear systems predates behavior theory. In particular one should note the contribution of Rosenbrock [1970], Blomberg and Ylinen [1983], Pernebo [1977], Hinrichsen and Prätzel-Wolters [1980a,b] and Prätzel-Wolters [1981].

We follow Willems [1991] in defining a *dynamical system* Σ as a triple

$$\Sigma = (T, W, \mathscr{B}), \tag{37}$$

where $T \subset \mathbf{R}$ is the *time axis*, W is an abstract set called the *signal alphabet* and $\mathscr{B} \subset W^{\mathrm{T}}$ is called the *behavior*. The elements of $\mathscr{B}$ are called the *trajectories* of the system. In this generality the definition has its origin in automata theory, see Eilenberg [1974].

This definition is very general and is representation free. In the context of this paper we will identify T with $\mathbf{Z}_+$, the set of positive integers, assume F is an arbitrary field and take $W = F^m$. We identify W^{T} with $z^{-1}F^m[[z^{-1}]]$. The space $z^{-1}F^m[[z^{-1}]]$ has a standard $F[z]$-module structure induced by the *left* or *backward shift operator* S_- or σ defined by

$$S_-h = \sigma h = \pi_- zh, \quad h \in z^{-1}F^m[[z^{-1}]]. \tag{38}$$

Recall that π_- is the projection of $F^m((z^{-1}))$ onto $z^{-1}F^m[[z^{-1}]]$ corresponding to the direct sum representation

$$F^m((z^{-1})) = F^m[z] \oplus z^{-1}F^m[[z^{-1}]] \tag{39}$$

and that the complementary projection is denoted by π_+.

Given a polynomial matrix $P(z) \in F^{p\times m}[z]$, it defines a Toeplitz map $\mathscr{T}_P$, usually denoted in the behavior literature as $P(\sigma)$, as the map $\mathscr{T}_P = P(\sigma) : z^{-1}F^m[[z^{-1}]] \to z^{-1}F^p[[z^{-1}]]$ via

$$\mathscr{T}_P h = P(\sigma)h = \pi_- Ph, \quad h \in z^{-1}F^m[[z^{-1}]]. \tag{40}$$

Clearly the operators of the form $P(\sigma)$ are a special class of Toeplitz operator and it is their kernels that are of interest to us. In fact we would like to characterize those subspaces of $z^{-1}F^p[[z^{-1}]]$ that are representable in the form $\operatorname{Ker} P(\sigma)$ for some polynomial matrix $P(z)$. This *kernel representation*, due to Willems [1986], is the key result for the study of behaviors. In what follows we shall describe a purely algebraic approach to this representation result. To this end, let X be a linear vector space over an arbitrary field F and let X^* be its algebraic dual. Given a subspace $M \subset X$, we denote by $M^\perp$ its *annihilator*, i.e.

$$M^\perp = \left\{h \in X^* | h|M = 0\right\}. \tag{41}$$

Similarly, given a subspace $V \subset X^*$, we denote by ${}^\perp V$ its *preannihilator*, i.e.

$${}^\perp V = \left\{x \in X | h(x) = 0 \ \forall h \in V\right\}. \tag{42}$$

We have the following characterization of preannihilators.

Proposition 3.1. *Let X be a linear vector space over a field F and let X^* be its algebraic dual. Then a subspace $V \subset X^*$ satisfies*

$$({}^\perp V)^\perp = V \tag{43}$$

if and only if for any $h_0 \notin V$ there exists an $x \in {}^\perp V$ such that $h_0(x) \neq 0$.

Proof. We prove both implications by contradiction.

Assume $({}^{\perp}V)^{\perp} = V$ holds but the other condition is not satisfied, i.e. there exists $h_0 \notin V$ such that for all $x \in {}^{\perp}V$ we have $h_0(x) = 0$. This implies $h_0 \in ({}^{\perp}V)^{\perp} = V$ which is a contradiction.

Conversely, assume $({}^{\perp}V)^{\perp} \neq V$. Since clearly $V \subset ({}^{\perp}V)^{\perp}$, it follows that there exists $h_0 \in ({}^{\perp}V)^{\perp} - V$ such that for all $x \in {}^{\perp}V$ we have $h_0(x) = 0$. Again we have obtained a contradiction. □

Since submodules of the space $F^m[z]$ of vector polynomial are well studied and have a nice representation in terms of polynomial matrices, it leads immediately to a nice representation of those submodules of $z^{-1}F^m[[z^{-1}]]$ that are annihilators of submodules of $F^m[z]$.

As an $F[z]$ module, the space $z^{-1}F^m[[z^{-1}]]$ has a multitude of submodules, i.e. linear, shift invariant subspaces. In this class we single out a special, small, subclass which is determined by the extra property of completeness.

Definition 3.1. In $z^{-1}F^m[[z^{-1}]]$ we define the projections P_n, $n \in \mathbf{Z}_+$, by

$$P_n \sum_{i=1}^{\infty} \frac{h_i}{z^i} = \sum_{i=1}^{n} \frac{h_i}{z^i}. \tag{44}$$

We say that a subset $\mathscr{B} \subset z^{-1}F^m[[z^{-1}]]$ is *complete* if for any $w = \sum_{i=1}^{\infty} w_i z^{-i} \in z^{-1}F^m[[z^{-1}]]$ and for each positive integer N, $P_N w \in P_N(\mathscr{B})$ implies $w \in \mathscr{B}$. A *behavior* in our context is defined as a linear, shift invariant and complete subspace of $z^{-1}F^m[[z^{-1}]]$.

Proposition 3.2. *Let F be a field and let $V \subset z^{-1}F^m[[z^{-1}]]$ be a subspace. Then V is complete if and only if*

$$({}^{\perp}V)^{\perp} = V. \tag{45}$$

Proof. Assume V is a complete subspace of $z^{-1}F^m[[z^{-1}]]$. Let $h \in z^{-1}F^m[[z^{-1}]] - V$. With the projections defined in (44), it is clear that $P_N(V)$ is a finite dimensional vector space, with $\dim P_N(V) \leqslant mN$. Since $h \notin V$ and V is assumed complete, there exists an index N_0 for which $P_{N_0}h \notin P_{N_0}(V)$. By elementary linear algebra, there exists a linear functional ϕ on $P_{N_0}(V)$ such that $\phi | P_{N_0}(V) = 0$ and $\phi(P_{N_0}h) \neq 0$. Extending the definition of ϕ to $P_{N_0}(z^{-1}F^m[[z^{-1}]])$ it is clear that we can identify ϕ with a polynomial vector $f \in P_{N_0}[z] \subset F^m[z]$. Thus we have $f \in {}^{\perp}V$ and $h(f) \neq 0$. Applying Proposition 3.1 we conclude that (45) holds. □

Propositions 3.1 and 3.2 are due to Fonf [2000]. The principal characterization of behaviors, due to Willems [1986], is now an easy corollary.

Theorem 3.1. *A subset $\mathscr{B} \subset z^{-1}F^m[[z^{-1}]]$ is a behavior if and only if it admits a* kernel representation, *i.e. there exists a $p \times m$ polynomial matrix $P(z)$ for which*

$$\mathscr{B} = \operatorname{Ker} P(\sigma) = \left\{h \in z^{-1}F^m[[z^{-1}]] \mid \pi_- P h = P(\sigma)h = 0\right\}. \tag{46}$$

Proof. Let $\mathscr{B} \subset z^{-1}F^m[[z^{-1}]]$ be a behavior. By completeness we have $({}^{\perp}\mathscr{B})^{\perp} = \mathscr{B}$. Clearly, ${}^{\perp}\mathscr{B}$ is a submodule of $F^m[z]$, hence of the form ${}^{\perp}\mathscr{B} = QF^m[z]$ for some polynomial matrix $Q(z)$. An elementary calculation yields $\mathscr{B} = (QF^m[z])^{\perp} = \operatorname{Ker} \tilde{Q}(\sigma)$ and we set $P(z) = \tilde{Q}(z)$.

Conversely, assuming $\mathscr{B} = \operatorname{Ker} P(\sigma)$. Clearly, $\mathscr{B}$ is linear and shift invariant. Moreover, we have ${}^{\perp}\mathscr{B} = \tilde{P}F^p[z]$ and hence $({}^{\perp}\mathscr{B})^{\perp} = \operatorname{Ker} P(\sigma) = \mathscr{B}$. By Proposition 3.2, $\mathscr{B}$ is complete. □

Since, given two submodules $M \subset N \subset F^m[z]$, we clearly have $N^{\perp} \subset M^{\perp}$, we can state the following all important result due, in slightly different form and in the multidimensional setting, to Oberst [1990].

Theorem 3.2. *The mapping $M \mapsto M^{\perp}$ establishes a bijective, inclusion reversing correspondence between submodules of $F^m[z]$ and behaviors in $z^{-1}F^m[[z^{-1}]]$.*

Theorem 3.2 is a key result inasmuch as it allows us to study behaviors in $z^{-1}F^m[[z^{-1}]]$ by studying submodules of $F^m[z]$. As a first step, we study the full lattice of submodules of $F^m[z]$ and its relation to the factorizations of rectangular polynomial matrices. This is a generalization of Theorem 2.1 which covered only the case of full submodules. Thus, omitting the standard proof, we have:

Theorem 3.3.

1. *Any submodule $M \subset F^m[z]$ has a representation of the form*

$$M = PF^l[z] \tag{47}$$

for some polynomial matrix $P \in F^{m\times l}[z]$ of full column rank. $P(z)$ is uniquely defined up to a right unimodular factor.

2. *For $P \in F^{m\times l}[z]$, we have*

$$PF^l[z] = F^m[z] \tag{48}$$

if and only if P is left prime.

3. *Given two submodules $M_i = P_iF^{l_i}[z]$, $i = 1, 2$, with $P_i \in F^{m\times l_i}[z]$ of full column rank, then $M_1 \subset M_2$ if and only if*

$$P_1 = P_2 Q \tag{49}$$

for some polynomial matrix $Q \in F^{l_2\times l_1}[z]$, i.e. P_2 is a left factor, or divisor, of P_1.

4. *Given submodules* $M_i = P_i F^{l_i}[z] \subset F^m[z]$, $i = 1, \dots k$, *then*

$$\bigcap_{i=1}^{k} M_i = P F^l[z], \tag{50}$$

where P is a greatest common left divisor (*g.c.l.d.*) *of the* P_i.

5. *Given submodules* $M_i = P_i F^{l_i}[z] \subset F^m[z]$, $i = 1, \dots, k$, *then*

$$\sum_{i=1}^{k} M_i = P F^l[z], \tag{51}$$

where P is a least common right multiple (*l.c.r.m.*) *of the* P_i.

Given a $k \times m$ polynomial matrix $P(z)$, then $\operatorname{Ker} P = \{f \in F^m[z] | Pf = 0\} \subset F^m[z]$ is clearly a submodule and hence has also an image representation $\operatorname{Ker} P = \operatorname{Im} Q$ for an essentially unique polynomial matrix Q of full column rank. However, not every submodule of $F^m[z]$ has a kernel representation. The following proposition characterizes submodules having a kernel representation. As we shall see later, this characterization is dual to the characterization of controllable behaviors.

Proposition 3.3.

1. *Let* $P \in F^{k \times m}[z]$ *and let* $E \in F^{k \times k}[z]$ *be nonsingular*. *Then*

$$\operatorname{Ker} E(z)P(z) = \operatorname{Ker} P(z). \tag{52}$$

2. *Given* $P \in F^{k \times m}[z]$. *Then* $\operatorname{Ker} P$ *has a representation of the form* $\operatorname{Ker} \overline{P}$ *with* $\overline{P} \in F^{\overline{k} \times m}[z]$ *left prime*.
3. *A submodule* $M = QF^l[z] \subset F^m[z]$ *with* $Q \in F^{m \times l}[z]$ *of full column rank has a kernel representation* $M = \operatorname{Ker} P$ *if and only if Q is right prime*.
4. *Given two left prime polynomial matrices* $P_i \in F^{k_1 \times m}[z]$. *Then*

$$\operatorname{Ker} P_1(z) \subset \operatorname{Ker} P_2(z) \tag{53}$$

if and only if $P_2(z) = A(z)P_1(z)$ *for some*, *necessarily left prime*, $A \in F^{k_2 \times k_1}[z]$. *Under the same assumptions*, *we have*

$$\operatorname{Ker} P_1(z) = \operatorname{Ker} P_2(z) \tag{54}$$

if and only if $P_2(z) = A(z)P_1(z)$ *for some*, *necessarily unimodular*, $A \in F^{k \times k}[z]$, *with* $k = k_1 = k_2$.

5. $\operatorname{Ker} P(z) = 0$ *if and only if* $P(z)$ *has full column rank*.

Proof.

1. Clear.
2. Let U be a unimodular polynomial matrix for which $UP = \binom{\hat{P}}{0}$ with $\hat{p}$ of full row rank. This shows $\operatorname{Ker} UP = \operatorname{Ker} P = \operatorname{Ker} \hat{P}$. Factor now $\hat{P} = E\overline{P}$ with E nonsingular and $\overline{P}$ left prime. By Part 1, we have $\operatorname{Ker} \hat{P} = \operatorname{Ker} \overline{P}$.

3. Assume that $M = QF^l[z]$ and that Q is right prime. Extend Q to a unimodular polynomial matrix $U = (Q \ \ \overline{Q})$ and set $U^{-1} = \binom{\overline{P}}{P}$. Then
$$\begin{pmatrix} \overline{P} \\ P \end{pmatrix} \begin{pmatrix} Q & \overline{Q} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$
and in particular $PQ = 0$, i.e. $\operatorname{Ker} P \supset QF^l[z]$. To prove the converse inclusion, assume $f \in \operatorname{Ker} P$. Then $f = Qf_1 + \overline{Q} f_2$ and hence $Pf = 0 = PQf_1 + P\overline{Q} f_2 = f_2$. So $f = Qf_1 \in \operatorname{Im} Q$ and we get $\operatorname{Ker} P \subset QF^l[z]$.
4. The factorization $P_2(z) = A(z)P_1(z)$ clearly implies the inclusion (53).
Conversely, the left primeness of the P_i implies the existence of unimodular completions $\binom{\overline{P}_i}{P_i}$. We let
$$\begin{pmatrix} Q_i & \overline{Q}_i \end{pmatrix} = \begin{pmatrix} \overline{P}_i \\ P_i \end{pmatrix}^{-1}.$$
Thus we have
$$\begin{pmatrix} \overline{P}_i \\ P_i \end{pmatrix} \begin{pmatrix} Q_i & \overline{Q}_i \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$
and hence also
$$\begin{pmatrix} \tilde{Q}_i \\ \tilde{\overline{Q}}_i \end{pmatrix} \begin{pmatrix} \tilde{\overline{P}}_i & \tilde{P}_i \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$
Since $\operatorname{Ker} P_i = \operatorname{Im} Q_i$, the inclusion (53) is equivalent to $\operatorname{Im} Q_i \subset \operatorname{Im} Q_2$ and hence, by Theorem 3.3, there exists a polynomial matrix T such that $Q_1 = Q_2 T$. Now the transposed unimodular product yields $\operatorname{Ker} \tilde{Q}_i = \operatorname{Im} \tilde{P}_i$. We also have $\tilde{Q}_1 = \tilde{T}\tilde{Q}_2$, and hence the inclusion $\operatorname{Ker} \tilde{Q}_2 \subset \operatorname{Ker} \tilde{Q}_1$. Thus in turn implies $\operatorname{Im} \tilde{P}_2 \subset \operatorname{Im} \tilde{P}_1$. Thus there exists an appropriately sized polynomial matrix A for which $\tilde{P}_2 = \tilde{P}_1 \tilde{A}$. Transposing the last equality leads to $P_2 = AP_1$.
The equality (54) is equivalent to the existence of A, B for which $P_2(z) = A(z)P_1(z)$ and $P_1(z) = B(z)P_2(z)$. Thus we have $P_2(z) = A(z)B(z)P_2(z)$, which by the left primeness of P_2 implies $A(z)B(z) = I$. That $B(z)A(z) = I$ follows by a similar argument.
5. This is immediate.

An important subclass of behaviors arises when we restrict the polynomial matrix in a kernel representation to be nonsingular. Following Willems, we say that a behavior $\mathscr{B}$ is *autonomous* if it is finite dimensional as a vector space over F. We have the following.

Proposition 3.4. *The following statements are equivalent*:
1. *The behavior $\mathscr{B}$ is autonomous*.
2. $\mathscr{B} = \operatorname{Ker} D(\sigma)$ *for some nonsingular polynomial matrix* $D(z)$.
3. $\mathscr{B}$ *is equal to the rational model* X^D.

4. *There exists an observable pair (C, A) for which*
$$\mathscr{B} = \left\{C(zI - A)^{-1}\xi \mid \xi \in F^n\right\}. \tag{55}$$

We omit the details of the proof.

3.1. Subbehaviors

Central results in the polynomial approach to the study of linear transformations and linear systems are the representation of submodules to the free module $F^p[z]$ and the transformation of the analysis of the lattice of submodules to the arithmetic of factorizations of polynomial matrices. Since there is, via duality theory as in Theorem 3.1, a bijective correspondence between behaviors in $z^{-1}F^m[[z^{-1}]]$ and submodules of the free module $F^p[z]$, we expect to use this correspondence for the study of the lattice of subbehaviors of a given behavior and relate it to factorizations. This we proceed to do, and we begin by defining subbehaviors.

We begin by defining subbehaviors.

Definition 3.2. A subset $\mathscr{B}_0 \subset \mathscr{B}$ is called a *subbehavior* if it is itself a behavior, i.e. it is linear, shift invariant and complete.

We wish to point out that not every linear, shift invariant subspace of a behavior is a subbehavior. Completeness is neccessary.

We can apply now Theorem 3.3 to the analysis of the lattice structure of behaviors, using duality as expressed in Theorem 3.2. As for submodules of $F^m[z]$, the geometric structure is reduced to the factorization theory of rectangular polynomial matrices.

Theorem 3.4.
1. *Given two behaviors $\mathscr{B}_1, \mathscr{B}_2 \subset z^{-1}F^m[[z^{-1}]]$ in kernel representations $\mathscr{B}_i =$ $\operatorname{Ker} P_i(\sigma)$. Then $\operatorname{Ker} P_1(\sigma) \subset \operatorname{Ker} P_2(\sigma)$ if and only if, for some polynomial matrix $Q(z)$, we have $P_2(z) = Q(z)P_1(z)$.*
2. *If P_i have full row rank, then $\operatorname{Ker} P_1(\sigma) = \operatorname{Ker} P_2(\sigma)$ if and only if $P_2(z) = U(z)P_1(z)$ for some unimodular polynomial matrix U.*
3. *$P \in F^{p\times m}[z]$ is right prime if and only if $\operatorname{Ker} P(\sigma) = 0$.*
4. *Given behaviors $\mathscr{B}_i \subset z^{-1}F^m[[z^{-1}]]$ in kernal representations $\mathscr{B}_i = \operatorname{Ker} P_i(\sigma)$, then $\sum_{i=1}^k \mathscr{B}_i$ is a behavior and has a kernal representation*
$$\sum_{i=1}^k \mathscr{B}_i = \operatorname{Ker} R(\sigma), \tag{56}$$
where R is a l.c.l.m of the P_i.
5. *Given behaviors $\mathscr{B}_i \subset z^{-1}F^m[[z^{-1}]]$ in kernel representations $\mathscr{B}_i = \operatorname{Ker} P_i(\sigma)$, then $\bigcap_{i=1}^k \mathscr{B}_i$ is a behavior and has a kernel representation*

$$\bigcap_{i=1}^{k} \mathscr{B}_i = \operatorname{Ker} Q(\sigma), \tag{57}$$

where Q is a g.c.r.d. of the P_i.

Proof. Follows by duality from Theorem 3.3. □

The previous results are not new, see Oberst [1990]. Theorem 3.4.1 is due to Hinrichsen and Prätzel-Wolters [1980b]. Next we analyze image representation of behaviors.

Definition 3.3. We say that a behavior $\mathscr{B}$ has an *image representation* if, for some polynomial matrix $Q \in F^{m\times l}[z]$, we have

$$\mathscr{B} = \operatorname{Im} Q(\sigma). \tag{58}$$

We can state next the following.

Proposition 3.5.
1. *Given $Q \in F^{m\times l}[z]$. Then* $\operatorname{Im} Q(\sigma)$ *is a behavior.*
2. *Let $Q \in F^{m\times l}[z]$, and let $R \in F^{l\times l}[z]$ be nonsingular. Then*
$$\operatorname{Im}(QR)(\sigma) = \operatorname{Im} Q(\sigma). \tag{59}$$
3. *Let $Q \in F^{m\times l}[z]$. Then there exists a right prime polynomial matrix $\overline{Q} \in F^{m\times k}[z]$ for which* $\operatorname{Im} Q(\sigma) = \operatorname{Im} \overline{Q}(\sigma)$.
4. *Let $P \in F^{m\times l}[z]$. Then*
$$\operatorname{Im} P(\sigma) = z^{-1}F^m[[z^{-1}]] \tag{60}$$
if and only if P has full row rank.
5. *Given $P \in F^{k\times m}[z]$ left prime and $M \in F^{m\times l}[z]$ right prime. Then*
$$\operatorname{Ker} P(\sigma) = \operatorname{Im} M(\sigma)$$
if and only if
$$\operatorname{Im} \tilde{P}(\sigma) = \operatorname{Ker} \tilde{M}(\sigma).$$

Proof. Clearly $\mathscr{B} = \operatorname{Im} Q(\sigma)$ is a submodule of $z^{-1}F^m[[z^{-1}]]$. Indeed, for any $h \in z^{-1}F^m[[z^{-1}]]$, we have $\sigma Q(\sigma)h$. That $\mathscr{B}$ is a linear space is obvious. To see closure it suffices to note that

$$\operatorname{Im} Q(\sigma) = (\operatorname{Ker} \tilde{Q}(z))^{\perp}.$$

This follows from the following computation:

$$[Q(\sigma)h, f] = [\pi_- Q(z)h, f] = [h, \tilde{Q}(z)f].$$

All other statements now follow from Proposition 3.3. □

3.2. Doubly unimodular embeddings

We proceed to prove a proposition that is the analog, in the behavioral setting of the doubly coprime factorizations the play such an important role in standard system theory. The importance is due to the fact, already apparent in the statement and its proof, that they provide the key to many duality results. For the use of doubly coprime factorizations in different settings, see Fuhrmann and Ober [1993] as well as Fuhrmann [1994].

Given a pair of polynomial matrices K_2, L_1 we say that there exists a *doubly unimodular embedding*, if there exist polynomial matrices K_1, L_2 such that

$$\begin{pmatrix} K_1(z) \\ K_2(z) \end{pmatrix} \begin{pmatrix} L_1(z) & L_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{61}$$

with both matrices on the left unimodular.

We start with a simple lemma, generalizing a well known result for linear transformations.

