
International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

# On a connection between spectral factorization and geometric control theory

A. Gombani; P. A. Fuhrmann

To link to this article: DOI: 10.1080/00207170210139511
URL: http://dx.doi.org/10.1080/00207170210139511

INT. J. CONTROL, 2002, VOL. 75, NO. 11, 824–859





# On a connection between spectral factorization and geometric control theory

A. GOMBANI†* and P. A. FUHRMANN‡

We investigate here how the geometric control theory of Basile, Marro and Wonham can be obtained in a Hilbert space context, as the byproduct of the factorization of a spectral density with no zeros on the imaginary axis. We show how controlled invariant subspaces can be obtained as images of orthogonal projections of co-invariant subspaces onto a semi-invariant (Markovian) subspace of the Hardy space of square integrable functions analytic in the right half-plane. Output nulling subspaces are then related to a particular spectral factorization problem. A similar construction is presented for controllability subspaces, and a new algorithm for the computation of these subspaces is presented.

## 1. Introduction

Perhaps one of the most appealing features of the geometric control theory developed by Basile and Marro (1969) and Wonham (1991) is the generality of its construction. First a characterization of *all* controlled invariant subspaces for a given controllable pair $(A, B)$, and then a characterization of the output nulling ones for a given minimal realization of the transfer function $W = C(sI - A)^{-1}B + D$. The key point for us is that a given stable, controllable pair $(A, B)$ determines uniquely an inner function. It is therefore quite enticing to try and lift such a construction to a natural space for inner functions, a Hardy space. It turns out that under mild coercivity conditions on $WW^*$, the formulation of the results in the Hardy space setting is very simple, although the same does not hold for some proofs, and that the solution is strictly connected with stochastic realization. Such a direction could already be guessed from the paper by Lindquist *et al.* (1995), where the transmission zeros are studied. So, if the problem we investigate here is quite old, in fact the first works of Basile and Marro, for example, is from 1968, and the first work on the strong stochastic realization problem is from the mid-seventies (see e.g. Ruckebusch 1975), the approach we take here is nevertheless new. We exploit the Hilbert space structure, as is done in classical stochastic realization theory, but instead of analysing the geometry in the stochastic domain, we carry this structure entirely into the frequency setting. The result is that there is no stochastics in this paper, but just Hilbert spaces and geometric control theory. It is nevertheless only in stochastic realization theory (see Lindquist and Picci 1991, Lindquist *et al.* 1995, Fuhrmann and Gombani 1998) that the factorizations we will use are derived. This explains the terminology. The main advantage of this approach is a computational one. In fact, in Fuhrmann and Gombani (1998), we reduce the characterization of spectral factors to the study of a set of stable, all-pass (inner) functions. Then the factorizations and the projections can all be computed using state space formulas and solving Lyapunov and Riccati equations. As a byproduct, we obtain a simple algorithm for determining output nulling and controllability subspaces for a given function $W$.

It has been shown (see Lindquist *et al.* 1995) that if $\Phi$ is a full rank $p \times p$ rational spectral density *without zeros on the extended imaginary axis* then $W$ is a minimal stable $p \times m$ spectral factor if and only if there exist essentially unique inner functions $Q'$ and $Q''$, determined up to a constant unitary factors, such that $W = [W_-, 0]Q' = [W_+, 0]Q''^*$, where $W_-$ and $W_+$ are the minimum and maximum phase spectral factors. Moreover, it can be shown (see Fuhrmann and Gombani 1998) that this result holds also when the matrix $\Phi$ does not have full rank and that the inner function $Q'$ and $Q''$ are, under mild assumptions, uniquely determined by $W$ and that

$$Q'Q'' = \begin{pmatrix} Q_+ & 0 \\ 0 & R \end{pmatrix}$$

where $Q_+$ is an invariant depending only on $\Phi$ and the matrix $R$ is equivalent to an inner divisor of $Q_+$.

Let now

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

be a fixed realization of $W$. We show here how the functions $Q'$ and $Q''$ associated to $W$ determine a maximal, inner antistabilizable subspace $\mathcal{V}_+^*$ and a maximal, inner stabilizable subspace $\mathcal{V}_-^*$ of $\mathbb{C}^n$ which are output nulling. Moreover, the supremal output nulling controllability subspace $\mathcal{R}$ is the intersection of $\mathcal{V}_+^*$ and $\mathcal{V}_-^*$. The step from $Q'$ to the supremal antistabilizable subspace is quite simple. Let $\bar{W}K$ be the Douglas–Shapiro–Shields factorization (see below) of $W$; then $\mathcal{V}_+^*$ is (isomorphic

Received 1 September 2000. Revised 1 February 2002.