Lemma 3.1. *Let $P(z) \in F^{p\times m}[z]$. We consider $P(z)$ as a multiplication map from $F^m(z)$ into $F^p(z)$ which is clearly $F(z)$-linear. Then* $\operatorname{Ker} P(z)$ *and* $\operatorname{Im} P(z)$ *are linear subspaces of $F^m(z)$ and $F^p(z)$ respectively. Both carry also a natural $F[z]$ module structure. Moreover:*
1. *We have*
$$m = \dim \operatorname{Ker} P + \dim \operatorname{Im} P. \tag{62}$$
2. *Given a nonsingular polynomial matrix $R \in F^{m\times m}[z]$, then*
$$\operatorname{Im} P(z) = \operatorname{Im} P(z)R(z). \tag{63}$$

Proof.
1. Follows from the fact that $F^m(z)$ is a finite dimensional vector space over the field $F(z)$ of rational functions and the multiplication operator $P(z)$ is $F(z)$-linear.
2. Since $R(z)$ is nonsingular, the corresponding multiplication operator in $F^m(z)$ is invertible. $\square$

Lemma 3.2. *Given a pair of polynomial matrices K_2, L_1. Then:*
1. *There exists a doubly unimodular embedding if any only if K_2 is left prime, L_1 right prime and*
$$\operatorname{Ker} K_2(z) = \operatorname{Im} L_1(z). \tag{64}$$
2. *There exists a doubly unimodular embedding for K_2 and L_1 if and only if there exists a doubly unimodular embedding for*
$$\begin{pmatrix} K_2(z) & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_1(z) \\ 0 \end{pmatrix}.$$

3. *Given polynomial matrices satisfying*

$$N_2M_1 = M_2N_1, \tag{65}$$

with M_1, M_2 *square and nonsingular. Then a doubly unimodular embedding for*

$$\begin{pmatrix} -N_2 & M_2 \end{pmatrix}, \quad \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}$$

exists if and only if M_1, N_1 *are right coprime and* M_2, N_2 *are left coprime.*

Proof.

1. If (61) is a doubly unimodular embedding, then clearly K_2 is left prime and L_1 right prime. The equality $K_2(z)L_1(z) = 0$ implies $\operatorname{Ker} K_2(z) \supset \operatorname{Im} L_1(z)$. Moreover, the assumption that all matrices in (61) are unimodular implies

$$\begin{pmatrix} L_1 & L_2 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = I.$$

Assuming $f \in \operatorname{Ker} K_2$, we have

$$f = \begin{pmatrix} L_1 & L_2 \end{pmatrix}\begin{pmatrix} K_1 f \\ K_2 f \end{pmatrix} = L_1(K_1 f) \in \operatorname{Im} L_1(z).$$

So $\operatorname{Ker} K_2(z) \subset \operatorname{Im} L_1(z)$ and (64) follows.
Conversely, assume K_2 is left prime, L_1 right prime and (64) holds. Let L_2' be a polynomial right inverse of K_2 and K_1 a polynomial left inverse of L_1. Thus

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}\begin{pmatrix} L_1 & L_2' \end{pmatrix} = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}.$$

We define $L_2 = L_2' - L_1 Q$. For concreteness, assume $L_i \in F^{\mu\times\lambda_i}[z]$, i.e.

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}\begin{pmatrix} L_1 & L_2 \end{pmatrix} = \begin{pmatrix} I_{\lambda_1} & 0 \\ 0 & I_{\lambda_2} \end{pmatrix}. \tag{66}$$

It remains to show that both matrices on the left are unimodular. Since the multiplication map $(L_1 \;\; L_2) : F^{\lambda_1+\lambda_2}[z] \to F^{\mu}[z]$ has a left inverse, it is injective and so are the multiplication maps given by L_1, L_2. So $\operatorname{rank} \operatorname{Im} L_i = \lambda_i$. Note also that, since $K_2L_2 = I_{\lambda_2}$, $K_2 : F^{\mu}[z] \to F^{\lambda_2}[z]$ is surjective, so $\operatorname{rank} \operatorname{Im} K_2 = \lambda_2$. Using now Lemma 3.1, we compute

$$\begin{aligned} \mu &= \operatorname{rank} \operatorname{Ker} K_2 + \operatorname{rank} \operatorname{Im} K_2 \\ &= \operatorname{rank} \operatorname{Im} L_1 + \operatorname{rank} \operatorname{Im} K_2 \\ &= \lambda_1 + \lambda_2. \end{aligned}$$

This shows that the polynomial matrices $(L_1 \;\; L_2)$ and $\binom{K_1}{K_2}$ are both square and hence, by (66), necessarily unimodular.

2. Assume there exists a doubly unimodular embedding for K_2, L_1 of the form

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}\begin{pmatrix} L_1 & L_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{67}$$

Then

$$\begin{pmatrix} K_1 & 0 \\ K_2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} L_1 & L_2 & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{68}$$

which is clearly a doubly unimodular embedding for

$$\begin{pmatrix} K_2(z) & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_1(z) \\ 0 \end{pmatrix}.$$

Conversely, if a doubly unimodular embedding exists for

$$\begin{pmatrix} K_2(z) & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_1(z) \\ 0 \end{pmatrix},$$

it is necessarily of the form (68). Eliminating the third rows and columns, we obtain (67).

3. Clearly, if a doubly unimodular embedding for $(-N_2 \;\; M_2)$, $\binom{M_1}{N_1}$ exists, then necessarily the coprimeness conditions hold.

   To prove the converse, we note that (65) implies the coprime factorizations $N_1 M_1^{-1} = M_2^{-1} N_2$. From equality (65) we immediately obtain the inclusion

$$\operatorname{Im} \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \subset \operatorname{Ker} \begin{pmatrix} -N_2 & M_2 \end{pmatrix}.$$

   To prove the reverse inclusion, assume $\binom{f_1}{f_2} \in \operatorname{Ker}(-N_2 \;\; M_2)$, i.e. $N_2 f_1 = M_2 f_2$. From this it follows that $f_2 = M_2^{-1} N_2 f_1 = N_1 M_1^{-1} f_1$. Defining $g = M_1^{-1} f_1$, we have

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} -N_2 & M_2 \end{pmatrix} = \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} g \in \operatorname{Im} \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}.$$

   So

$$\operatorname{Ker} \begin{pmatrix} -N_2 & M_2 \end{pmatrix} \subset \operatorname{Im} \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \subset \operatorname{Ker} \begin{pmatrix} -N_2 & M_2 \end{pmatrix},$$

   and equality follows. □

Proposition 3.6. *Let* $\binom{M_1}{M_2}$ *be a* $p \times k$ *right prime polynomial matrix. Let* $(N_1 \;\; N_2)$ *be a left prime polynomial matrix satisfying*

$$\operatorname{Ker} \begin{pmatrix} N_1(z) & N_2(z) \end{pmatrix} = \operatorname{Im} \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix}. \tag{69}$$

Then:

1. *There exist unimodular extensions of both matrices that satisfy*

$$\begin{pmatrix} V_1(z) & V_2(z) \\ N_1(z) & N_2(z) \end{pmatrix} \begin{pmatrix} M_1(z) & U_1(z) \\ M_2(z) & U_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{70}$$

   We call such an extension a doubly unimodular extension.

2. *We have also*

$$\begin{pmatrix} M_1(z) & U_1(z) \\ M_2(z) & U_2(z) \end{pmatrix} \begin{pmatrix} V_1(z) & V_2(z) \\ N_1(z) & N_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{71}$$

3. (a) *M_1 is a left prime polynomial matrix if and only if N_2 is.*
   (b) *M_2 is a left prime polynomial matrix if and only if N_1 is.*
   (c) *V_1 is a left prime polynomial matrix if and only if U_2 is.*
   (d) *U_1 is a left prime polynomial matrix if and only if V_2 is.*
4. *We have*

$$\begin{pmatrix} \tilde{M}_1(z) & \tilde{M}_2(z) \\ \tilde{U}_1(z) & \tilde{U}_2(z) \end{pmatrix} \begin{pmatrix} \tilde{V}_1(z) & \tilde{N}_1(z) \\ \tilde{V}_2(z) & \tilde{N}_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{72}$$

   and

$$\begin{pmatrix} \tilde{V}_1(z) & \tilde{N}_1(z) \\ \tilde{V}_2(z) & \tilde{N}_2(z) \end{pmatrix} \begin{pmatrix} \tilde{M}_1(z) & \tilde{M}_2(z) \\ \tilde{U}_1(z) & \tilde{U}_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{73}$$

   We refer to the above as the dual doubly unimodular extension.
5. *N_1 has full column rank if and only if M_2 has.*
6. *N_1 is nonsingular if and only if M_2 is.*
7. *N_1 is a right prime polynomial matrix if and only if M_2 is.*

Proof.

1. By the right primeness of

$$\begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix},$$

   there exist appropriately sized polynomial matrices V_1, V_2 such that $V_1(z)M_1(z) + V_2(z)M_2(z) = I$. Similarly, there exist U_1', U_2' such that $N_1(z)U_1'(z) + N_2(z)U_2'(z) = I$. Let $V_1(z)U_1'(z) + V_2(z)U_2'(z) = Q(z)$. Defining

$$\begin{pmatrix} U_1(z) \\ U_2(z) \end{pmatrix} = \begin{pmatrix} U_1'(z) \\ U_2'(z) \end{pmatrix} - \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} Q(z),$$

   the factorization (70) follows.
2. That (71) follows from (70) is clear.
3. Assume M_1 has full row rank. If N_2 does not have full row rank, we can apply a unimodular matrix on the left so that

$$U \begin{pmatrix} N_1 & N_2 \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & 0 \end{pmatrix}$$

   and this implies

$$\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & 0 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Now $N_{21}M_1 = 0$ together with the assumption that M_2 has full row rank yield $N_{21} = 0$. This contradicts the assumption that $(N_1\ N_2)$ is left prime. So necessarily N_2 has full row rank.

Conversely, assume N_2 has full row rank. We apply the first part to the equality

$$\begin{pmatrix} M_2 & U_2 \end{pmatrix} \begin{pmatrix} V_2 \\ N_2 \end{pmatrix} = 0 \tag{74}$$

that follows from (71). This implies M_1 has full row rank.

4. Assume that $M_1(z)$ is left prime. Then there exists a polynomial matrix $M_1^\sharp$ such that $M_1 M_1^\sharp = I$. Computing

$$\begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} M_1 M_1^\sharp \\ M_2 M_1^\sharp \end{pmatrix} = \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} I \\ M_2 M_1^\sharp \end{pmatrix}$$

which leads to $N_1 = -N_2 M_2 M_1^\sharp$. This in turn implies

$$\begin{pmatrix} N_1 & N_2 \end{pmatrix} = N_2 \begin{pmatrix} -M_2 M_1^\sharp & I \end{pmatrix}.$$

This shows that N_2 is left prime.
Conversely, assume that N_2 is left prime. Again, applying the first part to (74) yields the left primeness of $M_1(z)$.
5. Follows by transposing the equalities (70) and (71) respectively.
6. Follows from Part 3 using the duality provided by the dual doubly unimodular extensions.
7. Follows from Part 3(a) using the duality provided by the dual doubly unimodular extensions. □

A special case of Proposition 3.6 has been proved by Kuijper [1992,1994], see Lemma 3.24. Other results have been obtained independently by Bisiacco and Valcher [2001].

Proposition 3.7. *Let $P(z) \in F^{p\times m}[z]$ be left prime. Consider $F^p((z^{-1}))$ as a p-dimensional vector space over the field $F((z^{-1}))$ of truncated Laurent series. Let $\mathscr{V}$ be the subspace defined by*

$$\mathscr{V} = \operatorname{Ker} P(z) = \left\{ h \in F^p((z^{-1})) \mid P(z)h = 0 \right\}. \tag{75}$$

Let $\begin{pmatrix} P \\ P_1 \end{pmatrix}$ be a unimodular completion with inverse $\begin{pmatrix} Q_1 & Q \end{pmatrix}$. Then

$$\mathscr{V} = \operatorname{Ker} P(z) = \operatorname{Im} Q(z) = Q(z)F^{m-p}((z^{-1})). \tag{76}$$

Proof. Since $P(z)Q(z) = 0$ we clearly have $\operatorname{Ker} P(z) \supset Q(z)F^{m-p}((z^{-1}))$.
Conversely, let $h \in \operatorname{Ker} P(z)$. We have

$$h = \begin{pmatrix} Q_1 & Q \end{pmatrix} \begin{pmatrix} P \\ P_1 \end{pmatrix} h = Q_1 P h + Q P_1 h = Q_1 f_1 + Q f_2.$$

Applying P to this equality we obtain, with $h_1 = Q_1 f_1$,

$$0 = Ph = PQ_1 f_1 + PQ f_2 = PQ_1 f_1 = f_1.$$

So $h = Qf_2 \in Q(z)F^{m-p}((z^{-1}))$. $\square$

Remarks.

1. An analogous result holds true if the field $F((z^{-1}))$ is replaced by the field of rational functions $F(z)$.
2. The polynomial matrix Q in the representation (76) is uniquely determined up to a right unimodular factor. Thus without loss of generality, we can assume Q to be column proper with column indices $\delta_1 \geqslant \cdots \geqslant \delta_{m-p}$.

4. Behavior homomorphisms

In this section we introduce and study the very natural concept of behavior homomorphism. It seems that, in most of the literature on behaviors, the biggest missing item is indeed the study of these homomorphisms. At least as far as this author is concerned, the principal insight that was needed to gain a better understanding of behaviors is the fact that a behavior can be looked at as a generalization of a rational model, introduced in Fuhrmann [1976] and reviewed in Section 2. Given a nonsingular polynomial matrix D, the rational model X^D given by $X^D = \{h \in z^{-1}F^m[[z^{-1}]] | Dh \in F^m[z]\} = \operatorname{Ker} D(\sigma)$. As we saw in Proposition 3.4, rational models are identical to a subclass of behaviors, specifically to the subclass of autonomous behaviors.

Now a rational model X^D is isomorphic to the polynomial model X_D via a simple multiplication map (17). Thus the isomorphism of two polynomial models can be translated into the isomorphism of the corresponding rational models. However, the isomorphisms between polynomial models have been characterized in Fuhrmann [1976] and are quoted in Theorem 2.4. These results are easily translated to the setting of rational models. To this end, let us consider two nonsingular polynomial matrices $D, \overline{D}$. If $Z : X^D \to X^{\overline{D}}$ is an $F[z]$-module homomorphism, then $\overline{Z} : X_D \to X_{\overline{D}}$ defined by $\overline{Z}f = D_2 Z D_1^{-1} f$, with $f \in X_D$ is given by $\overline{Z}f = \pi_{D_2}\overline{N}f$ and the intertwining relation $\overline{N}D = \overline{D}N$ holds for some polynomial matrices $N, \overline{N}$. Now, for $h \in X^D$, we have

$$\begin{aligned} Zh &= \overline{D}^{-1}\overline{Z}Dh = \overline{D}^{-1}\pi_{\overline{D}}\overline{N}Dh \\ &= \overline{D}^{-1}\overline{D}\pi_{-}\overline{D}^{-1}\overline{N}Dh = \pi_{-}Nh \\ &= N(\sigma)h \end{aligned}$$

or $Zh = N(\sigma)h$, with $\overline{N}D = \overline{D}N$ holding. Thus a homomorphism $\overline{Z} : \operatorname{Ker} D(\sigma) \to \operatorname{Ker} \overline{D}(\sigma)$ is given by a Toeplitz map of the form $\mathscr{T}_N = N(\sigma)$ restricted to $X^D = \operatorname{Ker} D(\sigma)$. The invertibility properties of Z are the same as for $\overline{Z}$. Hence, using the results of Fuhrmann [1976], Z is injective if and only if N, D are right coprime and Z is surjective if and only if $\overline{N}, \overline{D}$ are left coprime.

In view of this, it would not come as a great surprise if an $F[z]$-homomorphism $Z : \operatorname{Ker} M(\sigma) \to \operatorname{Ker} \overline{M}(\sigma)$ turns out to be of the form $Z = V(\sigma)|\operatorname{Ker} M(\sigma)$ for some polynomial matrix $V(z)$, with an identity of the form $U(z)M(z) = \overline{M}(z)V(z)$ holding. Indeed, this turns out to be the case and is the content of Theorem 4.5. We proceed to study this problem via the use of the duality introduced in Section 2.

Applying duality considerations, it is easy to translate these results to the behavioral context. Given a behavior $\mathscr{B}$, it is natural to consider the map $\sigma^{\mathscr{B}}$ which is defined as the restriction of the (backward) shift σ to the behavior. Given two behaviors $\mathscr{B}_i$, $i = 1, 2$, a behavior homomorphism is defined to be a map $Z : \mathscr{B}_1 \to \mathscr{B}_2$ satisfying $Z\sigma^{\mathscr{B}_1} = \sigma^{\mathscr{B}_2} Z$. Thus behavior homomorphisms are intertwining maps and their analysis relate to the celebrated commutant lifting theorem of Sarason and Sz.-Nagy–Foias. Thus it is expected that the method presented in this paper will be found to be applicable in other contexts, most notably in the setting of Hardy spaces. Some of the relevant mathematics for this can be found in Fuhrmann [1981b] and Fuhrmann [1994].

Let the backward shift operator $S_- : z^{-1}F^m[[z^{-1}]] \to z^{-1}F^m[[z^{-1}]]$ be defined by

$$S_-h = \sigma h = \pi_- zh, \quad h \in z^{-1}F^m[[z^{-1}]]. \tag{77}$$

Definition 4.1. Given a behavior $\mathscr{B}$, we define the corresponding restricted shift operator $S^{\mathscr{B}}$ by

$$S^{\mathscr{B}} = S_-|\mathscr{B}. \tag{78}$$

If the behavior $\mathscr{B}_P$ is given in AR form as $\mathscr{B}_P = \operatorname{Ker} P(\sigma)$, then we will write also S^P for $S^{\mathscr{B}_P}$.

Next we introduce behavior homomorphisms.

Definition 4.2. Given two behaviors $\mathscr{B}_1$, $\mathscr{B}_2$. A map $Z : \mathscr{B}_1 \to \mathscr{B}_2$ is a *behavior homomorphism* if Z intertwines $S^{\mathscr{B}_1}$ and $S^{\mathscr{B}_2}$, i.e. if

$$ZS^{\mathscr{B}_1} = S^{\mathscr{B}_2} Z. \tag{79}$$

Two behaviors are *isomorphic* or *equivalent* if there exists an invertible behavior homomorphism $Z : \mathscr{B}_1 \to \mathscr{B}_2$.

Clearly, behavior equivalence is an equivalence relation.

With applications to behavior theory in mind, we want to extend Theorems 2.4 and 2.5 concerning homomorphisms of polynomial and rational models and their invertibility properties. Note that, for the case of a nonsingular polynomial matrix D, the polynomial model X_D is isomorphic to the quotient module $F^m[z]/DF^m[z]$, and this quotient module is a torsion module. Similarly, the rational model X^D is a torsion submodule of $Z^{-1}F^m[[z^{-1}]]$. We generalize these results by dropping the nonsingularity assumptions. An $F[z]$-submodule L of $F^p[z]$ has a, not necessarily unique, representation of the form $L = MF^m[z]$, with $M \in F^{p\times m}[z]$. Given a $p \times m$ polynomial matrix $M(z)$ and $f \in F^p[z]$, we shall denote by $[f]_M$ the equivalence class of f in the quotient module $F^p[z]/MF^m[z]$. We denote by π_M the canonical projection of $F^p[z]$ onto $F^p[z]/MF^m[z]$, i.e. $\pi_m f = [f]_M$.

Before analyzing behavior homomorphisms, we study in some detail factor modules of $F^p[z]$. We recall that an element f in any $F[z]$-module $\mathscr{M}$ is a *torsion element* if there exists a polynomial $0 \neq a \in F[z]$ for which $af = 0$. The set of all torsion elements in $\mathscr{M}$ is clearly a submodule $\mathscr{T}$ which we call the *torsion submodule*. The module $\mathscr{M}$ is called *torsion free* if $\mathscr{T} = \{0\}$. It is well known, see Hilton and Wu [1974, p. 174], that the factor module $\mathscr{M}/\mathscr{T}$ is a free module. Moreover, any finitely generated torsion free module over $F[z]$ is a free $F[z]$-module.

Let $M \in F^{p\times k}[z]$ be of full column rank. The quotient space $F^p[z]/MF^k[z]$ has a natural $F[z]$-module structure induced by the shift operator $S_M : F^p[z]/MF^m[z] \to F^p[z]/MF^m[z]$ defined by

$$S_M[f]_M = z \cdot [f]_M = [zf]_M, \tag{80}$$

i.e. for any polynomial $a \in F[z]$, we have

$$a \cdot [f]_M = [af]_M. \tag{81}$$

We proceed to study factor modules of $F^p[z]$.

Theorem 4.1. *Let $M \in F^{p\times k}[z]$ be of full column rank. Let*

$$M = \overline{M}E \tag{82}$$

be a factorization with $\overline{M}$ right prime and E nonsingular and nonunimodular. Then:

1. *The torsion submodule of $F^p[z]/MF^k[z]$ is given by $\overline{M}F^k[z]/MF^k[z]$. Moreover, we have the isomorphism*

   $$F^p[z]/\overline{M}F^k[z] \simeq (F^p[z]/MF^k[z])/(\overline{M}F^k[z]/MF^k[z]) \tag{83}$$

   with $F^p[z]/\overline{M}F^k[z]$ free.
2. *The factor module $F^p[z]/MF^k[z]$ is torsion free if and only if M is right prime.*
3. *The factor module $F^p[z]/MF^k[z]$ is a torsion module if and only if $p = k$, i.e. M is square and nonsingular.*

Proof.

1. The isomorphism (83) follows by standard arguments from the inclusions

   $$F^p[z] \supset \overline{M}F^k[z] \supset MF^k[z].$$

   Assume first that $M = \overline{M}E$ is a factorization with $\overline{M}$ right prime and E nonsingular and nonunimodular. We proceed to show that all elements of $\overline{M}F^k[z]/MF^k[z]$ are torsion elements. To this end, let $[f]_m \in \overline{M}F^k[z]/MF^k[z]$ are torsion elements. This is equivalent to the existence of $g \in F^k[z]$ such that $f = \overline{M}g$ and $[f]_M = [\overline{M}_g]_m$. Let $e = \det E$. Then, using the identity $(\det E)I = E \operatorname{adj} E$, we compute

$$
\begin{aligned}
e[f]_M &= [ef]_M = [e\overline{M}g]_M = [\overline{M}eg]_M \\
&= [\overline{M}E(\operatorname{adj} Eg)]_M = [M(\operatorname{adj} Eg)]_M = 0.
\end{aligned}
$$

So $[f]_M$ is a nonzero torsion element.
Conversely, let $\mathscr{M} \subset F^p[z]$ be the set of all $f \in F^p[z]$ for which $[f]_M$ is a torsion element of $F^p[z]/MF^k[z]$. Clearly $\mathscr{M}$ is a submodule of $F^p[z]$ which contains $MF^k[z]$. As a submodule, it has a representation of the form $\mathscr{M} = \overline{\overline{M}}F^l[z]$ for some $p \times l$ polynomial matrix $\overline{\overline{M}}$ of full column rank. The inclusion $MF^k[z] \subset \overline{\overline{M}}F^l[z]$, with $MF^k[z]$ and $\overline{\overline{M}}F^l[z]$ free submodules of rank k and l respectively, implies $k \leqslant l$ as well as a factorization $M = \overline{\overline{M}}\,\overline{\overline{E}}$ with $E \in F^{l\times k}[z]$. The torsion submodule of $F^p[z]/MF^k[z]$ is therefore $\overline{M}F^k[z]/MF^k[z]$. Moreover, an isomorphism result for modules implies the isomorphism

$$
F^p[z]/\overline{\overline{M}}F^l[z] \simeq (F^p[z]/MF^k[z])/(\overline{\overline{M}}F^l[z]/MF^k[z]). \tag{84}
$$

We show now that the right primeness of $\overline{M}$ implies that $F^p[z]/\overline{M}F^k[z]$ is torsion free. In fact if $[f]_M$ is a torsion element of $F^p[z]/\overline{M}F^k[z]$, then for some nonzero $a \in F[z]$ and $g \in F^k[z]$, we have $af = Mg$. Since M is right prime, it has a polynomial left inverse $M^\sharp$. Applying it to the equality $af = Mg$, we get $g = M^\sharp(af) = aM^\sharp f$. So $af = Mg = M(aM^\sharp f) = a(M\overline{f})$ with $\overline{f} = M^\sharp f$. Since a is nonzero, we have $f = M\overline{f}$ and hence $[f]_M = 0$ contrary to our assumption that $[f]_M$ is a nonzero element. Thus we have the inclusion $\overline{M}F^k[z] \supset \mathscr{M} = \overline{\overline{M}}F^l[z]$. Since both modules are free, of ranks k and l respectively, we must have $k = l$. The two factorizations $M = \overline{M}E$ and $M = \overline{\overline{M}}\,\overline{\overline{E}}$ imply now that $\overline{E}$ is also nonsingular and E and $\overline{E}$ differ by at most a left unimodular factor. So, without loss of generality, we can assume $E = \overline{E}$ and $\overline{M} = \overline{\overline{M}}$.

2. Follows from Part 1.
3. Assume M is nonsingular. In particular $k = p$. In this case $\overline{M} = I$ in the factorization (82) and the torsion submodule is equal $F^p[z]/MF^p[z]$.
   Conversely, assume the factor module $F^p[z]/MF^k[z]$ is a torsion module. This implies $F^p[z] = \overline{M}F^k[z]$. Necessarily $k = p$ and $\overline{M}$ is unimodular, which without loss of generality can assume to be I. So M, being square and of full column rank, is necessarily nonsingular. □

We proceed to study the module homomorphisms of polynomial factor modules.

Theorem 4.2. *Let $M \in F^{p\times m}[z]$ and $\overline{M} \in F^{\overline{p}\times\overline{m}}[z]$ be of full column rank. Then:*

1. *$Z : F^p[z]/M(z)F^m[z] \to F^{\overline{p}}[z]/\overline{M}(z)F^{\overline{m}}[z]$ is an $F[z]$-homomorphism if and only if there exist $\overline{U} \in F^{\overline{p}\times p}[z]$ and $U \in F^{\overline{m}\times m}[z]$ such that*

$$
\overline{U}M = \overline{M}U \tag{85}
$$

and

$$
Z[f]_M = [\overline{U}f]_{\overline{M}}. \tag{86}
$$

2. *The map dual to* Z *is the map* $Z^* : \operatorname{Ker} \tilde{\tilde{\overline{M}}}(\sigma) \to \operatorname{Ker} \tilde{M}(\sigma)$ *given by*

$$Z^* = \tilde{\tilde{\overline{U}}}(\sigma). \tag{87}$$

Proof.

1. If Z is defined as above, then we have

$$ZS_M[f]_M = Z[zf]_M = [\overline{U}zf]_{\overline{M}} = [z\overline{U}f]_{\overline{M}} = S_M[\overline{U}f]_{\overline{M}} = S_{\overline{M}}Z[f]_M,$$

i.e. Z is an $F[z]$-homomorphism.
Define the map $Z_1 : F^p[z] \to F^p[z]/\overline{M}(z)F^m[z]$ by $Z_1 f = Z[f]_M$ for $f \in F^p[z]$. Clearly

$$Z_1S_+f = Z_1[zf]_M = [\overline{U}zf]_{\overline{M}} = [z\overline{U}f]_{\overline{M}} = S_M[\overline{U}f]_{\overline{M}} = S_M Z_1 f,$$

i.e.

$$Z_1S_+ = S_M Z_1. \tag{88}$$

Let $e_1, \ldots, e_p$ be the standard basis elements in F^p. Let $Z_1 e_i = [u_i]_{\overline{M}}$ with $u_i \in F^m[z]$. The u_i are fixed but not uniquely determined. Let U be the $p \times m$ polynomial matrix whose columns are the u_i. It is easy to check that by defining $\overline{Z} : F^p[z] \to F^p[z]$ via $\overline{Z}f = Uf$ for $f \in F^p[z]$ we have obtained (86).
Finally, since we have $Z\pi_M = \pi_{\overline{M}}\overline{Z}$, it follows that $\overline{Z} \operatorname{Ker} \pi_M \subset \operatorname{Ker} \pi_{\overline{M}}$, or $\overline{Z}M(z)F^{p\times m}[z] \subset F^{p\times m}[z]$. This, by a standard argument, implies the existence of a polynomial matrix U for which (85) holds.

2. Given $f \in F^p[z]$ and $h \in \operatorname{Ker} \tilde{\tilde{\overline{M}}}(\sigma)$, we compute

$$\begin{aligned}
[Z[f]_M, h] &= [[\overline{U}f]_M, h] = [\overline{U}f, h] = [f, \tilde{\tilde{\overline{U}}}h] \\
&= [\pi_+ f, \tilde{\tilde{\overline{U}}}h] = [f, \pi_- \tilde{\tilde{\overline{U}}}h] = [f, \tilde{\tilde{\overline{U}}}(\sigma)h] \\
&= [[f]_M, Z^*h],
\end{aligned}$$

or $Z^* = \tilde{\tilde{\overline{U}}}(\sigma)$. $\square$

We proceed to analyze the invertibility of the module homomorphisms characterized by Theorem 4.2.

Theorem 4.3. *Let* $M \in F^{p\times m}[z]$ *and* $\overline{M} \in F^{\overline{p}\times\overline{m}}[z]$ *be of full column rank. Let* $Z : F^p[z]/M(z)F^m[z] \to F^{\overline{p}}[z]/\overline{M}(z)F^{\overline{m}}[z]$ *be an* $F[z]$*-homomorphism defined by* (86) *with* (85) *holding for some* $\overline{U} \in F^{\overline{p}\times p}[z]$ *and* $U \in F^{\overline{m}\times m}[z]$. *Then*:

1. *Z is injective if and only if* U, M *are right coprime and*

$$\operatorname{Ker} \begin{pmatrix} -\overline{U}(z) & \overline{M}(z) \end{pmatrix} = \operatorname{Im} \begin{pmatrix} M(z) \\ U(z) \end{pmatrix}. \tag{89}$$

2. *Z is surjective if and only if* $\overline{U}, \overline{M}$ *are left coprime.*
3. *Z as defined above is the zero map if and only if, for some appropriately sized polynomial matrix* $V(z)$, *we have*

$$\overline{U}(z) = \overline{M}(z)V(z). \tag{90}$$

4. *Z is invertible if and only if there exists a doubly unimodular embedding*

$$\begin{pmatrix} \overline{X} & -\overline{Y} \\ -\overline{U} & \overline{M} \end{pmatrix} \begin{pmatrix} M & Y \\ U & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{91}$$

of

$$\begin{pmatrix} -\overline{U}(z) & \overline{M}(z) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} M(z) \\ U(z) \end{pmatrix}.$$

5. *If Z is invertible, then in terms of the doubly unimodular embedding* (91), $Z^{-1}:$ $F^{\overline{p}}[z]/\overline{M}(z)F^{\overline{m}}[z] \to F^{p}[z]/M(z)F^{m}[z]$ *is given by*

$$Z^{-1}[g]_{\overline{M}} = -[Y_g]_M. \tag{92}$$

Proof.

1. Assume U, M are right coprime and (63) holds. By Lemma 3.2, there exists a doubly unimodular embedding (91). Let $[f]_M \in \operatorname{Ker} Z$, i.e. $[\overline{U}f]_{\overline{M}} = 0$. This implies the existence of a polynomial vector g satisfying

$$\overline{U}f = \overline{M}g \quad \text{or} \quad \begin{pmatrix} -\overline{U}(z) & \overline{M}(z) \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0.$$

Since

$$\begin{pmatrix} f \\ g \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} -\overline{U}(z) & \overline{M}(z) \end{pmatrix}$$

and (63) holds, there exists a rational function h for which

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} M \\ U \end{pmatrix} h. \tag{93}$$

We use now the Bezout equation $\overline{X}M - \overline{Y}U = I$ to obtain $h = \overline{X}f - \overline{Y}g$, which shows that h is actually a polynomial vector. From (93) we obtain $f = Mh$ and hence $[f]_M = 0$, i.e. Z is injective.

Conversely, assume the map Z defined in (86) is injective. Clearly, as by assumption M has full column rank, so has $\binom{M}{U}$. If $\binom{M}{U}$ is not right prime, we can write

$$\begin{pmatrix} M \\ U \end{pmatrix} = \begin{pmatrix} M_1 \\ U_1 \end{pmatrix} R$$

for some, nonsingular and nonunimodular, polynomial matrix R. Let $0 \neq g \in X_R$, i.e. $g = Rh$ for some strictly proper h. Let $f = M_1 g$. We show that $[f]_M \neq 0$. Indeed, $[f]_M = 0$ if and only if $f = M\overline{g}$ for some polynomial vector $\overline{g}$. Thus

$$f = M\overline{g} = M_1R\overline{g} = M_1 g = M_1Rh = Mh$$

or $M(\overline{g} - h) = 0$. Now M has full column rank and hence $\operatorname{Ker} M(z) = 0$. This implies $\overline{g} = h = 0$, contradicting our assumption that $g \neq 0$. Next we compute

$$\begin{aligned}[\overline{U}f]_{\overline{M}} &= [\overline{U}M_1 g]_{\overline{M}} = [\overline{U}M_1 Rh]_{\overline{M}} = [\overline{U}Mh]_{\overline{M}} \\ &= [\overline{M}Uh]_{\overline{M}} = [\overline{M}U_1 Rh]_{\overline{M}} = [\overline{M}(U_1 g)]_{\overline{M}} \\ &= 0,\end{aligned}$$

i.e. Z is not injective.
Next we show the necessity of condition (63). The intertwining condition $\overline{U}M = \overline{M}U$ shows that

$$\operatorname{Ker}\begin{pmatrix}-\overline{U}(z) & \overline{M}(z)\end{pmatrix} \supset \operatorname{Im}\begin{pmatrix}M(z)\\ U(z)\end{pmatrix}.$$

Hence we have the constraint $(p+\overline{k}) - \overline{p} \geqslant k$. Note now that the homomorphism Z maps to torsion elements into torsion elements. Let $M = M_0 E$ and $\overline{M} = \overline{M}_0\overline{E}$ be factorizations with $E, \overline{E}$ nonsingular and nonunimodular and $M_0, \overline{M}_0$ right prime. Therefore Z induces an injective map $\overline{Z} : F^p[z]/M_0F^k[z] \to F^{\overline{p}}[z]/\overline{M}_0F^k[z]$ between the free modules. These modules have rank $p-k$ and $\overline{p}-\overline{k}$ respectively. Since $\operatorname{Im}\overline{Z}$ is a free submodule of $F^{\overline{p}}[z]/\overline{M}_0F^{\overline{k}}[z]$, we have $p-k \leq \overline{p}-\overline{k}$. The two constraints imply the equality $(p+\overline{k}) - \overline{p} = k$ and as a result (63) holds.

2. Assume $\overline{U}, \overline{M}$ are left coprime. Thus there exist polynomial matrices X, Y for which the Bezout equation $\overline{M}X - \overline{U}Y = I$ is satisfied, Let $g \in F^{\overline{p}}[z]$. Then $g = \overline{M}Xg - \overline{U}Yg$ and hence $[g]_{\overline{M}} = [\overline{U}(-Yg)]_{\overline{M}} = [\overline{U}f]_{\overline{M}}$, with $f = -Yg$. Thus the map Z is clearly surjective.
Conversely, we show that if $\overline{U}, \overline{M}$ are not left coprime, then the map Z is not surjective. Indeed, if $\overline{U}, \overline{M}$ are not left coprime then there exists a nonsingular, nonunimodular polynomial matrix R such that $(-\overline{U}(z) \;\; \overline{M}(z)) = R(-\overline{U}_1(z) \;\; \overline{M}_1(z))$. (Note that if $(-\overline{U}(z) \;\; \overline{M}(z))$ does not have full row rank, we can find such R of arbitrary degree). Since R is nonunimodular, $RF^{\overline{p}}[z]$ is a proper submodule of $F^{\overline{p}}[z]$. Choose a $g \in F^{\overline{p}}[z]$ but not in $RF^{\overline{p}}[z]$. We claim that $[g]_{\overline{M}} \notin \operatorname{Im} Z$. To see this, assume the contrary, i.e. that there exists $f \in F^p[z]$ for which

$$Z[f]_M = [\overline{U}f]_{\overline{M}} = [R\overline{U}_1 f]_{\overline{M}} = [g]_{\overline{M}}.$$

This means that $g - R\overline{U}_1 f \in \overline{M}F^{\overline{m}}[z] = R\overline{M}_1F^{\overline{m}}[z]$, or $g \in R(\overline{U}_1F^m[z] + \overline{M}_1F^{\overline{m}}[z]) \subset RF^{\overline{p}}[z]$, contrary to our choice of g.

3. If (90) holds, then for every $[f]_M \in F^p[z]/M(z)F^m[z]$ we have $Z[f]_M = [\overline{U}f]_M = [\overline{M}(Vf)]_M = 0$.
Conversely, if $Z[f]_M = 0$ for every $[f]_M \in F^p[z]/M(z)F^m[z]$, then for every $f \in F^p[z]$ we have $\overline{U}f \in \overline{M}F^{\overline{m}}[z]$. This implies the factorization (90).

4. Follows from Parts 1 and 2, as the two coprimeness conditions and (63) are equivalent, by Lemma 3.2, to the existence of a doubly unimodular embedding (91).

5. Let $W : F^{\overline{p}}[z]/\overline{M}(z)F^{\overline{m}}[z] \to F^p[z]/M(z)F^m[z]$ be given by $W[g]_{\overline{M}} = -[Yg]_M$. From the doubly unimodular embedding (91) we have two Bezout equations $M\overline{X} - Y\overline{U} = I$ and $\overline{M}X - \overline{U}Y = I$. Using the first equation, we compute

$$
\begin{aligned}
[f]_M &= [(M\overline{X} - Y\overline{U})f]_M = [M\overline{X}f]_M - [Y\overline{U}f]_M = -[Y\overline{U}f]_M \\
&= W[\overline{U}f]_{\overline{M}} = WZ[f]_M,
\end{aligned}
$$

i.e. $WZ = I$. Similarly,

$$
\begin{aligned}
[g]_{\overline{M}} &= [(\overline{M}X - \overline{U}Y)g]_{\overline{M}} = [\overline{M}Xg]_{\overline{M}} - [\overline{U}Yg]_{\overline{M}} = -[\overline{U}Yg]_{\overline{M}} \\
&= z[Ug]_{\overline{M}} = ZW[g]_{\overline{M}}
\end{aligned}
$$

or $ZW = I$ and $W = Z^{-1}$. $\square$

The source of asymmetry in the first two statements of the theorem is the fact that we are dealing with rectangular polynomial matrices. In the case of two nonsingular polynomial matrices, condition (63) is an immediate consequence of the coprimeness conditions. This was proved in Lemma 3.2. The following corollary was proved in Fuhrmann [1976].

Corollary 4.1. *Let $T_1 \in F^{m\times m}[z]$, $T_2 \in F^{p\times p}[z]$ be nonsingular and let*

$$N_2T_1 = T_2N_1. \tag{94}$$

Then the map $Z : F^m[z]/T_1F^m[z] \to F^p[z]/T_2F^p[z]$ is invertible if and only if N_1, T_1 are right coprime and N_2, T_2 are left coprime.

Proof. The necessity of the coprimeness conditions follows from Theorem 4.3. To prove sufficiency, in view of the assumed coprimeness conditions, it suffices to show that

$$\operatorname{Ker}\begin{pmatrix} N_2(z) & T_2(z) \end{pmatrix} = \operatorname{Im}\begin{pmatrix} T_1(z) \\ N_1(z) \end{pmatrix}.$$

Clearly, we have the inclusion

$$\operatorname{Im}\begin{pmatrix} T_1(z) \\ -N_1(z) \end{pmatrix} \subset \operatorname{Ker}\begin{pmatrix} N_2(z) & T_2(z) \end{pmatrix}.$$

To prove the converse inclusion, assume f_i are rational vector functions with

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \operatorname{Ker}\begin{pmatrix} N_2(z) & T_2(z) \end{pmatrix},$$

i.e. $N_2f_1 + T_2f_2 = 0$. Noting that (94) implies $T_2^{-1}N_2 = N_1T_1^{-1}$, we compute

$$f_2 = -T_2^{-1}N_2f_1 = -N_1T_1^{-1}f_1.$$

Setting $g = T_1^{-1}f_1$, we have $f_1 = T_1g$ and $f_2 = -N_1g$, or

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} T_1(z) \\ N_1(z) \end{pmatrix} g \in \operatorname{Im}\begin{pmatrix} T_1(z) \\ -N_1(z) \end{pmatrix}. \qquad \square$$

As a corollary to Theorem 4.3, we get the structure theorem for finitely generated modules over $F[z]$.

Theorem 4.4. *Every finitely generated module $\mathscr{M}$ over $F[z]$ is isomorphic to a direct sum of cyclic modules and a finitely generated free module over $F[z]$, i.e. we have*

$$\mathscr{M} \simeq F[z]/d_1F[z] \oplus \cdots \oplus F[z]/d_kF[z] \oplus F^{(p-k)}[z], \tag{95}$$

with d_i, $i = 1, \ldots, k$, nonzero polynomials satisfying $d_i | d_{i-1}$ for $i = 2, \ldots, k$.

Proof. Let $\mathscr{M}$ be a finitely generated module over $F[z]$ and let $e_1, \ldots, e_p$ be a set of generators for $\mathscr{M}$. Define a map $\phi : F^p[z] \to \mathscr{M}$ by

$$\phi \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} = \sum_{i=1}^{p} f_i e_i. \tag{96}$$

Clearly ϕ is a surjective $F[z]$ homomorphism. Since $\operatorname{Ker}\phi$ is a submodule of $F^p[z]$, it has a representation $\operatorname{Ker}\phi = MF^k[z]$ for some $p \times k$ polynomial matrix M of full column rank. Thus we have the isomorphism $\mathscr{M} \simeq F^p[z]\operatorname{Ker}\phi = F^p[z]/MF^k[z]$. From Theorem 4.3 it follows that if U,V are appropriately sized unimodular matrices, then $F^p[z]/MF^k[z] \simeq F^p[z]/UMVF^k[z]$. Hence, without loss of generality, we can assume that M is in its Smith form, i.e.

$$M = \begin{pmatrix} m_1 & 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & m_k \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

with m_i, $i = 1, \ldots, k$, the invariant factors of M. From this the isomorphism (95) follows with $r = p - k$. $\square$

We note that $r = \operatorname{codim} \operatorname{Im} P(z)$ with $P(z)$ the multiplication operator from $F^m(z)$ into $F^p(z)$.

Corollary 4.2. *Given appropriately sized polynomial matrices $M, \overline{M}, N, \overline{N}$ with $M, \overline{M}$ of full row rank. Then there exists a doubly unimodular embedding for*

$$\begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix}, \quad \begin{pmatrix} M \\ N \end{pmatrix}$$

if and only if $M, \overline{M}$ *have the same nontrivial invariant factors and* $\operatorname{codim} M(z) = \operatorname{codim} \overline{M}(z)$.

The composition of two invertible $F[z]$-homomorphisms is an invertible $F[z]$-homomorphism. The following proposition investigates the corresponding doubly unimodular embedding.

Proposition 4.1. *Let* $M_i \in F^{p_i \times m_i}[z]$, $i = 1, 2, 3$, *be of full column rank. Let* $Z_1 : F^{p_1}[z]/M_1(z)F^{m_1}[z] \to F^{p_2}[z]/M_2(z)F^{m_2}[z]$ *be an invertible* $F[z]$-*homomorphism defined by*

$$Z_1[f]_{M_1} = [\overline{U}_2 f]_{M_2} \tag{97}$$

corresponding to the doubly unimodular embedding

$$\begin{aligned} \begin{pmatrix} \overline{X}_1 & -\overline{Y}_1 \\ bu_2 & \overline{M}_2 \end{pmatrix} \begin{pmatrix} M_1 & Y_2 \\ U_1 & X_2 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} M_1 & Y_2 \\ U_1 & X_2 \end{pmatrix} \begin{pmatrix} \overline{X}_1 & -\overline{Y}_1 \\ bu_2 & \overline{M}_2 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \end{aligned} \tag{98}$$

and $Z_2 : F^{p_2}[z]/M_2(z)F^{m_2}[z] \to F^{p_3}[z]/M_3(z)F^{m_3}[z]$ *be an invertible* $F[z]$-*homomorphism defined by*

$$Z_2[g]_{M_2} = [\overline{U}_3 g]_{M_3} \tag{99}$$

corresponding to the doubly unimodular embedding

$$\begin{aligned} \begin{pmatrix} \overline{X}_2 & -\overline{Y}_2 \\ bu_3 & \overline{M}_3 \end{pmatrix} \begin{pmatrix} M_2 & Y_3 \\ U_2 & X_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} M_2 & Y_3 \\ U_2 & X_3 \end{pmatrix} \begin{pmatrix} \overline{X}_2 & -\overline{Y}_2 \\ bu_3 & \overline{M}_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{100}$$

Then $Z = Z_2 Z_1 : F^{p_1}[z]/M_1(z)F^{m_1}[z] \to F^{p_3}[z]/M_3(z)F^{m_3}[z]$ *defined by*

$$Z[f]_{M_1} = [\overline{U}_3 \overline{U}_2 f]_{M_3} \tag{101}$$

is also an invertible $F[z]$-*homomorphism and it corresponds to the doubly unimodular embedding*

$$\begin{aligned} \begin{pmatrix} \overline{X}_1 - \overline{Y}_1 \overline{X}_2 \overline{U}_2 & \overline{Y}_1 \overline{Y}_2 \\ -\overline{U}_3 \overline{U}_2 & \overline{M}_3 \end{pmatrix} \begin{pmatrix} M_1 & -Y_2 Y_3 \\ V_2 V_1 & X_3 - V_2 X_2 Y_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} M_1 & -Y_2 Y_3 \\ V_2 V_1 & X_3 - V_2 X_2 Y_3 \end{pmatrix} \begin{pmatrix} \overline{X}_1 - \overline{Y}_1 \overline{X}_2 U_2 & \overline{Y}_1 \overline{Y}_2 \\ -\overline{U}_3 \overline{U}_2 & \overline{M}_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{102}$$

Proof. From the identities $\overline{U}_2 M_1 = M_2 V_1$ and $\overline{U}_3 M_2 = M_3 V_2$, we obtain $(\overline{U}_3 \overline{U}_2) M_1 = M_3(V_2 V_1)$. We also have

$$Z_2 Z_1 [f]_{M_1} = Z_2 [\overline{U}_2 f]_{M_2} = [\overline{U}_3 \overline{U}_2 f]_{M_3}.$$

So, to complete the proof, we show, using Theorem 4.3, that

$$\begin{pmatrix} -\overline{U}_3\overline{U}_2 & M_3 \end{pmatrix}, \quad \begin{pmatrix} M_1 \\ V_2V_1 \end{pmatrix}$$

have a doubly unimodular embedding. The doubly unimodular embeddings (98) and (100) yield 16 relations which we use in the rest of the proof. We begin by checking the first Bezout equation

$$\begin{aligned}
&(-U_3U_2)(-Y_2Y_3) + M_3(X_3 - V_2X_2Y_3)\\
&\quad = M_3X_3 + U_3U_2Y_2Y_3 - M_3V_2X_2Y_3\\
&\quad = M_3X_3 + U_3U_2Y_2Y_3 - U_3M_2X_2Y_3\\
&\quad = M_3X_3 + U_3(U_2Y_2 - M_2X_2)Y_3\\
&\quad = M_3X_3 - U_3Y_3 = I.
\end{aligned}$$

We proceed to check the other Bezout equation

$$\begin{aligned}
&(\overline{X}_1 - \overline{Y}_1\overline{X}_2U_2)M_1 + (\overline{Y}_1\overline{Y}_2)(V_2V_1)\\
&\quad = \overline{X}_1M_1 - \overline{Y}_1\overline{X}_2M_2V_1 + \overline{Y}_1\overline{Y}_2V_2V_1\\
&\quad = \overline{X}_1M_1 - \overline{Y}_1(\overline{X}_2M_2 - \overline{Y}_2V_2)V_1\\
&\quad = \overline{X}_1M_1 - \overline{Y}_1V_1 = I.
\end{aligned}$$

Finally we calculate

$$\begin{aligned}
&(\overline{X}_1 - \overline{Y}_1\overline{X}_2U_2)Y_2Y_3 - \overline{Y}_1\overline{Y}_2(X_3 - V_2X_2Y_3)\\
&\quad = \overline{X}_1Y_2Y_3 - \overline{Y}_1\overline{X}_2U_2Y_2Y_3 - \overline{Y}_1\overline{Y}_2X_3 + \overline{Y}_1\overline{Y}_2V_2X_2Y_3\\
&\quad = \overline{Y}_1X_2Y_3 - \overline{Y}_1\overline{X}_2U_2Y_2Y_3 - \overline{Y}_1\overline{X}_2Y_3 + \overline{Y}_1\overline{Y}_2V_2X_2Y_3\\
&\quad = \overline{Y}_1(X_2 - \overline{X}_2U_2Y_2 - \overline{X}_2 + \overline{Y}_2V_2X_2)Y_3\\
&\quad = \overline{Y}_1(X_2 - \overline{X}_2U_2Y_2 - \overline{X}_2 + (\overline{X}_2M_2 - I)X_2)Y_3\\
&\quad = \overline{Y}_1(-\overline{X}_2U_2Y_2 - \overline{X}_2 + \overline{X}_2M_2X_2)Y_3\\
&\quad = \overline{Y}_1\overline{X}_2(-U_2Y_2 - I + M_2X_2)Y_3 = 0.
\end{aligned}$$

The other relations in (102) are checked similarly. $\square$

Given a $p \times m$ polynomial matrix M, then Ker $M(\sigma)$ is a submodule of $z^{-1}F^m[[z^{-1}]]$. We define the restricted shift map $S^M : \operatorname{Ker} M(\sigma) \to \operatorname{Ker} M(\sigma)$ by $S^M h = \pi\_zh = \sigma h$. By a judicious use of duality we can state the analog of Theorem 2.6.