* Author for correspondence. e-mail: gombani@ladseb.pd.cnr.it

† LADSEB-CNR, Corso Stati Uniti 4, 35127 Padova, Italy.

‡ Department of Mathematics, Ben-Gurion University of Negev, Beer Sheva, Israel.

*International Journal of Control* ISSN 0020–7179 print/ISSN 1366–5820 online © 2002 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170210139511

to) the projection of the coinvariant subspace $H_r(Q')$ onto $H_r(K)$. The coinvariant subspace $H_r(K)$ is important in stochastic realization in this connection (see Lindquist and Picci 1991 or Ruckebusch 1980a) because it represents a natural state space for the realization of $W$.

Our work has been largely inspired by Lindquist *et al.* (1995). We believe that the present exposition is a natural extension of the results of that paper in the context of the geometric stochastic realization theory of Lindquist and Picci. Besides providing what we feel to be a better understanding of the theory, it presents a new algorithm for the computation of output nulling and controllability subspaces. This algorithm is based on spectral factorization and, unlike the existing ones (see Wonham 1991), does not need to invert the matrix $A$ of the state dynamics.

The paper is structured as follows: in § 2 we give some preliminary results. In § 3 we give a representation of controlled invariant subspaces in terms of co-invariant subspaces. We then show how output nulling subspaces are connected to spectral factorization theory for the case of stable spectral factors. In § 4 we give a similar two-steps representation for controllability subspaces. In § 5 we extend the results to non-stable factors. This approach is mainly due to expository reasons, since the main difficulties are already solved in the stable factors case. In § 6 we give a description of the algorithm for the construction of output nulling subspaces.

## 2. Preliminaries and notation

We work in the Hilbert space setting of the plane; we define (see Hoffman 1962) $L^2(\mathbb{I})$ to be the set of the vector or matrix valued (the proper dimension will be clear from the context) square integrable functions on the imaginary axis, and $H^2_+$ to be the subspace of $L^2$ of functions analytic in the right half-plane and such that

$$\sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr}\left(F^*(x+iy)F(x+iy)\right) \mathrm{d}y < \infty$$

where * denotes transposed conjugate. If $F$ and $G$ are column vectors, the inner product in $H^2_+$ is

$$\langle F, G\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(i\omega)F(i\omega)\,\mathrm{d}\omega$$

Analogously, $H^\infty_+$ is the subspace of $L^2$ of functions analytic in the right half-plane and such that

$$\sup_{\mathrm{Re}\, s>0} \|F(s)\| < \infty$$

where $\|F(x+iy)\|$ denotes the usual matrix norm. The spaces $H^2_-$ and $H^\infty_-$ are defined similarly on the left half-plane.

Let $F$ be a function of $L^2$: we denote by $\mathbf{P}^+F$ ($\mathbf{P}^-F$) the orthogonal projection of $L^2$ onto $H^2_+(H^2_-)$.

A function $Q \in H^\infty_+$ is **inner** if it is square and $Q^*Q = QQ^* = I$. It is well known that a column vector space $M$ in $H^2_+$ is invariant under multiplication by $\mathrm{e}^{-i\omega t}$ for $t \geq 0$ if and only if it is of the form $M = QH^2_+$ (Beurling's Theorem). Similarly, a row space $N$ is invariant if and only if it is of the form $N = H^2_+Q$. We set $H_c(Q) := (QH^2_+)^\perp$ and $H_r(Q) := (H^2_+Q)^\perp$. Clearly we have $\mathbf{P}^+\mathrm{e}^{i\omega t}H_c(Q) \subset H_c(Q)$, and similarly for $H_r(Q)$. Coinvariant subspaces are defined also in $H^2_-$. Given an inner function $Q \in H^\infty_+$, we define $\bar{H}_r(Q^*) = \{H^2_-Q^*\}^\perp = H^2_- \ominus H^2_-Q^*$. Similarly, a $p \times m$ function $Q$ is **row rigid** if $QQ^* = I_p$ (which entails $p \geq m$); it is **column rigid** if $Q^*Q = I_m$.

Clearly the concepts of inner and outer functions can be defined in a similar manner also in $H^2_-$. So, we will say that an $m \times m$ function $Q^* \in H^\infty_-$ is **conjugate** inner if $Q$ is inner and that an $m \times m_0$ function $\bar{F}_- \in H^2_-$ is conjugate outer if it is of full column rank as rational function and rank $\bar{F}_-(s) = m_0$ for Re $s < 0$.