Theorem 4.5. *Let $M \in F^{p\times m}[z]$ and $\overline{M} \in F^{\overline{p}\times\overline{m}}[z]$ be of full row rank. Then* $\operatorname{Ker} M(\sigma)$ *is an $F[z]$-submodule of $z^{-1}F^m[[z^{-1}]]$ and* $\operatorname{Ker} \overline{M}(\sigma)$ *is an $F[z]$-submodule of $z^{-1}F^{\overline{m}}[[z^{-1}]]$. Moreover $Z : \operatorname{Ker} M(\sigma) \to \operatorname{Ker} \overline{M}(\sigma)$ is an $F[z]$-homo-*

morphism, i.e. satisfies $ZS^M = S^{\overline{M}}Z$, *if and only if there exist* $\overline{U} \in F^{\overline{p}\times p}[z]$ *and* U *in* $F^{\overline{m}\times m}[z]$ *such that*

$$\overline{U}(z)M(z) = \overline{M}(z)U(z) \tag{103}$$

and

$$Zh = U(\sigma)h, \quad h \in \operatorname{Ker} M(\sigma). \tag{104}$$

Proof. Let $h \in \operatorname{Ker} M(\sigma)$. Then $M(\sigma)(\sigma h) = \sigma(M(\sigma)h) = 0$, i.e. $\sigma h \in \operatorname{Ker} M(\sigma)$ which shows that it is a submodule. Similarly for $\operatorname{Ker} \overline{M}(\sigma)$.

Let Z be defined by (104), with (103) holding. Then, for $h \in \operatorname{Ker} M(\sigma)$, $\overline{M}(\sigma)Zh = \overline{M}(\sigma)(U(\sigma)h) = \overline{U}(\sigma)(M(\sigma)h) = 0$, i.e. $Zh \in \operatorname{Ker} \overline{M}(\sigma)$. Moreover, we compute

$$ZS^M h = U(\sigma)\sigma h = \sigma U(\sigma)h = S^{\overline{M}}Zh,$$

that is Z is an $F[z]$-homomorphism.

Conversely, assume $Z : \operatorname{Ker} M(\sigma) \to \operatorname{Ker} \overline{M}(\sigma)$ is an $F[z]$-homomorphism. For a linear space X and a subspace $\mathscr{V} \subset X$, we have the isomorphism $\mathscr{V}^* \simeq X^*/\mathscr{V}^\perp$. We note that

$$(\operatorname{Ker} M(\sigma))^\perp = \tilde{M}(z)F^p[z],$$

and this leads to

$$(\operatorname{Ker} M(\sigma))^* = F^m[z]/\tilde{M}F^p[z]. \tag{105}$$

The identity $ZS^M = S^{\overline{M}}Z$ leads to $Z^*S_{\tilde{\overline{M}}} = S_{\tilde{M}}Z^*$, that is Z^* is an $F[z]$-module homomorphism. By Theorem 4.2, there exist polynomial matrices $\overline{U} \in F^{\overline{p}\times p}$ and $U \in F^{\overline{m}\times m}$, satisfying $\tilde{U}\tilde{\overline{M}} = \tilde{M}\tilde{\overline{U}}$, which is equivalent to (103), and for which

$$Z^*[f]_{\tilde{\overline{M}}} = [f]_{\tilde{M}}.$$

We can easily check now that necessarily $Z : \operatorname{Ker} M(\sigma) \to \operatorname{Ker} \overline{M}(\sigma)$ is given by (104). $\square$

Note that we cannot expect a direct proof of this, as we have to use completeness and the easiest, maybe the only way, to do this is by using duality, i.e. the fact that $({}^\perp V)^\perp = V$ as in Proposition 3.2 and Theorem 3.1.

Both Theorems 4.2 and 4.5 have an interpretation as lifting homomorphism results. Theorem 4.2 can be restated as follows.

Theorem 4.6. *Let* $M \in F^{p\times m}[z]$ *and* $\overline{M} \in F^{\overline{p}\times\overline{m}}[z]$ *have full row rank.*

Then any $F[z]$*-homomorphism* $Z : F^p[z]/M(z)F^m[z] \to F^{\overline{p}}[z]/\overline{M}(z)F^{\overline{m}}[z]$ *can be lifted to an* $F[z]$*-homomorphism* $\overline{Z} : F^p[z] \to F^{\overline{p}}[z]$ *such that the following diagram is commutative*:

$$\begin{tikzcd} F^p[z] \arrow[r, "\overline{Z}"] \arrow[d, "\pi_M"'] & F^{\overline{p}}[z] \arrow[d, "\pi_{\overline{M}}"] \\ F^p[z]/MF^m[z] \arrow[r, "Z"] & F^{\overline{p}}[z]/\overline{M}F^{\overline{m}}[z] \end{tikzcd}$$

Here π_M is the canonical projection defined by

$$\pi_M f = [f]_M, \quad f \in F^p[z]. \tag{106}$$

Proof. For the homomorphism Z given by (104), we define $\overline{Z}f = Uf$ for $f \in F^p[z]$. □

In much the same way, Theorem 4.5 can be restated as follows.

Theorem 4.7. *Let $M \in F^{p\times m}[z]$ and $\overline{M} \in F^{\overline{p}\times\overline{m}}[z]$ be nonsingular. Then any $F[z]$-homomorphism, $Z : \mathrm{Ker}\, M(\sigma) \to \mathrm{Ker}\, \overline{M}(\sigma)$ can be lifted to an $F[z]$-homomorphism $\overline{Z} : z^{-1}F^m[[z^{-1}]] \to z^{-1}F^{\overline{m}}[[z^{-1}]]$ such that the following diagram is commutative*:

$$\begin{tikzcd} z^{-1}F^m[[z^{-1}]] \arrow[r, "\overline{Z}"] & z^{-1}F^{\overline{m}}[[z^{-1}]] \\ \mathrm{Ker}\, M(\sigma) \arrow[u, "i_M"] \arrow[r, "Z"] & \mathrm{Ker}\, \overline{M}(\sigma) \arrow[u, "i_{\overline{M}}"'] \end{tikzcd}$$

Here i_M and $i_{\overline{M}}$ are the natural embedding maps.

Proof. Define $\overline{Z} = V(\sigma)$. □

Next we discuss the invertibility properties of behavior homomorphisms.

Theorem 4.8. *Given two full row rank polynomial matrices $M \in F^{p\times m}[z]$, $\overline{M} \in F^{\overline{p}\times\overline{m}}[z]$ describing the behaviors $\mathscr{B} = \mathrm{Ker}\, M(\sigma)$ and $\overline{\mathscr{B}} = \mathrm{Ker}\, \overline{M}(\sigma)$ respectively. Let $\overline{U}$, U be appropriately sized polynomial matrices satisfying*

$$\overline{U}(z)M(z) = \overline{M}(z)U(z), \tag{107}$$

and let $Z : \operatorname{Ker} M(\sigma) \to \operatorname{Ker} \overline{M}(\sigma)$ *be defined by*

$$Zh = U(\sigma)h = \pi\_Uh, \quad h \in \operatorname{Ker} M(\sigma). \tag{108}$$

Then:

1. *Z is injective if and only if* M, U *are right coprime*.
2. *Z is surjective if and only if* $\overline{U}, \overline{M}$ *are left coprime and*

$$\operatorname{Ker}\begin{pmatrix} -\tilde{U}(z) & \tilde{M}(z) \end{pmatrix} = \operatorname{Im}\begin{pmatrix} \tilde{\overline{M}}(z) \\ \tilde{\overline{U}}(z) \end{pmatrix}. \tag{109}$$

3. *Z as defined above is the zero map if and only if*, *for some appropriately sized polynomial matrix* $L(z)$, *we have*

$$U(z) = L(z)M(z). \tag{110}$$

4. *Z defined in* (108) *is invertible if and only if there exists a doubly unimodular embedding*

$$\begin{pmatrix} \overline{X} & -\overline{Y} \\ -\overline{U} & \overline{M} \end{pmatrix}\begin{pmatrix} M & Y \\ U & X \end{pmatrix} = \begin{pmatrix} M & Y \\ U & X \end{pmatrix}\begin{pmatrix} \overline{X} & -\overline{Y} \\ -\overline{U} & \overline{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{111}$$

of

$$\begin{pmatrix} -\overline{U}(z) & \overline{M}(z) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} M(z) \\ U(z) \end{pmatrix}.$$

5. *If Z is invertible, then in terms of the doubly unimodular embedding* (61), *its inverse* $Z^{-1} : \operatorname{Ker} \overline{M}(\sigma) \to \operatorname{Ker} M(\sigma)$ *is given by*

$$Z^{-1} = -\overline{Y}(\sigma). \tag{112}$$

Proof.

1. Transposing (107), we have $\tilde{U}\tilde{\overline{M}} = \tilde{M}\tilde{\overline{U}}$. The left invertibility of Z is equivalent, by Theorem 4.2, to the right invertibility of $Z^* : F^{\overline{m}}[z]/\tilde{\overline{M}}F^{\overline{p}}[z] \to F^m[z]/\tilde{M}F^p[z]$ given by

$$Z^*[f]_{\tilde{\overline{M}}} = [\tilde{U}f]_{\tilde{\overline{M}}}, \quad [f]_{\tilde{\overline{M}}} \in F^{\overline{m}}[z]/\tilde{\overline{M}}F^{\overline{p}}[z]. \tag{113}$$

Applying Theorem 4.3, this in turn is equivalent to the left coprimeness of $\tilde{U}$, $\tilde{M}$, hence to the right coprimeness of U, M.

1. The right invertibility of Z is equivalent, again by Theorem 4.2, to the left invertibility of Z^* defined in (113). Applying Theorem 4.3, this in turn is equivalent to the right coprimeness of $\tilde{\overline{M}}$ and $\tilde{\overline{U}}$ together with the condition

$$\operatorname{Ker}\begin{pmatrix} -\tilde{U}(z) & \tilde{M}(z) \end{pmatrix} = \operatorname{Im}\begin{pmatrix} \tilde{\overline{M}}(z) \\ \tilde{\overline{U}}(z) \end{pmatrix}.$$

The statement now follows.

2. We use the fact that $Z = 0$ if and only if $Z^* = 0$.
3. Follows from Parts 1 and 2 as in the proof of Theorem 4.3.
4. We use the fact that $(Z^*)^{-1} = (Z^{-1})^*$. $\square$

Proposition 4.1 has a counterpart in the behavioral setting that follows from Theorem 4.8. We state it without a proof as the proof can be obtained from Proposition 4.1 by duality considerations or directly using the same calculations on the doubly unimodular embeddings.

Proposition 4.2. *Let $M_i \in F^{p_i \times m_i}[z]$, $i = 1, 2, 3$, be of full column rank. Let $Z_1 : \operatorname{Ker} M_1(\sigma) \to \operatorname{Ker} M_2(\sigma)$ be an invertible behavior isomorphism defined by*

$$Z_1 h = V_1(\sigma) h, \quad h \in \operatorname{Ker} M_1(\sigma), \tag{114}$$

corresponding to the doubly unimodular embedding

$$\begin{aligned} \begin{pmatrix} \overline{X}_1 & -\overline{Y}_1 \\ -U_2 & M_2 \end{pmatrix} \begin{pmatrix} M_1 & Y_2 \\ V_1 & X_2 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} M_1 & Y_2 \\ V_1 & X_2 \end{pmatrix} \begin{pmatrix} \overline{X}_1 & -\overline{Y}_1 \\ -U_2 & M_2 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{115}$$

Let $Z_2 : \operatorname{Ker} M_2(\sigma) \to \operatorname{Ker} M_3(\sigma)$ be an invertible behavior isomorphism defined by

$$Z_2 h = V_2(\sigma) h, \quad h \in \operatorname{Ker} M_2(\sigma), \tag{116}$$

corresponding to the doubly unimodular embedding

$$\begin{aligned} \begin{pmatrix} \overline{X}_2 & -\overline{Y}_2 \\ -U_3 & M_3 \end{pmatrix} \begin{pmatrix} M_2 & Y_3 \\ V_2 & X_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} M_2 & Y_3 \\ V_2 & X_3 \end{pmatrix} \begin{pmatrix} \overline{X}_2 & -\overline{Y}_2 \\ -U_3 & M_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{117}$$

Then $Z = Z_2 Z_1 : \operatorname{Ker} M_1(\sigma) \to \operatorname{Ker} M_3(\sigma)$ defined by

$$Z_2 Z_1 h = V_2(\sigma) V_1(\sigma) h, \quad h \in \operatorname{Ker} M_1(\sigma), \tag{118}$$

is also a behavior isomorphism that corresponds to the doubly unimodular embedding

$$\begin{aligned} \begin{pmatrix} \overline{X}_1 - \overline{Y}_1 \overline{X}_2 U_2 & \overline{Y}_1 \overline{Y}_2 \\ -\overline{U}_3 \overline{U}_2 & \overline{M}_3 \end{pmatrix} \begin{pmatrix} M_1 & -Y_2 Y_3 \\ V_2 V_1 & X_3 - V_2 X_2 Y_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} M_1 & -Y_2 Y_3 \\ V_2 V_1 & X_3 - V_2 X_2 Y_3 \end{pmatrix} \begin{pmatrix} \overline{X}_1 - \overline{Y}_1 \overline{X}_2 U_2 & \overline{Y}_1 \overline{Y}_2 \\ -\overline{U}_3 \overline{U}_2 & \overline{M}_3 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{119}$$

5. Representation of behaviors

Linear systems may have many different representations, and to each representation corresponds a permissible set of trajectories. The following definition names the most notable ones. While we are using the term behavior, the fact that these are indeed behaviors according to Willems' definition will follow only when we show that each system listed can be reduced to an AR representation. Since every representation reduces to a NARMA representation, it suffices to show how NARMA representations are reducible to AR ones. This will be taken up in Proposition 5.1. In the following definition, although autoregressive or kernel representations have been previously defined, they are included for completeness.

Definition 5.1.

1. An *autoregressive*, denoted by *AR*, or *kernel representation* of a behavior is a representation
$$\mathscr{B}_P = \operatorname{Ker} P(\sigma) = \left\{w \in z^{-1}F^m[[z^{-1}]] \mid P(\sigma)w = 0\right\}, \tag{120}$$
with $P \in F^{p\times m}[z]$.
2. An *autoregressive moving average*, or *ARMA*, representation of a behavior is a representation
$$\mathscr{B} = \left\{w \in z^{-1}F^q[[z^{-1}]] \mid \exists \xi \in z^{-1}F^n[[z^{-1}]],\, P(\sigma)w = M(\sigma)\xi\right\}. \tag{121}$$
The *full behavior* $\mathscr{B}_{\text{full}}$ is defined by
$$\begin{aligned}\mathscr{B}_{\text{full}} &= \operatorname{Ker}\bigl(P(\sigma) - M(\sigma)\bigr)\\ &= \left\{(w,\xi) \in z^{-1}F^q[[z^{-1}]] \times z^{-1}F^n[[z^{-1}]] \mid P(\sigma)w = M(\sigma)\xi\right\}.\end{aligned} \tag{122}$$
Clearly, defining $\pi_W(w,\xi) = w$, we have
$$\mathscr{B} = \pi_W \mathscr{B}_{\text{full}}. \tag{123}$$
$\mathscr{B}$ is also referred to as the *manifest behavior*.
3. A behavior $\mathscr{B}$ has a *normalized ARMA representation*, or *NARMA representation*, if it satisfies
$$\begin{pmatrix}0\\ I\end{pmatrix} w = \begin{pmatrix}M_1(\sigma)\\ M_2(\sigma)\end{pmatrix}\xi \tag{124}$$
for $M_1 \in F^{r\times m}[z]$, $M_2 \in F^{q\times m}[z]$. Generally, we will assume that M_1 has full row rank and M_1, M_2 are right coprime.
4. A behavior $\mathscr{B}$ has a *driving variable polynomial matrix description*, or *DVPMD* description, if it satisfies
$$\begin{pmatrix}0\\ I\end{pmatrix} w = \begin{pmatrix}T & -U\\ V & W\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \tag{125}$$
with T nonsingular and V, T right coprime.

5. A *moving average MA* or *image representation* of a behavior is a representation
$$\mathscr{B} = \operatorname{Im} Q(\sigma) \tag{126}$$
for some polynomial matrix Q.
6. A behavior $\mathscr{B}$ with split variables $w = \binom{y}{u}$ has a *polynomial matrix description*, or *PMD*, if
$$\begin{cases} T(\sigma)\xi = U(\sigma)u, \\ y = V(\sigma)\xi + W(\sigma)u. \end{cases} \tag{127}$$
With (127) we associate the Rosenbrock polynomial system matrix
$$\mathscr{P} = \begin{pmatrix} T(z) & -U(z) \\ V(z) & W(z) \end{pmatrix}. \tag{128}$$
7. A behavior has a *pencil realization*, or *P-representation*, if there exist constant matrices G, F, H such that
$$\begin{cases} (\sigma G - F)\xi = 0, \\ w = H\xi. \end{cases} \tag{129}$$
8. A behavior has a *dual pencil realization*, or *DP-representation*, if there exist constant matrices K, L, M such that
$$\sigma K\xi = L\xi + Mw. \tag{130}$$
9. A behavior with split variables $w = \binom{y}{u}$ has a *descriptor representation* or *D-representation*, if
$$\begin{cases} \sigma E\xi = A\xi + Bu, \\ y = C\xi + Du. \end{cases} \tag{131}$$
10. A behavior with split variables has a *state space realization* if
$$\begin{cases} \sigma\xi = A\xi + Bu, \\ y = C\xi + Du. \end{cases} \tag{132}$$
11. An *output nulling*, *state space representation*, or *ONSTSP*, of a behavior $\mathscr{B}$ is a representation of the form
$$\mathscr{B} = \mathscr{B}_{\text{ONSTSP}} = \left\{ w \;\middle|\; \exists x \in z^{-1}F^n[[z^{-1}]], \begin{cases} \sigma x = Ax + Bw, \\ 0 = Cx + Dw. \end{cases} \right\} \tag{133}$$
12. An *output nulling*, *PMD representation*, or *ONPMD*, of a behavior $\mathscr{B}$ is a representation of the form
$$\begin{aligned} \mathscr{B} &= \mathscr{B}_{\text{ONPMD}} \\ &= \left\{ w \in z^{-1}F^m[[z^{-1}]] \;\middle|\; \exists \xi \in z^{-1}F^k[[z^{-1}]], \begin{cases} T(\sigma)\xi = U(\sigma)w, \\ 0 = V(\sigma)\xi + W(\sigma)w, \end{cases} \right\} \end{aligned} \tag{134}$$
with $T \in F^{k\times k}[z], U \in F^{k\times p}[z], V \in F^{p\times k}[z], W \in F^{p\times m}[z]$ and T nonsingular.

13. An *output nulling, left matrix fraction representation*, or *ONLMF*, of a behavior $\mathscr{B}$ is a representation of the form

$$\begin{aligned}\mathscr{B} &= \mathscr{B}_{\mathrm{ONLMF}} \\ &= \left\{ w \in z^{-1}F^m[[z^{-1}]] \,\middle|\, \exists \xi \in z^{-1}F^p[[z^{-1}]], \begin{cases} Q(\sigma)\xi = P(\sigma)w, \\ 0 = I\xi, \end{cases} \right\}\end{aligned} \tag{135}$$

with $Q \in F^{p\times p}[z]$, $P \in F^{p\times m}[z]$ and Q nonsingular.

Remarks.
1. In Schumacher [1989], NARMA systems are denoted by AR/MA systems in contradiction to ARMA systems. We prefer to distinguish between the two in a stronger way.
2. We followed Weiland and Stoorvogel [1997] in making the definition of output nulling representations.

We will single out the NARMA representation (124) of a behavior as the focal point of our study. The reason is that essentially all other representations are easily transformed into an NARMA representation. This is summed up in Table 1.

We have introduced a large number of possible representations of linear systems. While in all cases it is clear that the set of trajectories is a linear shift invariant subspace, it remains to show that they are also complete, i.e. that indeed they describe behaviors. We do this by showing that all these system representations are reducible to AR ones. However, as any of the listed system representations is reducible to the NARMA representation, it suffices to show that the set of trajectories of a system in NARMA form is a behavior.

Proposition 5.1. *Let*

$$\mathscr{B} = \left\{ w \,\middle|\, \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \right\}.$$

Then $\mathscr{B}$ is a behavior.

Proof. In view of Theorem 3.1, it suffices to show that $\mathscr{B}$ so defined has a kernel representation. Assuming

$$\begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix}$$

is right prime and let

$$\begin{pmatrix} M_1(z) & X_1(z) \\ M_2(z) & X_2(z) \end{pmatrix}$$

be its embedding in a unimodular matrix having

$$\begin{pmatrix} Y_1(z) & Y_2(z) \\ N_2(z) & N_2(z) \end{pmatrix}$$

Table 1

System	Representation	NARMA representation
AR	$P(\sigma)w=0$	$\begin{pmatrix}0\\I\end{pmatrix}w=\begin{pmatrix}P(\sigma)\\I\end{pmatrix}\xi$
ARMA	$P(\sigma)w=M(\sigma)\xi$	$\begin{pmatrix}0\\I\end{pmatrix}w=\begin{pmatrix}P(\sigma) & -M(\sigma)\\I & 0\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
MA	$w=M(\sigma)\xi$	
NARMA		$\begin{pmatrix}0\\I\end{pmatrix}w=\begin{pmatrix}M_1(\sigma)\\M_2(\sigma)\end{pmatrix}\xi$
DVPMD	$\begin{cases}T(\sigma)\xi_1=U(\sigma)\xi_2\\w=V(\sigma)\xi_1+W(\sigma)\xi_2\end{cases}$	$\begin{pmatrix}0\\I\end{pmatrix}w=\begin{pmatrix}T(\sigma) & -U(\sigma)\\V(\sigma) & W(\sigma)\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
PMD	$\begin{cases}T(\sigma)\xi=U(\sigma)u\\y=V(\sigma)\xi+W(\sigma)u\end{cases}$	$\begin{pmatrix}0 & 0\\I & 0\\0 & I\end{pmatrix}\begin{pmatrix}y\\u\end{pmatrix}=\begin{pmatrix}T(\sigma) & -U(\sigma)\\V(\sigma) & W(\sigma)\\0 & I\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
STATE SPACE	$\begin{cases}\sigma\xi=A\xi+Bu\\y=C\xi+Du\end{cases}$	$\begin{pmatrix}0 & 0\\I & 0\\0 & I\end{pmatrix}\begin{pmatrix}y\\u\end{pmatrix}=\begin{pmatrix}\sigma I-A & -B\\C & D\\0 & I\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
DESCRIPTOR	$\begin{cases}\sigma E\xi=A\xi+Bu\\y=C\xi+Du\end{cases}$	$\begin{pmatrix}0 & 0\\I & 0\\0 & I\end{pmatrix}\begin{pmatrix}y\\u\end{pmatrix}=\begin{pmatrix}\sigma E-A & -B\\C & D\\0 & I\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
P	$\begin{cases}(\sigma G-F)\xi=0\\w=H\xi\end{cases}$	$\begin{pmatrix}0\\I\end{pmatrix}w=\begin{pmatrix}\sigma G-F\\H\end{pmatrix}\xi$
DP	$\sigma K\xi=L\xi+Mw$	$\begin{pmatrix}0\\I\end{pmatrix}w=\begin{pmatrix}\sigma K-L & -M\\I & 0\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
ONLMF	$\begin{cases}Q(\sigma)\xi=P(\sigma)w\\0=I\xi\end{cases}$	$\begin{pmatrix}0\\0\\I\end{pmatrix}w=\begin{pmatrix}P(\sigma) & -Q(\sigma)\\0 & I\\I & 0\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
ONPMD	$\begin{cases}T(\sigma)\xi=U(\sigma)w\\0=V(\sigma)\xi+W(\sigma)w\end{cases}$	$\begin{pmatrix}0\\0\\I\end{pmatrix}w=\begin{pmatrix}T(\sigma) & -U(\sigma)\\V(\sigma) & W(\sigma)\\0 & I\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$
ONSTSP	$\begin{cases}\sigma x=Ax+Bw\\0=Cx+Dw\end{cases}$	$\begin{pmatrix}0\\0\\I\end{pmatrix}w=\begin{pmatrix}\sigma I-A & -B\\C & D\\0 & I\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$

as its inverse, i.e.

$$\begin{pmatrix}Y_1(z) & Y_2(z)\\N_1(z) & N_2(z)\end{pmatrix}\begin{pmatrix}M_1(z) & X_1(z)\\M_2(z) & X_2(z)\end{pmatrix}=\begin{pmatrix}I & 0\\0 & I\end{pmatrix}.$$

We will show that an AR representation of $\mathscr{B}$ is given by

$$\mathscr{B} = \operatorname{Ker} N_2(\sigma), \tag{136}$$

and hence $\mathscr{B}$ is indeed a behavior. To see this, let $w \in \mathscr{B}$, and hence

$$\begin{aligned} N_2(\sigma)w &= \begin{pmatrix} N_1(\sigma) & N_2(\sigma) \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} w \\ &= \begin{pmatrix} N_1(\sigma) & N_2(\sigma) \end{pmatrix} \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi = 0, \end{aligned}$$

i.e. $\mathscr{B} \subset \operatorname{Ker} N_2(\sigma)$.

To prove the converse, we use

$$\begin{pmatrix} M_1(z) & X_1(z) \\ M_2(z) & X_2(z) \end{pmatrix} \begin{pmatrix} Y_1(z) & Y_2(z) \\ N_1(z) & N_2(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and in particular the identity

$$\begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} Y_2(z) + \begin{pmatrix} X_1(z) \\ X_2(z) \end{pmatrix} N_2(z) = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Now, given $w \in \operatorname{Ker} N_2(\sigma)$, then

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} Y_2(\sigma)w,$$

i.e.

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} h,$$

with $h = Y_2(\sigma)w$. This implies $w \in \mathscr{B}$ and hence $\operatorname{Ker} N_2(\sigma) \subset \mathscr{B}$. $\square$

It is important to provide a guide to the transformations between different representations. The reader is advised to consult Schumacher [1988, 1989] and Kuijper [1992,1994] for more on this. We will restrict ourselves to showing how the shift realization, given by (32), can be used to pass from an AR representation to a state space one. This is a slight variation on the construction in Kuijper and Schumacher [1990] or Kuijper [1992,1994]. In this connection see also Rosenthal and Schumacher [1997].

Theorem 5.1. *Given a behavior $\mathscr{B}$ in minimal AR representation*

$$\mathscr{B} = \operatorname{Ker} P(\sigma), \tag{137}$$

where $P(z)$ is a $p \times m$, full row rank polynomial matrix. Then a minimal P-representation exists.

Proof. Assume first that $P(z) = (D(z) \;\; -N(z))$ with D $p \times p$ nonsingular and $D^{-1}N$ proper. Under these assumptions, it follows that $X_D = X_P = X_{(D(z) \;\; -N(z))}$. We choose this space as the state space of a P-representation. Without loss of generality we can assume that the polynomial matrix $P(z)$ is row proper with row degrees $\nu_1 \geqslant \cdots \geqslant \nu_p \geqslant 0$. We set $\sum_{i=1}^{p} \nu_i = n$. Thus

$$X_D = \left\{ \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} \middle| \deg f_i < \nu_i \right\}.$$

The standard basis for X_D is given by the n vectors

$$\left\{ \begin{pmatrix} z^{\nu_1 - 1} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ z^{\nu_p - 1} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The corresponding standard basis matrix of X_D is given by

$$\Phi = \begin{pmatrix} z^{\nu_1 - 1} & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 & z^{\nu_p - 1} & \cdot & \cdot & \cdot & 1 \end{pmatrix}. \tag{138}$$

Write now $D_\infty = \pi_+ D^{-1}N$, then $D(z)^{-1}(N(z) - D(z)D_\infty)$ is strictly proper and hence there exists a uniquely determined $n \times (m - p)$ constant matrix B for which

$$N(z) - D(z)D_\infty = \Phi(z)B.$$

Next we consider the shift realization

$$\left(\begin{array}{c|c} A_D & N(z) - D(z)D_\infty \\ \hline C_D & D_\infty \end{array} \right)$$

with C_D, A_D defined by (32). Let C, A be the respective matrix representations of C_D, A_D with respect to the standard basis. Thus we get the state space representation given by

$$\begin{aligned} \sigma x &= Ax + Bu, \\ y &= Cx + D_\infty u, \end{aligned} \tag{139}$$

which can be written in the P-representation

$$\begin{aligned} &\begin{pmatrix} \sigma I - A & -B \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = 0, \\ &\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}. \end{aligned} \tag{140}$$

Here $\xi = \binom{x}{u}$ is the vector of latent variables, and the representation is with

$$G = \begin{pmatrix} I & 0 \end{pmatrix}, \quad F = \begin{pmatrix} A & B \end{pmatrix}, \quad H = \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}. \tag{141}$$

In order to see that this is indeed a representation of the behavior $\mathscr{B}$, we note that $C(zI - A)^{-1} = D(z)^{-1}\Phi(z)$ and clearly D, Φ are left coprime. The previous equality can be rewritten as $D(z)C = \Phi(z)(zI - A)$ and therefore

$$\begin{aligned} 0 &= \begin{pmatrix} -\Phi(\sigma) & D(\sigma) \end{pmatrix} \begin{pmatrix} \sigma I - A \\ C \end{pmatrix} \xi = \begin{pmatrix} -\Phi(\sigma) & D(\sigma) \end{pmatrix} \begin{pmatrix} 0 & B \\ I & -D \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} \\ &= \begin{pmatrix} D(\sigma) & -(\Phi(\sigma)B + D(\sigma)D_\infty) \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} D(\sigma) & -N(\sigma) \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}. \end{aligned}$$

Applying Proposition 5.1, we conclude that the behavior is indeed $\mathscr{B} = \operatorname{Ker} P(\sigma)$. $\square$

6. Controllability

Controllability and observability are fundamental properties of linear systems that relate to minimality of realizations. These properties were introduced by Kalman, see Kalman et al. [1969] and the further references therein. These notions were extended by Willems [1986] to the behavioral setting, and not surprisingly they relate to the minimality of various representations of behaviors. This we proceed to discuss. Here we modify the definition of controllability in order to consider also the cases of controllability to zero and reachability in the behavioral context. Due to space constraints, we will omit a discussion of observability.

Definition 6.1.

1. Let $\mathscr{B}$ be a behavior defined on $\mathbf{Z}_+$.
   (a) A trajectory $w \in \mathscr{B}$ is *reachable* if there exists a $k \in \mathbf{Z}_+$, a polynomial vector $\sum_{i=0}^{k-1} f_i z^i \in \mathbf{F}^m[z]$ such that for all $T \in \mathbf{Z}_+$, there exists a $\overline{w} \in \mathscr{B}$ for which

$$\overline{w}_t = \begin{cases} 0, & 1 \leqslant t \leqslant T, \\ f_{T+k-t}, & T+1 \leqslant t \leqslant T+k, \\ w_{t-T-k}, & T+k+1 \leqslant t. \end{cases} \tag{142}$$

   (b) The behavior $\mathscr{B}$ is *reachable* if every trajectory $w \in \mathscr{B}$ is reachable.

   (c) The set of all reachable trajectories is a linear subspace and will be denoted by $\mathscr{B}_{\mathrm{r}}$.
2. (a) Let $\mathscr{B}$ be a behavior defined on $\mathbf{Z}_+$. A trajectory $w \in \mathscr{B}$ is *controllable to zero* if there exists a $k \in \mathbf{Z}_+$, a polynomial vector $\sum_{i=0}^{k-1} f_i z^i \in \mathbf{F}^m[z]$ such that for all $T \in \mathbf{Z}_+$, there exists a $\overline{w} \in \mathscr{B}$ for which

$$\overline{w}_t = \begin{cases} w_t, & 1 \leqslant t \leqslant T, \\ f_{T+k-t}, & T+1 \leqslant t \leqslant T+k, \\ 0, & T+k+1 \leqslant t. \end{cases} \tag{143}$$

   (b) The behavior $\mathscr{B}$ is *controllable to zero* if every trajectory $w \in \mathscr{B}$ is controllable to zero.
   (c) The set of all controllable to zero trajectories is a linear subspace and will be denoted by $\mathscr{B}_{\mathrm{cz}}$.
3. A system is *controllable* if for every two trajectories $w^\alpha, w^\beta \in \mathscr{B}$ and $T \in \mathbf{Z}_+$, there exists a $k \in \mathbf{Z}_+$ and a trajectory $\overline{w} \in \mathscr{B}$ such that

$$\overline{w}_t = \begin{cases} w_t^\alpha, & 1 \leqslant t \leqslant T, \\ w_{t-T-k}^\beta, & T+k+1 \leqslant t. \end{cases} \tag{144}$$

Clearly, if a system is reachable then it is controllable to zero. The converse is not true. To see this consider the autonomous behavior $\mathscr{B} = \operatorname{Ker} \sigma^n$. Obviously it is controllable to zero but is not reachable. In fact, given any nonsingular polynomial matrix $D(z)$, then the autonomous behavior $X^D = \operatorname{Ker} D(\sigma)$ is nonreachable.

Proposition 6.1. *A behavior $\mathscr{B}$ is controllable if and only if it is reachable.*

Proof. Assume $\mathscr{B}$ is controllable and let $w^\beta \in \mathscr{B}$. Taking w^α to be the zero trajectory it is clear that w^β is reachable. Since it is arbitrary, it follows that $\mathscr{B}$ is a reachable behavior.

Conversely, assume $\mathscr{B}$ is reachable. Let $w^\alpha, w^\beta \in \mathscr{B}$ be any pair of trajectories. Clearly, $\sigma^{k+T} w^\alpha \in \mathscr{B}$ and we have

$$(\sigma^{k+T} w^\alpha)_t = w_{t+k+T}^\alpha, \quad t \geqslant 1.$$

We consider the trajectory $w^\beta - \sigma^{k+T} w^\alpha$. By the assumption of reachability, there exists a trajectory $v \in \mathscr{B}$ such that

$$v_t = \begin{cases} 0, & 1 \leqslant t \leqslant T, \\ w_{t-T-k}^\beta - w_{t+k+T}^\alpha, & T+k+1 \leqslant t. \end{cases}$$

By linearity, $w = v + w^\alpha \in \mathscr{B}$. Clearly

$$w_t = \begin{cases} w_t^\alpha, & 1 \leqslant t \leqslant T, \\ w_t^\beta, & T+k+1 \leqslant t. \end{cases}$$

Thus $\mathscr{B}$ is controllable. □

Lemma 6.1. *Let $\mathscr{B} = \operatorname{Ker} P(\sigma)$ and $\overline{\mathscr{B}} = \operatorname{Ker} \overline{P}(\sigma)$ be isomorphic behaviors. Then $\mathscr{B}$ is reachable if and only if $\overline{\mathscr{B}}$ is reachable.*

Proof. Since the behaviors are isomorphic, we apply Theorems 4.5 and 4.8 to conclude that there exist polynomial matrices U, V satisfying $UP = \overline{P}V$, with $U, \overline{P}$ left coprime and P, V right coprime, in terms of which the map $V(\sigma) : \mathscr{B} \to \overline{\mathscr{B}}$ is a behavior isomorphism. Assume $\deg V(z) = l$ and let $\mathscr{B}$ be reachable. Let $\overline{w} \in \overline{\mathscr{B}}$. Then there exists a unique $w \in \mathscr{B}$ such that $\overline{w} = V(\sigma)w$. By the assumption of reachability there exists $v \in \mathscr{B}$ such that

$$v_t = w_{t-k-l}, \quad t > k + l.$$

Define $\overline{v} = V(\sigma)v$. A simple computation yields

$$\overline{v}_t = (V(\sigma)v)_t = \overline{w}_{t-k}, \quad t > k.$$

Since behavior isomorphism is an equivalence relation, the converse implication holds also. □

Lemma 6.2. *Let $D(z)$ be a $p \times p$ nonsingular polynomial matrix. Then the autonomous behavior $X^D = \operatorname{Ker} D(\sigma)$ is nonreachable.*

Proof. Let $d(z) = \det D(z)$ and $n = \deg d$. We show that $X^D \cap z^{-n-1}F^p[[z^{-1}]] = \{0\}$. Let h be any element in the above intersection. Write $h = z^{-n}h'$ with $h' \in z^{-1}F^p[[z^{-1}]]$. Since $X^D \subset X^{dI}$, we compute

$$h = \pi^d h = \pi_- d^{-1}\pi_+ dh = \pi^d h = \pi_- d^{-1}\pi_+ dz^{-n}h' = 0,$$

as dz^{-n} is proper. □

As a result of Proposition 6.1, the concepts of controllability and reachability for behaviors coincide. The usefulness of using reachability rather than controllability is due to the greater ease in using the former.

We proceed to characterize reachability in terms of various behavior representations.

Theorem 6.1. *Given a behavior $\mathscr{B} \subset z^{-1}F^p[[z^{-1}]]$. Then:*

1. *Let $\mathscr{B}$ be given in the AR representation $\mathscr{B} = \operatorname{Ker} P(\sigma)$, where $P \in F^{p\times m}[z]$ has full row rank and let $P(z) = E(z)\overline{P}(z)$ be an* internal/external factorization, *i.e. with E nonsingular and $\overline{P}$ left prime. Then*
   (a) *$\mathscr{B}_{\mathrm{r}}$, the set of all reachable trajectories in $\mathscr{B}$, is a linear subspace for which we have*

   $$\mathscr{B}_{\mathrm{r}} = \left\{\pi_- h | h \in \operatorname{Ker} P(z) \mid F^m((z^{-1}))\right\}. \tag{145}$$

   (b) *We have*

   $$\mathscr{B}_{\mathrm{r}} = \operatorname{Ker} \overline{P}(\sigma). \tag{146}$$

In particular, $\mathscr{B}_{\mathrm{r}}$ *is a subbehavior of* $\mathscr{B}$.
We have $w \in \mathscr{B}_{\mathrm{r}}$ *if and only if* $P(z)w \in X_P \cap PF^m[z]$.

(c) $\mathscr{B}$ *is reachable if and only if* $P(z)$ *is left prime*.

(d) *Assume* $P(z) = (T(z) \quad -U(z))$ *with T nonsingular and* $T^{-1}U$ *proper*. *Let* $\mathscr{R}$ *be the reachable subspace for the shift realization corresponding to the matrix fraction* $T^{-1}U$. *Then* $\binom{y_-}{u_-} \in \mathscr{B}_{\mathrm{r}}$ *if and only if* $Ty_- - Uu_- \in \mathscr{R}$. *We have* $\mathscr{B}_{\mathrm{r}} = \mathscr{B}$ *if and only if* $\mathscr{R} = X_T$. *The behavior* $\mathscr{B}$ *is reachable if and only if the shift realization* (32) *corresponding to the left matrix fraction* $T^{-1}U$ *is reachable*, *hence* $\mathscr{B}$ *is reachable if and only if* T, U *are left coprime*.

(e) *Let* $Q(z)$ *be any nonsingular polynomial matrix for which* $Q^{-1}P$ *is proper*. *Then* $\mathscr{B}$ *is reachable if and only if*, *with respect to the shift realization* (32) *corresponding to the matrix fraction* $Q^{-1}P$, *we have* $\mathscr{V}^* = \mathscr{R}^*$.

2. *The behavior* $\mathscr{B}$ *has an MA representation if and only if it is reachable*.
3. *Let the behavior* $\mathscr{B}$ *be given in the NARMA representation*

$$\mathscr{B} = \left\{ w \,\middle|\, \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \right\}, \tag{147}$$

where we assume that the polynomial matrix $\binom{M_1(z)}{M_2(z)}$ *is right prime and that* $M_1(z)$ *has full row rank*. *Let*

$$M_1(z) = E(z)\overline{M}_1(z) \tag{148}$$

be an internal/external factorization of M_1. *Then*:

(a) $\mathscr{B}_{\mathrm{r}}$, *the reachable sub-behavior*, *is given by the NARMA equation*

$$\mathscr{B}_{\mathrm{r}} = \left\{ w \,\middle|\, \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \overline{M}_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \right\}. \tag{149}$$

(b) $\mathscr{B}$ *is reachable if and only if* $M_1(z)$ *is left prime*.

4. *Given a system in ARMA form*

$$P(\sigma)w = M(\sigma)\xi, \tag{150}$$

where we assume that $(P(z) \quad -M(z))$ *has full row rank and* $M(z)$ *is right prime*. *Let E be a g.c.l.d. of P and M*, *i.e.*

$$\begin{aligned} P(z) &= E(z)\overline{P}(z), \\ M(z) &= E(z)\overline{M}(z), \end{aligned} \tag{151}$$

with $\overline{P}, \overline{M}$ *left coprime*. *Then*:

(a) *The reachable subspace* $\mathscr{B}_{\mathrm{r}}$ *has the ARMA representation*

$$\mathscr{B}_{\mathrm{r}} = \mathrm{Ker}\big(\overline{P}(z) \quad -\overline{M}(z)\big). \tag{152}$$

(b) $\mathscr{B}$ *is reachable if and only if* P, M *are left coprime*.

(c) *The full behavior* $\mathscr{B}_{\mathrm{full}} = \mathrm{Ker}(P(\sigma) \quad -M(\sigma))$ *is reachable if and only if the* (*manifest*) *behavior* $\mathscr{B} = \left\{ w \mid \exists \xi \in z^{-1}F^m[[z^{-1}]], P(\sigma)w = M(\sigma)\xi \right\}$ *is*.

5. *Let $\mathscr{B}$ be a behavior with split variables $w = \binom{y}{u}$, given in the PMD representation*

$$\mathscr{B} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \middle| \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} T & -U \\ V & W \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\}, \tag{153}$$

where we assume that V, T are right coprime. Let E be a g.c.l.d. of T and U, i.e.

$$\begin{aligned} T(z) &= E(z)T_1(z), \\ U(z) &= E(z)U_1(z), \end{aligned} \tag{154}$$

with T_1, U_1 left coprime. Then:

(a) *The reachable subbehavior $\mathscr{B}_{\mathrm{r}} \subset \mathscr{B}$ is given by the PMD form*

$$\mathscr{B}_{\mathrm{r}} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \middle| \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} T_1 & -U_1 \\ V & W \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\}. \tag{155}$$

(b) *Let $\overline{T}, \overline{V}$ be defined via a left coprime factorization*

$$\overline{T}^{-1}\overline{V} = VT_1^{-1}. \tag{156}$$

The reachable subbehavior $\mathscr{B}_{\mathrm{r}} \subset \mathscr{B}$ is given by

$$\mathscr{B}_{\mathrm{r}} = \operatorname{Ker}\big(\overline{T}(\sigma) \quad -(\overline{T}(\sigma)W(\sigma) + \overline{V}(\sigma)U(\sigma))\big). \tag{157}$$

(c) *$\mathscr{B}$ is reachable if and only if $(T \quad -U)$ is left prime.*

6. *A behavior $\mathscr{B}$ with a pencil representation*

$$\mathscr{B} = \left\{ w \middle| \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma G - F \\ H \end{pmatrix} \xi \right\}, \tag{158}$$

where we assume that $zG - F$ has full row rank and $\binom{zG-F}{H}$ is right prime. Then $\mathscr{B}$ is reachable if and only if $zG - F$ is left prime.

7. *Let a behavior $\mathscr{B}$ be given in a dual pencil representation*

$$\mathscr{B} = \big\{ w | (\sigma K - L)\xi = Mw \big\}, \tag{159}$$

where we assume without loss of generality that $(zK - L \quad M)$ is of full row rank and $zK - L$ is right prime. Then $\mathscr{B}$ is reachable if and only if $(zK - L \quad M)$ if left prime.

8. *Let $\mathscr{B}$ be a behavior with split variables $w = \binom{y}{u}$, given in the state space representation*

$$\begin{cases} \sigma\xi = A\xi + Bu, \\ y = C\xi + Du, \end{cases} \tag{160}$$

which is assumed observable. Then $\mathscr{B}$ is reachable if and only if $(zI - A \quad B)$ is left prime.

Proof.

1. (a) Let $h_- \in \mathscr{B}_{\mathrm{r}}$. By the definition of reachability, there exists a polynomial vector h_+ such that for every time τ we have $\pi_- P(z)z^{-\tau}(h_- + h_+) = 0$. Choosing τ large enough so that $P(z)z^{-\tau}(h_- + h_+)$ is strictly proper, it follows that necessarily $P(z)(h_- + h_+) = 0$, i.e. $h_- \in \{\pi_- h \mid h \in \operatorname{Ker} P(z) \mid F^m((z^{-1}))\}$.
Conversely, if $P(z)(h_- + h_+) = 0$, also $P(z)z^{-\tau}(h_- + h_+) = 0$ and in particular, for $\tau > \deg h_+$ it follows that $h_- \in \mathscr{B}_{\mathrm{r}}$.

(b) Assume $w \in \operatorname{Ker} \overline{P}(\sigma)$, i.e. $\pi_- \overline{P} w = 0$ or for some $f \in F^p[z]$, we have $\overline{P}(z)w = f(z)$. Since $\overline{P}$ is left prime, there exists a polynomial matrix $\overline{Q}$ such that $\overline{P}(z)\overline{Q}(z) = I$. Write $f = \overline{P}\,\overline{Q} g = -\overline{P} g$ with $g = -\overline{Q} f$. It follows that $\overline{P} w = -\overline{P} g$ or $\overline{P}(z)(w + f) = 0$. So, with $h = w + g$ we have $w = \pi_- h$ and $h \in \operatorname{Ker} \overline{P}(z)$. Since $\operatorname{Ker} \overline{p}(z) = \operatorname{Ker} P(z)$, it follows that $\operatorname{Ker} \overline{p}(z) \subset \{\pi_- h \mid h \in \operatorname{Ker} P(z) \mid F^m((z^{-1}))\}$.
Conversely, assume $w \in \{\pi_- h | h \in \operatorname{Ker} P(z) \mid F^m((z^{-1}))\}$, i.e. there exists an $h \in F^m((z^{-1}))$ for which $w = \pi_- h$ and $P(z)h = 0$. Defining $g = \pi_+ h$, it follows that $\overline{P}(z)h = \overline{P}(z)(w + g) = 0$. This clearly implies that $\overline{P}(\sigma)w = 0$, and so $\{\pi_- h | h \in \operatorname{Ker} P(z) | F^m((z^{-1}))\} \subset \operatorname{Ker} \overline{P}(z)$ and the equality (145) follows.

(c) Assume $P(z)$ is left prime and $w \in \mathscr{B}$. Thus $\pi_- P w = 0$, i.e. $f(z) = P(z)\, w(z)$ is a polynomial. Let $Q(z)$ be any right inverse of $P(z)$ and set $g = Qf$. Then $f = Pg$ and $Pg = Pw$ or equivalently $P(g - w) = 0$. If $\deg g = k$, then $\overline{w} = z^{-k-1}(g - w) \in z^{-1}F^m[[z^{-1}]]$ and it clearly satisfies (142).
To prove the converse we show now that if P is not left prime then $\mathscr{B}$ is not reachable. By an application of Lemma 6.1, we may assume without loss of generality that

$$P(z) = \begin{pmatrix} r_1(z) & & & & 0 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & r_p(z) & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

where r_i are the invariant factors of R with at least one nontrivial. This implies

$$\operatorname{Ker} P(\sigma) = [X^{r_1} \oplus \cdots \oplus X^{r_p}] \oplus z^{-1}F^{m-p}[[z^{-1}]]$$

and, as pointed out before, no nonzero element in the autonomous behavior $[X^{r_1} \oplus \cdots \oplus X^{r_p}]$ is reachable.

(d) Of course this follows from the first two parts of this statement. However, we prove this directly and thus provide an independent proof of Part 1(c).
Assume $Ty_- - Uu_- \in \mathscr{R}$, i.e. for some polynomial vectors u_+, y_+, we have $Ty_- - Uu_- = \pi_T U u_+ = Uu_+ - Ty_+$. Write this as $T(y_- + y_+) = U(u_- + u_+)$ or

$$\begin{pmatrix} T(z) & -U(z) \end{pmatrix} \begin{pmatrix} y_- + y_+ \\ u_- + u_+ \end{pmatrix} = 0. \tag{161}$$

So we conclude $\binom{y_-}{u_-} \in \mathscr{B}_{\mathrm{r}}$.

Conversely, if $\binom{y_-}{u_-} \in \mathscr{B}_{\mathrm{r}}$, then by Part 1, there exist polynomial vectors u_+, y_+, for which (161) holds. From this we compute

$$y_- = \pi_-(y_- + y_+) = \pi_- T^{-1}U(u_- + u_+) = T^{-1}Uu_- + \pi_- T^{-1}Uu_+.$$

This in turn implies

$$\begin{pmatrix} T(z) & -U(z) \end{pmatrix} \begin{pmatrix} y_- \\ u_- \end{pmatrix} \in \mathscr{R}.$$

Let E be the greatest common left divisor of T and U. We write $(T(z) \; - U(z)) = E(\overline{T}(z) \; - \overline{U}(z))$ and note that $EX_{\overline{T}} \subset X_T$ is the reachable subspace of the shift realization. Clearly, the equality $\mathscr{B}_{\mathrm{r}} = \mathscr{B}$ implies $\mathscr{R} = X_T$. Conversely, assume $\mathscr{R} = X_T$. This means that T, U are left coprime. If $\binom{y_-}{u_-} \in \mathscr{B}$, then for some polynomial vector f_+ we have $Ty_- - Uu_- = f_+$. Since T, U are left coprime, there exist polynomial vectors u_+, y_+ for which $f_+ = -(Ty_+ - Uu_+)$. This leads to (161) and hence $\binom{y_-}{u_-} \in \mathscr{B}_{\mathrm{r}}$ and the equality $\mathscr{B} = \mathscr{B}_{\mathrm{r}}$ follows. This also shows that $\mathscr{B}$ is reachable if and only if T, U are left coprime.

(e) Assume that $\mathscr{B}$ is reachable and hence $P(z)$ is left prime. We know, from Theorem 2.10, that, with respect to the shift realization corresponding to $Q^{-1}P$, we have $\mathscr{V}^* = X_P$ and $\mathscr{R}^* = X_P \cap PF^m[z]$. Since $P(z)$ is left prime, then $PF^m[z] = F^p[z]$ and so

$$\mathscr{R}^* = X_P \cap PF^m[z] = X_P \cap F^m[z] = X_P = \mathscr{V}^*.$$

Conversely, assume $\mathscr{V}^* = \mathscr{R}^*$. Given the factorization $P = E\overline{P}$ with E nonsingular and $\overline{P}$ left prime, then we know, see Fuhrmann [1981], that $\mathscr{V}^*/\mathscr{R}^* \simeq X_E$. Hence the equality $\mathscr{V}^* = \mathscr{R}^*$ is equivalent to the unimodularity of E and hence also to the left primeness of P. So $\mathscr{B}$ is reachable.

2. Assume $\mathscr{B}$ has a MA representation, $\mathscr{B} = \operatorname{Im} Q(\sigma)$, with $Q(z)$ right prime. Extend $Q(z)$ to a unimodular polynomial matrix $(Q_1(z) \;\; Q(z))$ with inverse $\binom{P(z)}{P_1(z)}$. Then, by Proposition 5.1, $\mathscr{B} = \operatorname{Ker} P(\sigma)$ and, since $P(z)$ is left prime, it follows that $\mathscr{B}$ is reachable.

   Conversely, assume $\mathscr{B}$ is reachable. Then $\mathscr{B} = \operatorname{Ker} P(\sigma)$ with $P(z)$ left prime. Let $\binom{P(z)}{P_1(z)}$ be a unimodular extension with inverse $(Q_1(z) \;\; Q(z))$. Clearly, $\operatorname{Im} Q(\sigma)$ is a behavior which, by the same result quoted above, is given by $\operatorname{Ker} P(\sigma)$, i.e. $\mathscr{B}$ has an image, or an MA, representation.

3. (a) Clearly the factorization (148) implies

$$\mathscr{B}' = \left\{ w \;\middle|\; \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \overline{M}_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \right\} \subset \mathscr{B}.$$

The right primeness of $\binom{M_1(z)}{M_2(z)}$ implies the right primeness of $\binom{\overline{M}_1(z)}{M_2(z)}$ so there exist unique, up to unimodular left factors, left prime polynomial matrices $(N_1 \;\; N_2)$ and $(\overline{N}_1 \;\; \overline{N}_2)$ such that

$$\operatorname{Ker}\begin{pmatrix} N_1 & N_2 \end{pmatrix} = \operatorname{Im}\begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} \tag{162}$$

and

$$\operatorname{Ker}\begin{pmatrix} \overline{N}_1 & \overline{N}_2 \end{pmatrix} = \operatorname{Im}\begin{pmatrix} \overline{M}_1(z) \\ M_2(z) \end{pmatrix}. \tag{163}$$

By Proposition 5.1, the behavior $\mathscr{B}$ is given in the AR form $\mathscr{B} = \operatorname{Ker} N_2(\sigma)$ whereas $\mathscr{B}' = \operatorname{Ker} \overline{N}_2(\sigma)$. By Proposition 3.6, we have that N_2 is left prime, i.e. $\mathscr{B}'$ is reachable. Now (162) implies

$$\begin{pmatrix} N_1 E & N_2 \end{pmatrix}\begin{pmatrix} \overline{M}_1(z) \\ M_2(z) \end{pmatrix} = 0.$$

In turn this implies that for some polynomial matrix K, we have $(N_1 E \;\; N_2) = K(\overline{N}_1 \;\; \overline{N}_2)$, or $N_2 = K\overline{N}_2$. However this is an internal/external factorization and so $\mathscr{B}_{\mathrm{r}} = \operatorname{Ker} \overline{N}_2(\sigma)$ which proves the statement.

(b) Embed $\binom{M_1(z)}{M_2(z)}$ in a doubly coprime extension (70). The result follows from Proposition 3.6.3a.

4. (a) We rewrite the ARMA equation (150) in the NARMA form

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} P & -M \\ I & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

By Part 3, the reachable subbehavior is given by the NARMA equation

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \overline{P} & -\overline{M} \\ I & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

which in turn is equivalent to the ARMA equation

$$\overline{P}(\sigma)w = \overline{M}(\sigma)\xi.$$

(b) Follows from the previous part.

(c) Assume $\mathscr{B}_{\text{full}} = \operatorname{Ker}(P(\sigma) \;\; -M(\sigma))$, or $P(\sigma)w = M(\sigma)\xi$. Let

$$U(z) = \begin{pmatrix} U_1(z) \\ U_2(z) \end{pmatrix}$$

be unimodular with $\operatorname{Ker} U_1(z) = \operatorname{Im} M(z)$. Then, by Proposition 5.1, the manifest behavior is given by the AR representation $\operatorname{Ker} U_1(\sigma)P(\sigma)$. By our assumption there exist polynomial matrices X, Y such that the Bezout identity $PX - MY = I$ is satisfied. Applying U_1 to this we get $U_1PX - U_1MY = U_1PX = U_1$. Let $U_1^\sharp$ be a polynomial right inverse of U_1. Then $(U_1P)(XU_1^\sharp) = I$, which shows that U_1P is right unimodular.

5. (a) This follows from Part 3.
   (b) Using (155), it follows from an application of Proposition 5.1.
   (c) Follows from Part 5(a).
6. Follows from Part 3.
7. Rewrite the equation $(\sigma K - L)\xi = Mw$ in the NARMA form

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma K - L & -M \\ 0 & I \end{pmatrix} \tag{164}$$

   and apply Part 3.
8. We rewrite Eq. (160) in the form

$$\begin{pmatrix} 0 & B \\ I & -D \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} zI - A \\ C \end{pmatrix} \xi$$

   and apply Part 5. Note that

$$\begin{pmatrix} 0 & B & zI - A \\ I & -D & C \end{pmatrix}$$

   is left prime if and only if $(zI - A \quad B)$ is. □

The definition of controllability in the behavioral setting as well as the characterization of controllable AR representations is due to Willems [1986]. The characterization of controllability in Part 5 of Theorem 6.1 and its connection to the shift realization is due to Fuhrmann [1976,1977], see Section 2.3. The controllability test given in Part 8 is known as the *Hautus test*. In this connection, see also Section 2.3.

We study next how the controllability property is preserved under behavior homomorphisms. A preliminary result was given in Lemma 6.1.

Corollary 6.1. *Let $\mathscr{B}_i = \operatorname{Ker} P_i(\sigma)$ be two behaviors with P_i of full row rank. Then*:
1. *If $Z : \mathscr{B}_1 \to \mathscr{B}_2$ is a surjective behavior homomorphism, then the controllability of $\mathscr{B}_1$ implies the controllability of $\mathscr{B}_2$.*
2. *If $Z : \mathscr{B}_1 \to \mathscr{B}_2$ is an injective behavior homomorphism, then the controllability of $\mathscr{B}_2$ implies the controllability of $\mathscr{B}_1$.*
3. *If $Z : \mathscr{B}_1 \to \mathscr{B}_2$ is a behavior isomorphism, then $\mathscr{B}_1$ is controllable if and only if $\mathscr{B}_2$ is.*

Proof. By Theorem 4.5, there exit polynomial matrices U, V, satisfying $U(z)P_1(z) = P_2(z)V(z)$ in terms of which $Z = V(\sigma)$. By the assumed surjectivity we have the left coprimeness of U, P_2. We apply Proposition 3.6 to get the result. It is easy to see that, with a slight modification of the proof, the right coprimeness of P_1, V is not necessary for the conclusion. The other statements are proved similarly. □

In most parts of the definition we follow Kuijper [1992,1994] whose terminology, in turn, follows Willems'. In making the definition of output nulling representations we are motivated by Weiland and Stoorvogel [1997]. These behavior representations open up the possibility of studying deeper underlying connections between the behavioral theory and geometric control theory. Of particular interest is using factorization theory as the unifying tool. This will be the subject of additional publications.

To conclude this section we digress briefly on the question of stability and stabilizability of behaviors. Up to now we have worked over an arbitrary field. In this context it seems that the only meaningful way to introduce stability is to say that a trajectory $w \in z^{-1}F^m[[z^{-1}]]$ is *stable* if it is eventually zero, i.e. there exists an index n_0 such that, with $w(z) = \sum_{i=1}^{\infty} w_i z^{-i}$, we have $w_i = 0$ for $i \geqslant n_0$. On the other hand, if the field is the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers, then we have alternative definitions. Again, we follow Weiland and Stoorvogel [1997] in making the following definition.

Definition 6.2.

1. A trajectory $w \in \mathscr{B}$ is *stable* if $\lim_{n\to\infty} w_n = 0$.
2. An autonomous behavior $\mathscr{B}$ over $\mathbf{R}, \mathbf{C}$ is *stable* if every trajectory $w \in \mathscr{B}$ is stable.
3. A behavior $\mathscr{B}$ is *stabilizable* if given any trajectory $w^{(1)} \in \mathscr{B}$ and an integer $n_0 > 0$, there exits a stable trajectory $w^{(2)} \in \mathscr{B}$ satisfying $w_j^{(1)} = w_j^{(2)}$ for $j \leqslant n_0$.

We recall that a nonsingular polynomial matrix in $\mathbf{R}^{m\times m}[z]$ or $\mathbf{C}^{m\times m}[z]$ is stable if $\det P(z)$ is a stable polynomial, i.e. has all its zeros in the interior of the unit disk. The following proposition, which we give without proof, is a characterization of stable and stabilizable behaviors.

Proposition 6.2. *Given a real or complex behavior* $\mathscr{B} = \operatorname{Ker} P(\sigma)$, *with* $P(z)$ *of full row rank. Then*:

1. $\mathscr{B}$ *is stable if and only if* $P(z)$ *is a nonsingular and stable polynomial matrix*.
2. $\mathscr{B}$ *is stabilizable if and only if there exists a factorization* $P(z) = E(z)\overline{P}(z)$ *with* $E(z)$ *a nonsingular and stable polynomial matrix and* $\overline{P}(z)$ *left prime*.
3. *The set* $\mathscr{B}_{\mathrm{st}}$ *of all stabilizable trajectories in* $\mathscr{B}$ *is a subbehavior*. *Moreover*, *if*
$$P(z) = E_{\mathrm{as}}(z)E_{\mathrm{st}}(z)\overline{P}(z) \tag{165}$$
is a factorization with $E_{\mathrm{as}}(z)$ *antistable and* $E_{\mathrm{st}}(z)$ *stable*, *then*
$$\mathscr{B}_{\mathrm{st}} = \operatorname{Ker} E_{\mathrm{st}}(\sigma)\overline{P}(\sigma). \tag{166}$$
4. *If a behavior* $\mathscr{B}$ *is reachable*, *then it is stabilizable.*
5. *Given a behavior* $\mathscr{B}$ *in the NARMA form*
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi. \tag{167}$$

Let $M_1 = E_{as}(z)E_{st}(z)\overline{M}_1(z)$ be a factorization with $E_{as}(z)$ antistable and $E_{st}(z)$ stable. Then the stabilizable subbehavior is given by

$$\mathscr{B}_{st} = \left\{ w \,\middle|\, \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} E_{st}(\sigma)\overline{M}_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \right\}. \tag{168}$$

The characterization in Proposition 6.2.2 is essentially the Hautus test for stability.

7. Equivalence of behavior representations

This section is devoted to the central theme of this paper, namely the unified derivation of equivalence results for different classes of behavior representations. More specifically, in terms of the system representation's data, we want to establish necessary and sufficient conditions for the corresponding behaviors to coincide. In a way this is a far reaching variation on the theme of equivalence and similarity, that is the standard results that two matrices A, B are similar if and only if the pencils $zI - A, zI - B$ are equivalent. Other examples of results of the same nature are the Kalman state space isomorphism theorem and the analysis of strict system equivalence in the context of polynomial matrix descriptions.

In fact, we begin our analysis of the problem for the class of behaviors given in NARMA representation that was introduced in Section 5. We do this by extending the authors' version of strict system equivalence, referred to by Kailath [1980] and Özgüler [1994] as Fuhrmann system equivalence (FSE) to distinguish it from Rosenbrock's original definition, by introducing the concept of NARMA equivalence of two systems given in NARMA representation. We show directly that NARMA equivalence is indeed a bona fide equivalence relation. We proceed to show that FSE turns out to be a special case of NARMA equivalence. The principal result, namely Theorem 7.1, characterizes equivalence for different classes of behavior representations. First and foremost in importance is that of NARMA representations. This provides the key for all other cases. To analyze NARMA equivalence, we bring to bear all the machinery of behavior isomorphism which in turn is based on doubly unimodular embeddings. It is worthwhile to note that, in the characterization of similarity of behaviors, contrary to the case of rational models, coprimeness conditions are necessary but, due to the use of rectangular polynomial matrices, are not sufficient and have to be replaced by the stronger condition of the existence of doubly unimodular embeddings.

One another thing to point out is that all equivalence results are derived under conditions weaker than minimality. This is not surprising inasmuch as in the case of strict system equivalence, no minimality constraints were imposed, see Fuhrmann [1977].

We introduce now the new concept of NARMA equivalence. This is related to another concept of equivalence, namely that of strict system equivalence. For ease of reference, we also recall the definition of Fuhrmann system equivalence.

Definition 7.1.
1. Given two NARMA representations
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \tag{169}$$
with the behavior $\mathscr{B}$, and
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \overline{M}_1(\sigma) \\ \overline{M}_2(\sigma) \end{pmatrix} \xi \tag{170}$$
with the behavior $\overline{\mathscr{B}}$, we say that the representations are *NARMA equivalent* if there exists polynomial matrices $\overline{U}$, V, $\overline{X}$ of appropriate size such that
$$\begin{pmatrix} \overline{U}(z) & 0 \\ -\overline{X}(z) & I \end{pmatrix} \begin{pmatrix} \overline{M}_1(z) \\ \overline{M}_2(z) \end{pmatrix} = \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} V(z), \tag{171}$$
$\overline{U}$, $\overline{M}_1$ are left coprime and $\binom{M_1(z)}{M_2(z)}$, V right coprime and
$$\operatorname{Ker} \begin{pmatrix} \overline{U}(z) & 0 & \overline{M}_1(z) \\ -\overline{X}(z) & I & \overline{M}_2(z) \end{pmatrix} = \operatorname{Im} \begin{pmatrix} M_1(z) \\ M_2(z) \\ -V(z) \end{pmatrix} \tag{172}$$
holds, i.e. there exists a doubly unimodular embedding of the polynomial matrices
$$\begin{pmatrix} \overline{U}(z) & 0 & \overline{M}_1(z) \\ -\overline{X}(z) & I & \overline{M}_2(z) \end{pmatrix}, \qquad \begin{pmatrix} M_1(z) \\ M_2(z) \\ -V(z) \end{pmatrix}. \tag{173}$$
2. Two polynomial system matrices
$$\mathscr{P}_i = \begin{pmatrix} T_i & -U_i \\ V_i & W_i \end{pmatrix}, \quad i = 1, 2,$$
are called *Fuhrmann system equivalent*, or *FSE*, if there exist polynomial matrices M, N, X, Y, with M, T_2 left coprime T_1, N right coprime and for which
$$\begin{pmatrix} M & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{pmatrix} = \begin{pmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{pmatrix} \begin{pmatrix} N & Y \\ 0 & I \end{pmatrix}. \tag{174}$$

Remarks.
1. Note that the left coprimeness of $\overline{U}$ and $\overline{M}_1$ is equivalent to the left coprimeness of
$$\begin{pmatrix} \overline{U}(z) & 0 \\ -\overline{X}(z) & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overline{M}_1(z) \\ \overline{M}_2(z) \end{pmatrix}.$$

2. There are two variations on the definition of NARMA equivalence. We say the two systems are *weakly NARMA equivalent* if (171) is replaced by

$$\begin{pmatrix} \overline{U}(z) & 0 \\ -\overline{X}(z) & P \end{pmatrix} \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} = \begin{pmatrix} \overline{M}_1(z) \\ \overline{M}_2(z) \end{pmatrix} V(z), \tag{175}$$

with P a nonsingular constant matrix, i.e. we allow also a change of basis in the space W. If P is restricted to a permutation matrix, we will say that the systems are *permutation NARMA equivalent*. In this special case of weak NARMA equivalence, we are allowing only a reordering of the external variables.

Proposition 7.1.
1. *NARMA equivalence is an equivalence relation.*
2. *FSE is an equivalence relation.*

Proof. Both statements can be proved directly, however the computations are somewhat tedious and will be omitted. One can easily avoid them. In fact, that FSE is an equivalence relation proved in Fuhrmann [1977], by showing that two polynomial system matrices are FSE if and only if the associated shift realizations are similar. Since system similarity is an equivalence relation, so is FSE. In the same way, one can show that NARMA equivalence is indeed an equivalence relation by showing that, under the assumed right primeness conditions, two NARMA systems are NARMA equivalent if and only if they represent the same behavior. This is proved in Theorem 7.1.3. □

Clearly if a NARMA system

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \overline{M}_1(\sigma) \\ \overline{M}_2(\sigma) \end{pmatrix} \xi$$

is obtained from

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi$$

through

$$\begin{pmatrix} \overline{M}_1(z) \\ \overline{M}_2(z) \end{pmatrix} = \begin{pmatrix} \overline{U}(z) & 0 \\ -\overline{X}(z) & I \end{pmatrix} \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} V(z)$$

with $U(z)$, $V(z)$ appropriately sized unimodular matrices, then the two systems are NARMA equivalent. This easy observation is useful in the reduction of representations.

The following theorem is the central result for this section. It characterizes the conditions, in terms of behavior representation, that two different representations have the same behavior. The key result is that of NARMA representations and we derive most other characterizations from this one. Thus we present a uniform derivation for the problems of system equivalence. Part 1 is identical to Theorem 3.4.2 and is included for completeness.

Theorem 7.1.

1. *Let $P, Q \in F^{p\times m}[z]$ have full row rank. Then*
$$\operatorname{Ker} P(\sigma) = \operatorname{Ker} Q(\sigma) \tag{176}$$
if and only if $Q(z) = U(z)P(z)$ for some unimodular polynomial matrix U.
2. *Two behaviors in MA representations*
$$\mathscr{B}_i = \operatorname{Im} M_i(\sigma), \quad i = 1, 2, \tag{177}$$
under the assumption that $M_i(z)$ are right prime, are equal if and only if $M_2(z) = M_1(z)V(z)$ for some unimodular V.
3. *Given two behaviors $\mathscr{B}$ and $\overline{\mathscr{B}}$ in the NARMA representations*
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \xi \tag{178}$$
and
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \overline{M}_1(\sigma) \\ \overline{M}_2(\sigma) \end{pmatrix} \xi \tag{179}$$
respectively. We assume that both
$$\begin{pmatrix} M_1(\sigma) \\ M_2(\sigma) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overline{M}_1(\sigma) \\ \overline{M}_2(\sigma) \end{pmatrix}$$
are right prime. Then $\mathscr{B} = \overline{\mathscr{B}}$ if and only if the two representations are NARMA equivalent.
4. *Given two behaviors $\mathscr{B}_i$, $i = 1, 2$ in DVPMD form*
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} T_i & -U_i \\ V_i & W_i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad i = 1, 2, \tag{180}$$
under the assumption that the polynomial system matrices
$$\begin{pmatrix} T_i & -U_i \\ V_i & W_i \end{pmatrix}$$
are right prime. Then the two behaviors coincide if and only if there exist appropriately sized polynomial matrices $M, X, N_{11}, N_{12}, N_{21}, N_{22}$ for which
$$\begin{pmatrix} M & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{pmatrix} = \begin{pmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{181}$$
and there exists a doubly unimodular embedding for
$$\begin{pmatrix} M & 0 & T_2 & -U_2 \\ -X & I & V_2 & W_2 \end{pmatrix}, \quad \begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \\ -N_{11} & -N_{12} \\ -N_{21} & -N_{22} \end{pmatrix}.$$

5. *Given two behaviors $\mathscr{B}_i$, $i = 1, 2$, with split variables $w = \binom{y}{u}$ in PMD form*

$$\begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} T_i & -U_i \\ V_i & W_i \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad i = 1, 2, \tag{182}$$

under the assumption that T_i, V_i are right coprime. Associate with such a system a polynomial system matrix

$$\mathscr{P}_i = \begin{pmatrix} T_i & -U_i \\ V_i & W_i \end{pmatrix}, \quad i = 1, 2.$$

Then:

(a) *The associated behaviors are given by*

$$\mathscr{B}_i = \operatorname{Ker}\begin{pmatrix} \overline{T}_i(\sigma) & -(\overline{T}_i(\sigma)W_i(\sigma) + \overline{V}_i(\sigma)U_i(\sigma)) \end{pmatrix}. \tag{183}$$

(b) *The two behaviors are equal if and only if the two polynomial system matrices $\mathscr{P}_i$ are Fuhrmann system equivalent* (*FSE*).

6. *Given two behaviors in observable state space representations*

$$\begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \sigma I - A_i & -B_i \\ C_i & D_i \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad i = 1, 2. \tag{184}$$

Then the corresponding behaviors are equal if and only if the two systems are isomorphic, i.e. there exists a constant, *invertible matrix M*, *such that*

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zI - A_1 & -B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} zI - A_2 & -B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}. \tag{185}$$

7. *Given two behaviors in DV state space representations*

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} zI - \overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{186}$$

where we assume that the two polynomial matrices are right prime and D, $\overline{D}$ are both injective. Then the two behaviors coincide if and only if there exist constant matrices U, P, K, with U, P invertible and a permutation matrix Π, such that

$$\begin{pmatrix} U & 0 \\ 0 & \Pi \end{pmatrix} \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix} = \begin{pmatrix} zI - \overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} \begin{pmatrix} U & 0 \\ K & P \end{pmatrix}, \tag{187}$$

i.e. the two state space systems are state feedback equivalent.

8. *Given two behaviors $\mathscr{B}$ and $\overline{\mathscr{B}}$ in pencil representations*

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma G - F \\ H \end{pmatrix} \xi \tag{188}$$

and

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma\overline{G} - \overline{F} \\ \overline{H} \end{pmatrix} \xi \tag{189}$$

respectively. We assume that
(a) *G has full row rank.*
(b) $\binom{G}{H}$ *has full column rank.*
(c) $\binom{zG-F}{H}$ *is right prime.*
Then the two behaviors coincide if and only if there exist unique, nonsingular, constant polynomial matrices S and T for which

$$\begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zG - F \\ H \end{pmatrix} = \begin{pmatrix} z\overline{G} - \overline{F} \\ \overline{H} \end{pmatrix} T. \tag{190}$$

Proof.
1. See Theorem 3.4.2.
2. Assume first that $\operatorname{Im} M_1(\sigma) = \operatorname{Im} M_2(\sigma)$, i.e. the behaviors are equal. By assumption, the polynomial matrices $M_i(z)$ are right prime. By Proposition 3.5, we have $\operatorname{Im} M_i(\sigma) = \{\operatorname{Ker} \tilde{M}_i(z)\}^{\perp}$, and hence the equality $\operatorname{Ker} \tilde{M}_1(z) = \operatorname{Ker} \tilde{M}_2(z)$. Since the $\tilde{M}_i$ are left prime, it follows from Proposition 3.3.4 that $\tilde{M}_1 = \tilde{V}\tilde{M}_2$, for some unimodular V. Thus $M_2 = M_1 V$.
Conversely, if $M_2(z) = M_1(z)V(z)$ with V unimodular, it follows that $\operatorname{Im} M_2(\sigma) \subset \operatorname{Im} M_1(\sigma)$. Equality follows by symmetry.
3. Assume first that the representations are NARMA equivalent. Let $(N_1(z) \quad N_2(z))$ and $(\overline{N}_1(z) \quad \overline{N}_2(z))$ be left prime polynomial matrices for which

$$\operatorname{Ker}\begin{pmatrix} N_1(z) & N_2(z) \end{pmatrix} = \operatorname{Im} \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix},$$
$$\operatorname{Ker}\begin{pmatrix} \overline{N}_1(z) & \overline{N}_2(z) \end{pmatrix} = \operatorname{Im} \begin{pmatrix} \overline{M}_1(z) \\ \overline{M}_2(z) \end{pmatrix}.$$

By Theorem 5.1, we have $\mathscr{B} = \operatorname{Ker} N_2(\sigma)$ and $\overline{\mathscr{B}} = \operatorname{Ker} \overline{N}_2(\sigma)$. We compute

$$\begin{aligned} 0 &= \begin{pmatrix} \overline{N}_1(z) & \overline{N}_2(z) \end{pmatrix} \begin{pmatrix} \overline{M}_1(z) \\ \overline{M}_2(z) \end{pmatrix} V \\ &= \begin{pmatrix} \overline{N}_1(z) & \overline{N}_2(z) \end{pmatrix} \begin{pmatrix} U(z) & 0 \\ -X(z) & I \end{pmatrix} \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} \\ &= \begin{pmatrix} (\overline{N}_1(z)U(z) - \overline{N}_2(z)X(z)) & \overline{N}_2(z) \end{pmatrix} \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix}, \end{aligned}$$

i.e.

$$\operatorname{Ker}\begin{pmatrix} (\overline{N}_1(z)U(z) - \overline{N}_2(z)X(z)) & \overline{N}_2(z) \end{pmatrix} \supset \operatorname{Ker}\begin{pmatrix} N_1(z) & N_2(z) \end{pmatrix}.$$

By Proposition 3.3, there exists a polynomial matrix $L(z)$ for which

$$\begin{pmatrix} \overline{N}_1(z)U(z) - \overline{N}_2(z)X(z) & \overline{N}_2(z) \end{pmatrix} = L(z)\begin{pmatrix} N_1(z) & N_2(z) \end{pmatrix}.$$

This implies the equality $\operatorname{Ker}\overline{N}_2(\sigma) \supset \operatorname{Ker} N_2(\sigma)$ or $\overline{\mathscr{B}} \supset \mathscr{B}$. By Proposition 7.1, NARMA equivalence is an equivalence relation, and in particular a symmetric relation. The equality $\overline{\mathscr{B}} = \mathscr{B}$ follows by symmetry.
Conversely, assume the behaviors $\mathscr{B}$ and $\overline{\mathscr{B}}$ are equal. Clearly we have

$$\mathscr{B} = M_2(\sigma)\operatorname{Ker} M_1(\sigma) = \overline{M}_2(\sigma)\operatorname{Ker}\overline{M}_1(\sigma).$$

The right coprimeness of M_1, M_2 implies that $M_2(\sigma)|\operatorname{Ker} M_1(\sigma)$ is injective and so $M_2(\sigma)$ as a map from $\operatorname{Ker} M_1(\sigma)$ onto $\mathscr{B}$ is bijective. Moreover, it is an $F[z]$-homomorphism. In the same way $\overline{M}_2(\sigma)|\operatorname{Ker}\overline{M}_1(\sigma) : \operatorname{Ker}\overline{M}_1(\sigma) \to \overline{\mathscr{B}}$ is a behavior isomorphism. We define now a map $Z : \operatorname{Ker} M_1(\sigma) \to \operatorname{Ker}\overline{M}_1(\sigma)$ by

$$Zh = \overline{M}_2(\sigma)^{-1}M_2(\sigma)h, \quad h \in \operatorname{Ker} M_1(\sigma). \tag{191}$$

Clearly Z is an $F[z]$-isomorphism, i.e. satisfies $Z\sigma^{M_1} = \sigma^{\overline{M}_1}Z$ and is invertible. Since $M_1(z), \overline{M}_1(z)$ have both full row rank, we can apply Theorem 4.5 to conclude the existence of appropriately sized polynomial matrices U and V for which $U, \overline{M}_1$ are left coprime, M_1, V are right coprime, they satisfy the following equality:

$$\operatorname{Ker}\begin{pmatrix} U(z) & \overline{M}_1(z)\end{pmatrix} = \operatorname{Im}\begin{pmatrix} M_1(z) \\ -V(z)\end{pmatrix} \tag{192}$$

in terms of which $Z = V(\sigma)$. Note that the previous conditions are equivalent to the existence of a doubly unimodular embedding of

$$\begin{pmatrix} U(z) & \overline{M}_1(z)\end{pmatrix}, \quad \begin{pmatrix} M_1(z) \\ V(z)\end{pmatrix}.$$

Thus we have $\overline{M}_2(\sigma)^{-1}M_2(\sigma)h = V(\sigma)h$ for all $h \in \operatorname{Ker}M_1(\sigma)$. So

$$\operatorname{Ker}(M_2(\sigma) - \overline{M}_2(\sigma)V(\sigma)) \supset \operatorname{Ker} M_1(\sigma).$$

By Theorem 3.4, we conclude the existence of a polynomial matrix $X(z)$ such that

$$M_2(z) - \overline{M}_2(z)V(z) = X(z)M_1(z). \tag{193}$$

The equalities (192) and (193), taken together, imply

$$\begin{pmatrix} U(z) & 0 & \overline{M}_1(z) \\ 0 & I & 0\end{pmatrix}\begin{pmatrix} M_1(z) \\ 0 \\ V(z)\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}.$$

It remains to show that there exists a doubly unimodular embedding for

$$\begin{pmatrix} U(z) & 0 & \overline{M}_1(z) \\ -X(z) & I & \overline{M}_2(z)\end{pmatrix}, \quad \begin{pmatrix} M_1(z) \\ M_2(z) \\ -V(z)\end{pmatrix}.$$

First, we note that there exists a doubly unimodular embedding for

$$\begin{pmatrix} U(z) & 0 & \overline{M}_1(z) \\ 0 & I & 0 \end{pmatrix}, \quad \begin{pmatrix} M_1(z) \\ 0 \\ V(z) \end{pmatrix}.$$

This follows from Lemma 3.2 and the fact that there exists a doubly unimodular embedding of

$$\begin{pmatrix} U(z) & \overline{M}_1(z) \end{pmatrix}, \quad \begin{pmatrix} M_1(z) \\ V(z) \end{pmatrix}.$$

We note that

$$\begin{pmatrix} I & 0 & 0 \\ -X(z) & I & \overline{M}_2(z) \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ X(z) & I & -\overline{M}_2(z) \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

with both matrices unimodular. Now

$$\begin{pmatrix} U(z) & 0 & \overline{M}_1(z) \\ -X(z) & I & \overline{M}_2(z) \end{pmatrix} = \begin{pmatrix} U(z) & 0 & \overline{M}_1(z) \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -X(z) & I & \overline{M}_2(z) \\ 0 & 0 & I \end{pmatrix}$$

and, using Eq. (193), we have

$$\begin{pmatrix} I & 0 & 0 \\ X(z) & I & -\overline{M}_2(z) \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} M_1(z) \\ 0 \\ V(z) \end{pmatrix} = \begin{pmatrix} M_1(z) \\ X(z)M_1(z) + \overline{M}_2(z)V(z) \\ -V(z) \end{pmatrix}$$
$$= \begin{pmatrix} M_1(z) \\ M_2(z) \\ V(z) \end{pmatrix},$$

and (172) follows.

4. This is a straight application of Part 3.
5. Note that the right coprimeness of T_i, V_i is equivalent to the right primeness of

$$\begin{pmatrix} T_i & -U_i \\ V_i & W_i \\ 0 & I \end{pmatrix}.$$

Assume first that the two polynomial system matrices are FSE. Let, for $i = 1, 2$, $\overline{T}_i^{-1}\overline{V}_i$ be a left coprime factorization of $V_i T_i^{-1}$. By Proposition 5.1, the associated behaviors are given in AR representations by

$$\mathscr{B}_i = \operatorname{Ker}\begin{pmatrix} \overline{T}_i(\sigma) & -(\overline{T}_i(\sigma)W_i(\sigma) + \overline{V}_i(\sigma)U_i(\sigma)) \end{pmatrix}. \tag{194}$$

Since the two polynomial system matrices are assumed to be FSE, there exist appropriately sized polynomial matrices M, N, X, Y, with M, T_2 left coprime T_1, N right coprime and for which

$$\begin{pmatrix} M & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{pmatrix} = \begin{pmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{pmatrix} \begin{pmatrix} N & Y \\ 0 & I \end{pmatrix}. \tag{195}$$

Note that, by Proposition 7.1, this is equivalent to the existence of a doubly unimodular embedding. Eq. (195) implies the four identities

$$\begin{aligned} MT_1 &= T_2N, \\ MU_1 &= U_2 - T_2Y, \\ XT_1 - V_1 &= -V_2N, \\ XU_1 + W_1 &= V_2Y + W_2. \end{aligned} \tag{196}$$

Let, for $i = 1, 2$, $\overline{T}_i^{-1}\overline{V}_i$ be a left coprime factorization of $V_iT_i^{-1}$. We compute now, using (196),

$$\begin{aligned} \overline{T}_1^{-1}\overline{V}_1 &= V_1T_1^{-1} = X + V_2NT_1^{-1} = V_2T_2^{-1}M \\ &= X + \overline{T}_2^{-1}\overline{V}_2M = \overline{T}_2^{-1}(\overline{T}_2X + \overline{V}_2M). \end{aligned} \tag{197}$$

Thus $\overline{T}_2\overline{T}_1^{-1}\overline{V}_1$ is a polynomial matrix. By the left coprimeness of $\overline{T}_1, \overline{V}_1$, it follows that $\overline{T}_2 = S\overline{T}_1$ for some, necessarily nonsingular, polynomial matrix S. Now, by Fuhrmann [1976], the equality $MT_1 = T_2N$, taken together with the assumed coprimeness conditions imply that the invariant factors, and hence also the determinants, of T_1, T_2 are equal. This shows that S is necessarily unimodular. We continue, using (197),

$$\overline{T}_1^{-1}\overline{V}_1 = \overline{T}_2^{-1}(\overline{T}_2X + \overline{V}_2M)$$

or

$$S\overline{V}_1 = \overline{T}_2X + \overline{V}_2M.$$

Since

$$\overline{V}_1U_1 + \overline{T}_1W_1 = \overline{T}_1\overline{T}_2^{-1}(\overline{T}_2X + \overline{V}_2M)U_1 + \overline{T}_1W_1,$$

we have

$$\begin{aligned} S(\overline{V}_1U_1 + \overline{T}_1W_1) &= (\overline{T}_2X + \overline{V}_2M)U_1 + \overline{T}_2W_1 \\ &= \overline{T}_2(XU_1 + W_1) + \overline{V}_2MU_1 \\ &= \overline{T}_2(V_2Y + W_2) + \overline{V}_2MU_1 \\ &= \overline{T}_2W_2 + \overline{T}_2V_2Y + \overline{V}_2(U_2 - T_2Y) \\ &= \overline{T}_2W_2 + \overline{V}_2U_2 + (\overline{T}_2V_2 - \overline{V}_2T_2)Y \\ &= \overline{T}_2W_2 + \overline{V}_2U_2. \end{aligned}$$

So, with S unimodular, we have

$$\begin{aligned} &S(z)\big(\overline{T}_1(z) - (\overline{T}_1(z)W_1(z) + \overline{V_1}(z)U_1(z))\big) \\ &\quad = \big(\overline{T}_2(z) - (\overline{T}_2(z)W_2(z) + \overline{V_2}(z)U_2(z))\big). \end{aligned} \tag{198}$$

Applying Theorem 3.4.2, we have obtained

$$\begin{aligned}&\operatorname{Ker}\bigl(\overline{T}_1(\sigma)-(\overline{T}_1(\sigma)W_1(\sigma)+\overline{V_1}(\sigma)U_1(\sigma))\bigr)\\&\quad=\operatorname{Ker}\bigl(\overline{T}_2(\sigma)-(\overline{T}_2(\sigma)W_2(\sigma)+\overline{V_2}(\sigma)U_2(\sigma))\bigr).\end{aligned}$$

We conclude that the two behaviors are equal.
Conversely, assume the two behaviors are equal. Considering in a natural way the behavior equations (182) to be NARMA representations. Then, noting that $V_i, T_i, i = 1, 2$, are right coprime and applying Part 4, we conclude that there exist appropriately sized polynomial matrices $M_1, X_1, X_2, N_{11}, N_{12}, N_{21}, N_{22}$ satisfying

$$\begin{pmatrix}M_1 & 0 & 0\\ -X_1 & I & 0\\ -X_2 & 0 & I\end{pmatrix}\begin{pmatrix}T_1 & -U_1\\ V_1 & W_1\\ 0 & I\end{pmatrix}=\begin{pmatrix}T_2 & -U_2\\ V_2 & W_2\\ 0 & I\end{pmatrix}\begin{pmatrix}N_{11} & N_{12}\\ N_{21} & N_{22}\end{pmatrix}, \tag{199}$$

and for which there exists a doubly unimodular embedding of

$$\begin{pmatrix}M_1 & 0 & 0 & T_2 & -U_2\\ -X_1 & I & 0 & V_2 & W_2\\ -X_2 & 0 & I & 0 & I\end{pmatrix},\quad\begin{pmatrix}T_1 & -U_1\\ V_1 & W_1\\ 0 & I\\ -N_{11} & -N_{12}\\ -N_{21} & -N_{22}\end{pmatrix}.$$

Using the unimodular matrices in

$$\begin{pmatrix}I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ X_2 & 0 & I & 0 & -I\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\end{pmatrix}\begin{pmatrix}I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ -X_2 & 0 & I & 0 & I\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\end{pmatrix}=\begin{pmatrix}I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\end{pmatrix} \tag{200}$$

it follows that there exists also a doubly unimodular embedding of

$$\begin{pmatrix}M_1 & 0 & 0 & T_2 & -U_2\\ -X_1 & I & 0 & V_2 & W_2\\ 0 & 0 & I & 0 & 0\end{pmatrix},\quad\begin{pmatrix}T_1 & -U_1\\ V_1 & W_1\\ -X_2T_1-N_{21} & X_2U_1+I-N_{22}\\ -N_{11} & -N_{12}\\ -N_{21} & -N_{22}\end{pmatrix}.$$

However, from (199) it immediately follows that

$$N_{21}=-X_2T_1,$$
$$N_{22}=I+X_2U_1.$$

So we get, applying Lemma 3.2.2, that there exists also a doubly unimodular embedding of

$$\begin{pmatrix} M_1 & 0 & T_2 & -U_2 \\ -X_1 & I & V_2 & W_2 \end{pmatrix}, \quad \begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \\ -N_{11} & -N_{12} \\ -N_{21} & -N_{22} \end{pmatrix}.$$

In particular, we get

$$\begin{pmatrix} M_1 & 0 \\ -X_1 & I \end{pmatrix}\begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{pmatrix} = \begin{pmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{pmatrix}\begin{pmatrix} N_{11} & N_{12} \\ -X_2T_1 & I + X_2U_1 \end{pmatrix}, \tag{201}$$

which can be rewritten as

$$\begin{pmatrix} M_1 - U_2X_2 & 0 \\ -(X_1 - W_2X_2) & I \end{pmatrix}\begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{pmatrix} = \begin{pmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{pmatrix}\begin{pmatrix} N_{11} & N_{12} \\ 0 & I \end{pmatrix}. \tag{202}$$

By appropriately defining M, X, N, Y, Eq. (202) can be rewritten in the form (195) and the unimodular embeddability condition still holds. Thus we proved that the two polynomial system matrices are FSE.

6. As shown in Fuhrmann [1977], the isomorphism of the two systems is equivalent to the strict system equivalence of the associated polynomial system matrices

$$\begin{pmatrix} zI - A_i & -B_i \\ C_i & D_i \end{pmatrix}.$$

By Part 5, this is equivalent to the coincidence of the behaviors.
Alternatively, we can argue as follows. We apply Part 5 and conclude that the polynomial system matrices

$$\begin{pmatrix} zI - A_i & -B_i \\ C_i & D_i \end{pmatrix}$$

are FSE. Thus there exist polynomial matrices $M(z), X(z), N(z), Y(z)$ with M, T_2 left coprime T_1, N right coprime and for which

$$\begin{pmatrix} M(z) & 0 \\ -X(z) & I \end{pmatrix}\begin{pmatrix} zI - A_1 & -B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} zI - A_2 & -B_2 \\ C_2 & D_2 \end{pmatrix}\begin{pmatrix} N(z) & Y(z) \\ 0 & I \end{pmatrix}. \tag{203}$$

Decompose, in a unique way, the polynomial matrices $M(z), N(z)$ in the form

$$\begin{aligned} M(z) &= M + (zI - A_2)M'(z), \\ N(z) &= N + N'(z)(zI - A_1). \end{aligned} \tag{204}$$

Substituting in (203), we have

$$\begin{aligned} &\begin{pmatrix} M + (zI - A_2)M'(z) & 0 \\ -X(z) & I \end{pmatrix}\begin{pmatrix} zI - A_1 & -B_1 \\ C_1 & D_1 \end{pmatrix} \\ &\quad = \begin{pmatrix} zI - A_2 & -B_2 \\ C_2 & D_2 \end{pmatrix}\begin{pmatrix} N + N'(z)(zI - A_1) & Y(z) \\ 0 & I \end{pmatrix}, \end{aligned} \tag{205}$$

i.e. we have the following equations:

$$\begin{aligned}&M(zI-A_1)+(zI-A_2)M'(z)(zI-A_1)\\&\quad=(zI-A_2)N+(zI-A_2)N'(z)(zI-A_1)-MB_1-(zI-A_2)M'(z)B_1\\&\quad=(zI-A_2)Y(z)-B_2-X(z)(zI-A_1)+C_1\\&\quad=C_2N+C_2N'(z)(zI-A_1)-X(z)B_1+D_1=C_2Y(z)+D_2.\end{aligned}\tag{206}$$

Moreover, M, $(zI-A_2)$ are left coprime and $(zI-A_1)$, M are right coprime. These coprimeness conditions imply the nonsingularity of M.
From the first equation we obtain $M'(z)=N'(z)$ and $M(zI-A_1)=(zI-A_2)N$ which translates into

$$\begin{aligned}&M=N,\\&MA_1=A_2M.\end{aligned}\tag{207}$$

The second equation gives $Y(z)=M'(z)B_1$ and

$$B_2=MB.\tag{208}$$

From the third equation we infer $X(z)=-C_2N'(z)$ and

$$C_1=C_2M.\tag{209}$$

We compute now

$$-X(z)B_1=C_2N'(z)B_1=C_2M'(z)B_1=C_2Y(z).$$

Using this in the last equality of (206), we have

$$D_1=D_2.\tag{210}$$

Putting (207)–(210) in matrix form, we obtain (185).

7. If such matrices exist, then by Lemma 3.2.3, we obtain the existence of a doubly unimodular embedding for

$$\begin{pmatrix}U&0&zI-\overline{A}&-\overline{B}\\0&I&\overline{C}&\overline{D}\end{pmatrix},\quad\begin{pmatrix}zI-A&-B\\C&D\\-U&0\\-K&-P\end{pmatrix}.$$

By Part 3, the two behaviors coincide.
To prove the converse we reduce it to the case of state space representations treated in Part 6. we choose a basis in the signal space W for which

$$D=\begin{pmatrix}H_{11}&H_{12}\\H_{21}&H_{22}\end{pmatrix},\quad\overline{D}=\begin{pmatrix}\overline{H}_{11}&\overline{H}_{12}\\\overline{H}_{21}&\overline{H}_{22}\end{pmatrix},$$

where H_{22}, $\overline{H}_{22}$ are both nonsingular. With respect to this basis we have that the systems in (186) are permutation NARMA equivalent respectively to

$$\begin{aligned}\begin{pmatrix}0 & 0\\ I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} &= \begin{pmatrix}zI-A & -B\\ H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix},\\ \begin{pmatrix}0 & 0\\ I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} &= \begin{pmatrix}zI-\overline{A} & -\overline{B}\\ \overline{H}_{11} & \overline{H}_{12}\\ \overline{H}_{21} & \overline{H}_{22}\end{pmatrix}.\end{aligned} \tag{211}$$

Now

$$\begin{aligned}\begin{pmatrix}zI-A & -B\\ H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}\begin{pmatrix}I & 0\\ H_{22}^{-1}H_{21} & H_{22}^{-1}\end{pmatrix} &= \begin{pmatrix}zI-A+BH_{22}^{-1}H_{21} & -BH_{22}^{-1}\\ H_{11}-H_{12}H_{22}^{-1}H_{21} & H_{12}H_{22}^{-1}\\ 0 & I\end{pmatrix}\\ &= \begin{pmatrix}zI-F & -G\\ H & J\\ 0 & I\end{pmatrix}\end{aligned} \tag{212}$$

and similarly for the other system. The new systems, now in observable state space form, are still NARMA equivalent. Applying Part 6, there exists a constant invertible matrix U satisfying

$$\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}zI-F & -G\\ H & J\end{pmatrix} = \begin{pmatrix}zI-\overline{F} & -\overline{G}\\ \overline{H} & \overline{J}\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix}.$$

Putting all this together, we have

$$\begin{aligned}&\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}zI-A & -B\\ C & D\end{pmatrix}\begin{pmatrix}I & 0\\ H_{22}^{-1}H_{21} & H_{22}^{-1}\end{pmatrix}\\ &\quad= \begin{pmatrix}U & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix}zI-F & -G\\ H & J\\ 0 & I\end{pmatrix} = \begin{pmatrix}zI-\overline{F} & -\overline{G}\\ \overline{H} & \overline{J}\\ 0 & I\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix}\\ &\quad= \begin{pmatrix}zI-\overline{A} & -\overline{B}\\ \overline{H}_{11} & \overline{H}_{12}\\ \overline{H}_{21} & \overline{H}_{22}\end{pmatrix}\begin{pmatrix}I & 0\\ \overline{H}_{22}^{-1}\overline{H}_{21} & \overline{H}_{22}^{-1}\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix} = \begin{pmatrix}zI-\overline{A} & -\overline{B}\\ \overline{C} & \overline{D}\end{pmatrix}\end{aligned}$$

and so

$$\begin{aligned}&\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}zI-A & -B\\ C & D\end{pmatrix}\\ &\quad= \begin{pmatrix}zI-\overline{A} & -\overline{B}\\ \overline{C} & \overline{D}\end{pmatrix}\begin{pmatrix}I & 0\\ \overline{H}_{22}^{-1}\overline{H}_{21} & \overline{H}_{22}^{-1}\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I & 0\\ -H_{21} & H_{22}\end{pmatrix}\\ &\quad= \begin{pmatrix}zI-\overline{A} & -\overline{B}\\ \overline{C} & \overline{D}\end{pmatrix}\begin{pmatrix}U & 0\\ \overline{H}_{22}^{-1}\overline{H}_{21}U-\overline{H}_{22}^{-1}H_{21} & \overline{H}_{22}^{-1}H_{22}\end{pmatrix}.\end{aligned}$$

This proves the statement by defining $K = \overline{H}_{22}^{-1}\overline{H}_{21}U - \overline{H}_{22}^{-1}H_{21}$ and $P = \overline{H}_{22}^{-1}H_{22}$.

8. Assuming nonsingular polynomial matrices S and T exist for which (190) holds, then applying Lemma 3.2.3, we obtain the existence of a doubly unimodular embedding for

$$\begin{pmatrix} S & z\overline{G} - \overline{F} \end{pmatrix}, \quad \begin{pmatrix} zG - F \\ -T \end{pmatrix}.$$

By Lemma 3.2.2, so does

$$\begin{pmatrix} S & 0 & z\overline{G} - \overline{F} \\ 0 & I & 0 \end{pmatrix}, \quad \begin{pmatrix} zG - F \\ 0 \\ -T \end{pmatrix}.$$

Using the identity

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & \overline{H} \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & I & -\overline{H} \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

and the equality $H = \overline{H}T$ that follows from (190), we infer the existence of a doubly unimodular embedding for

$$\begin{pmatrix} S & 0 & z\overline{G} - \overline{F} \\ 0 & I & \overline{H} \end{pmatrix}, \quad \begin{pmatrix} zG - F \\ H \\ -T \end{pmatrix}.$$

By Part 3, the two behaviors coincide.

In order to prove the converse, we reduce it to Part 7. By our assumption that G has full row rank, there exist nonsingular constant matrices U, W such that $UGW = (I \;\; 0)$. So

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} zG - F \\ H \end{pmatrix} W = \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}. \tag{213}$$

Similarly, we have

$$\begin{pmatrix} \overline{M} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} z\overline{G} - \overline{F} \\ \overline{H} \end{pmatrix} W = \begin{pmatrix} zI - \overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} W. \tag{214}$$

These transformations imply the injectivity of $D, \overline{D}$ and the right primeness of the polynomial system matrices

$$\begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}, \quad \begin{pmatrix} zI - \overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{pmatrix}.$$

Thus the two DV state space systems

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} zI - \overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

satisfy the conditions of Part 7. Hence, there exists an invertible matrices U, P and a matrix K for which

$$\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix} = \begin{pmatrix} zI - \overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} \begin{pmatrix} U & 0 \\ K & P \end{pmatrix} \tag{215}$$

holds. Using (213)–(215), we compute

$$\begin{aligned} &\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zG - F \\ H \end{pmatrix} \\ &\quad = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix} W^{-1} \\ &\quad = \begin{pmatrix} zI - \overline{A} & -\overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} \begin{pmatrix} U & 0 \\ K & P \end{pmatrix} W^{-1} \\ &\quad = \begin{pmatrix} \overline{M} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} z\overline{G} - \overline{F} \\ \overline{H} \end{pmatrix} \begin{pmatrix} U & 0 \\ K & P \end{pmatrix} W^{-1}, \end{aligned}$$

i.e.

$$\begin{pmatrix} \overline{M}^{-1} U M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} zG - F \\ H \end{pmatrix} = \begin{pmatrix} z\overline{G} - \overline{F} \\ \overline{H} \end{pmatrix} \begin{pmatrix} U & 0 \\ K & P \end{pmatrix} W^{-1}.$$

So defining $S = \overline{M}^{-1} U M$ and $T = \begin{pmatrix} U & 0 \\ K & P \end{pmatrix} W^{-1}$, (187) follows. □

Remarks.

1. The equivalence notion given in Part 4 is not new and relates to the study of state feedback in the context of polynomial matrix descriptions. This has been studied in great detail in Prätzel-Wolters [1981]. The equivalence of two systems given in PMD form, given in Theorem 7.1.5, is due to Hinrichsen and Prätzel-Wolters [1980b].
2. The assumption in Part 7 that in the DV representation (186) the matrix D is injective is equivalent to the fact that the matrix

$$\begin{pmatrix} zI - F & -G \\ H & J \\ 0 & I \end{pmatrix}$$

in (212) is in column Kronecker–Hermite form. This, taken together with the assumption of right primeness is equivalent to the minimality of the representation, see Schumacher [1989] and Kuijper [1992,1994]. In fact, the proof of

Part 7 is adapted from Kuijper. Minimality will be discussed in the following subsection.

3. The three assumptions in Part 8 are equivalent to the minimality of the P-representation (188), see Kuijper [1992,1994]. Assumptions (a) and (b) by themselves again relate to the reduction to column Kronecker–Hermite form.

7.1. Minimality of representations

We end the paper with a brief discussion of minimality of representations. The results are standard, see Kuijper [1992,1994], however the derivation seems to be much more elementary as it is based mostly on operations done on polynomial matrices.

Definition 7.2.

1. We say that a state space representation (184) of a behavior $\mathscr{B}$, with $A : X \to X$ and $B : U \to X$, is minimal if the dimension of the state space X is minimal.
2. We say that a DV-representation (186) of a behavior $\mathscr{B}$, with $A : X_1 \to X_1$ and $B : X_2 \to X_1$, is minimal if both the dimensions of the spaces X_1, X_2 are minimal.
3. We say that a P-representation (188) of a behavior $\mathscr{B}$, with $G, F : Z \to X$ and $H : Z \to W$, is minimal if both the dimensions of the spaces X, Z are minimal.

Note that Part 3 of the definition says that in the representation (188) we use the minimal number of auxiliary variables (dim Z) and a minimal number of equations (dim X). The other statements are special cases. The following gives the characterization of minimality for the first order systems under discussion.

Theorem 7.2.

1. *Necessary and sufficient conditions for the minimality of a P-representation*
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma G - F \\ H \end{pmatrix} \xi \tag{216}$$
are
   (a) *G has full row rank.*
   (b) *$\begin{pmatrix} G \\ H \end{pmatrix}$ has full column rank.*
   (c) *$\begin{pmatrix} zG-F \\ H \end{pmatrix}$ is right prime.*
2. *A necessary and sufficient condition for the minimality of a DV-representation*
$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{217}$$
is that
   (a) *D has full column rank.*
   (b) *$\begin{pmatrix} zI-A \\ C \end{pmatrix}$ is right prime.*

3. *A necessary and sufficient condition for the minimality of a state space representation*

$$\begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \sigma I - A & -B \\ C & D \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{218}$$

is that the pair (C, A) *is observable.*

Proof.

1. We begin by proving the necessity of conditions 1(a)–(c). If G is not of full row rank, then, by applying constant elementary row operations on $zG - F$, we can assume without loss of generality that there exists a constant row in $zG - F$. It cannot be a zero row as it contradicts minimality. If it is a nonzero row, say $(\alpha_1 \quad \cdots \quad \alpha_m)$, we can assume without loss of generality that $\alpha_m \neq 0$. This means $\alpha_1\xi_1 + \cdots + \alpha_m\xi_m = 0$ and hence $\xi_m = -\alpha_m^{-1}(\alpha_1\xi_1 + \cdots + \alpha_{m-1}\xi_{m-1})$. Thus the number of variables could be reduced, contrary to minimality. This proves the necessity of 1(a).

   Applying an appropriate constant nonsingular matrix on the right, which clearly does not change the behavior, we can assume without loss of generality that

$$\begin{pmatrix} zG - F \\ H \end{pmatrix} = \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}. \tag{219}$$

   Now $\begin{pmatrix} G \\ H \end{pmatrix}$ has full column rank if and only if

$$\begin{pmatrix} I & 0 \\ C & D \end{pmatrix}$$

   has full column rank. If D fails to have full column rank, we can assume without loss of generality that

$$\begin{pmatrix} zG - F \\ H \end{pmatrix} = \begin{pmatrix} zI - A & -B_1 & -B_2 \\ C & D_1 & 0 \end{pmatrix}$$

   with D_1 of full row rank. Now, if $B_2 = 0$ then clearly the number of auxiliary variables can be reduced. If $B_2 \neq 0$, then by elementary column operations we can eliminate the variables in at least one of the rows of $(zI - A \quad -B_1 \quad -B_2)$, which contradicts the assumption of minimality. This proves the necessity of 1(b).

   Since we have the equality (219), and D has full column rank, the right primeness of $\begin{pmatrix} zG-F \\ H \end{pmatrix}$ is equivalent to the right primeness of $\begin{pmatrix} zI-A \\ C \end{pmatrix}$. If the last matrix is not right prime, then the pair C, A is not observable, hence in some basis has the representation

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix},$$

$$C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}.$$

The behavior equations are now

$$\begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma I - A_1 & -A_3 & -B_1 \\ 0 & \sigma I - A_2 & -B_2 \\ C_1 & C_2 & D \end{pmatrix} \xi.$$

The injectivity of D allows us to assume without loss of generality that $C_1 = 0$. Since obviously $zI - A_1$ has full row rank, we can apply Proposition 3.5 and reduced the behavior equations to

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma I - A_2 & -B_2 \\ C_2 & D \end{pmatrix} \xi,$$

which contradicts minimality. Thus the necessity of 1(c) is proved.
To prove the converse, let us assume that we have two P-representations of the same behavior

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} \sigma G_i - F_i \\ H_i \end{pmatrix} \xi, \quad i = 1, 2. \tag{220}$$

Let the first P-representation be minimal whereas the second P-representation satisfies assumptions 1(a)–(c). Since these conditions are necessary for minimality, they are satisfied for both systems. Hence, by Theorem 7.1.8, they are isomorphic, i.e. there exist nonsingular, constant polynomial matrices S and T for which

$$\begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} sG_1 - F_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} sG_2 - F_2 \\ H_2 \end{pmatrix} T. \tag{221}$$

2. In this case $G = (I \;\; 0)$ if of full row rank and the result follows from Part 1.
3. In this case $G = (I \;\; 0)$ is of full row rank and

$$\begin{pmatrix} G \\ H \end{pmatrix} = \begin{pmatrix} I & 0 \\ C & D \\ 0 & I \end{pmatrix}$$

has full column rank. Also the right primeness of

$$\begin{pmatrix} zI - A & -B \\ C & D \\ 0 & I \end{pmatrix}$$

is equivalent to the right primeness of $\begin{pmatrix} zI - A \\ C \end{pmatrix}$, i.e. to the observability of the pair (C, A). So the result follows from Part 1. $\square$

References

[1980] A.C. Antoulas, Ph.D. thesis, ETH Zürich, 1980.

[1973] G. Basile, G. Marro, Controlled and conditioned invariant subspaces in linear system theory, J. Optim. Theory Appl. 3 (1973) 306–316.

[2001] M. Bisiacco, M.E. Valcher, Behavior decompositions and two sided diophantine equations, Automatica, 2001.

[1983] H. Blomberg, R. Ylinen, Algebraic Theory for Multivariable Linear Systems, Academic Press, London, 1983.

[1982] F.M. Callier, C. Desoer, Multivariable Feedback Systems, Springer, New York, 1982.

[1974] S. Eilenberg, Automata, Languages and Machines, Academic Press, New York, 1974.

[1980] E. Emre, M.L.J. Hautus, A polynomial characterization of (A, B)-invariant and reachability subspaces, SIAM J. Control Optim. 18 (1980) 420–436.

[2000] V. Fonf, Private communication, 2000.

[1976] P.A. Fuhrmann, Algebraic system theory: an analyst's point of view, J. Franklin Inst. 301 (1976) 521–540.

[1977] P.A. Fuhrmann, On strict system equivalence and similarity, Int. J. Control 25 (1977) 5–10.

[1981a] P.A. Fuhrmann, Duality in polynomial models with some applications to geometric control theory, IEEE Trans. Automat. Control AC-26 (1981a) 284–295.

[1981b] P.A. Fuhrmann, Linear Systems and Operators in Hilbert Space, McGraw-Hill, New York, 1981b.

[1994] P.A. Fuhrmann, A duality theory for robust control and model reduction, Linear Algebra Appl. 203–204 (1994) 471–578.

[1993] P.A. Fuhrmann, R. Ober, On Coprime Factorizations, T. Ando volume, Birkhauser, Basel, 1993.

[1979] P.A. Fuhrmann, J.C. Willems, Factorization indices at infinity for rational matrix functions, Integral Equations Operator Theory 2 (1979) 287–301.

[1980] P.A. Fuhrmann, J.C. Willems, A study of (A, B)-invariant subspaces via polynomial models, Int. J. Control 31 (1980) 467–494.

[1980] M.L.J. Hautus, P.A. Fuhrmann, On the zero module of rational matrix functions, in: Proceedings CDC Conferemce, Albuquerque, 1980.

[1978] M.L.J. Hautus, M. Heymann, Feedback—an algebraic approach, SIAM J. Control 16 (1978) 83–105.

[1976] M. Heymann, Controllability subspaces and feedback simulation, SIAM J. Control Optim. 14 (1976) 769–789.

[1974] P.J. Hilton, Y. Wu, A Course in Modern Algebra, Wiley–Interscience, New York, 1974.

[1980a] D. Hinrichsen, D. Prätzel-Wolters, Module theoretic characterization of system equivalence in time domain, in: Algebraic and Geometric Methods in Linear System Theory, AMS Lectures in Applied Mathematics, vol. 18, 1980a, pp. 137–156.

[1980b] D. Hinrichsen, D. Prätzel-Wolters, Solution modules and system equivalence, Int. J. Control 32 (1980b) 777–802.

[1983] D. Hinrichsen, D. Prätzel-Wolters, Generalized Hermite matrices and complete invariants of strict system equivalence, SIAM J. Control Optim. 21 (1983) 289–305.

[1997] M. Hou, A.C. Pugh, G.E. Hayton, Generalized transfer functions and input/output equivalence, Int. J. Control 68 (1997) 1163–1178.

[1980] T. Kailath, Linear Systems, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[1969] R.E. Kalman, P. Falb, M. Arbib, Topics in Mathematical System Theory, McGraw-Hill, New York, 1969.

[1982] P.P. Khargonekar, E. Emre, Further results on polynomial characterizations of (F, G)-invariant and reachability subspaces, IEEE Trans. Automat. Control AC-27 (1982) 352–366.

[1992] M. Kuijper, First order representations of linear systems, Ph.D. thesis, Catholic University of Brabant, 1992.

[1994] M. Kuijper, First Order Representations of Linear Systems, Birkhauser, Basel , 1994.

[1990] M. Kuijper, J.M. Schumacher, Realization of autoregressive equations in pencil and descriptor form, SIAM J. Control Optim. 28 (1990) 1162–1189.
[1990] U. Oberst, Multidimensional constant linear systems, Acta Appl. Math. 20 (1990) 1–175.
[1986] A.B. Özgüler, Polynomial characterizations of (H, F)-invariant subspaces with applications, Linear Algebra Appl. 73 (1986) 1–31.
[1994] A.B. Özgüler, Linear Multichannel Control, Prentice-Hall, Englewood Cliffs, NJ, 1994.
[1977] L. Pernebo, Notes on strict system equivalence, Int J. Control 25 (1977) 21–38.
[1997] J.W. Polderman, J.C. Willems, Introduction to Mathematical System Theory, Springer, New York, 1997.
[1981] D. Prätzel-Wolters, Feedback morphism between linear systems—a unified approach to state space systems, transfer functions and system matrices, Ph.D. dissertation, University of Bremen, 1981.
[1970] H.H. Rosenbrock, State-Space and Multivariable Theory, John Wiley, New York, 1970.
[1997] J. Rosenthal, J.M. Schumacher, Realization by inspection, IEEE Trans. Automat. Control AC-42 (1997) 1257–1263.
[2000] J. Rosenthal, Connections between linear systems and convolutional coding theory, preprint, 2000.
[1988] J.M. Schumacher, Transformations of linear systems under external equivalence, Linear Algebra Appl. 102 (1988) 1–34.
[1989] J.M. Schumacher, Linear system representations, in: H. Nijmeijer, J.M.S. Schumacher (Eds.), Three Decades of Mathematical System Theory. A Collection of Surveys at the Occasioon of the Fiftieth Birthday of Jan C. Willems, Lecture Notes in Control and Information Sciences, vol. 135, Springer, Berlin, 1989, pp. 382–408.
[2000] M.E. Valcher, A behavioral approach to classic polynomial matrix descriptions, Systems and Control Lett. 40 (2000) 279–288.
[1997] S. Weiland, A. Stoorvogel, Rational representation of behaviors: interconnectability and stabilizability, Math. Control Signals Systems 10 (1997) 125–164.
[1986] J.C. Willems, From time series to linear systems. Part I: Finite-dimensional linear time invariant systems, Automatica 22 (1986) 561–580.
[1989] J.C. Willems, Models for dynamics, in: U. Kirchgraber, H.O. Walther (Eds.), Dynamics Reported, vol. 2, Wiley–Teubner, 1989, pp. 171–269.
[1991] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Trans. Automat. Control AC-36 (1991) 259–294.
[1979] H. Wimmer, A Jordan factorization theorem for polynomial matrices, Proc. Amer. Math. Soc. 75 (1979) 201–206.
[1979] W.M. Wonham, Linear Multivariable Control, second ed., Springer, New York, 1979.
[1981] B.F. Wyman, M.K. Sain, The zero module and essential inverse systems, IEEE Trans. Circuits and Systems 28 (1981) 112–126.
[1989] B.F. Wyman, M.K. Sain, G. Conte, A. Perdon, On the zeros and poles of a transfer function, Linear Algebra Appl. 122–124 (1989) 123–144.
[2000] E. Zerz, On strict system equivalence for multidimensional systems, Int. J. Control, 2000.