A full column-rank $p \times m_0$ rational matrix function $F_-$ in $H^\infty_+$ is said to be **minimum-phase** or **outer** (on the right) if rank $F_-(s) = m_0$ for Re $s > 0$. It is well known that a $p \times m$ rational function $F$ in $H^\infty_+$ of essential rank $m_0$, it admits an outer-rigid factorization

$$F = F_-Q \tag{1}$$

where $F_-$ is $p \times m_0$ outer and $Q$ is an $m_0 \times m$ row rigid function. This factorization is unique up to a unitary constant matrix.

Similarly, the factorization

$$F = \bar{F}K \tag{2}$$

of $F$ in $H^2_+$ is a Douglas–Shapiro–Shields (**DSS**) factorization of $F$ if $\bar{F} \in H^2_-$, $K$ is inner and the degree of $K$ is as small as possible. Also this factorization is unique up to a unitary constant matrix.

A full column-rank $p \times m_0$ rational matrix function $G_+$ is said to be **maximum-phase** if its DSS factor $\bar{G}_+$ is (conjugate) outer in $H^2_-$. Given a $p \times m$ rational function $G$ of essential rank $m_0$ in $H^\infty_+$, it admits a maximum-phase conjugate-rigid factorization

$$G = G_+Q^* \tag{3}$$

where $G_+$ is $p \times m_0$ maximum-phase and $Q$ is an $m \times m_0$ column rigid function. Also this factorization is unique up to a unitary constant matrix.

As already mentioned in the introduction, as $G$ we can take a stable spectral factor $W$ of a spectral density $\Phi$ of dimension $p \times p$ and rank $m_0$. Then $W_-$ is the minimum-phase spectral factor and $W_+$ is the maximum-phase spectral factor of the spectral density $\Phi$; the DSS factors $\bar{W}_-$ and $\bar{W}_+$ are the so-called conjugate maximum phase and conjugate outer spectral factors, respectively.

Given two inner functions $P$, $Q$, we say that $\overline{QP}$ is a **skew-prime factorization** (see Khargonekar *et al*. 1983), if $PQ = \overline{QP}$, $P$ and $\bar{Q}$ are left coprime and $Q$ and $\bar{P}$ are right coprime.

From now on we assume all the functions to be rational. The notation

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

indicates that the quadruple $(A, B, C, D)$ is a realization of $W$. By $A^{\#}$ we denote the Moore–Penrose pseudo-inverse of a matrix $A$.

**Note:** throughout the paper, when we write '$P_Y|X$ is injective', we mean that the restricted projection operator $(P_Y)_{|X}$ is injective.

### 2.1. *Rigid and inner functions*

In this section we give a state space characterization of rigid functions and study their embedding in a square inner function. We begin by giving a state space characterization of rigid functions. This is a generalization of the well known characterization of inner functions (see Genin *et al*. 1983 or Fuhrmann and Ober 1993).

**Proposition 1:**

(1) *Let $\hat{R}$ be a $p \times m_0$ proper stable rational matrix function and let*

$$\hat{R} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{4}$$

*be a not necessarily minimal realization. Then $\hat{R}$ is column rigid, i.e. $\hat{R}^*\hat{R} = I$, if and only if*

(a) *We have*

$$D^*D = I$$

(b) *We can choose*

$$B = -YC^*D \tag{5}$$

*where $Y$ is the maximal non-negative definite solution of the homogeneous Riccati equation*

$$AY + YA^* + YC^*CY = 0 \tag{6}$$

(2) *Let $\hat{R}$ be a $m_0 \times p$ proper stable rational matrix function and let*

$$\hat{R} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{7}$$

*be a not necessarily minimal realization. Then $\hat{R}$ is row rigid, i.e. $\hat{R}\hat{R}^* = I$, if and only if*

(a) *We have*

$$DD^* = I$$

(b) *We can choose*

$$C = -DB^*X$$

*where $X$ is the maximal non-negative definite solution of the homogeneous Riccati equation*

$$A^*X + XA + XBB^*X = 0 \tag{8}$$

**Proof:**

(1) From the realization of $\hat{R}$ we obtain

$$\hat{R}^* = \left(\begin{array}{c|c} -A^* & C^* \\ \hline -B^* & D^* \end{array}\right)$$

The equality $\hat{R}^*\hat{R} = I$ implies

$$\left(\begin{array}{c|c} -A^* & C^* \\ \hline -B^* & D^* \end{array}\right) \times \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} A & 0 & B \\ C^*C & -A^* & C^*D \\ \hline D^*C & -B^* & D^*D \end{array}\right)$$

Necessarily we must have $D^*D = I$.

Since $A$ is stable, there exists a unique non-negative definite solution to Lyapunov equation

$$A^*Q + QA + C^*C = 0 \tag{9}$$

Applying the similarity transformation

$$\begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}$$

we compute

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}\begin{pmatrix} A & 0 \\ C^*C & -A^* \end{pmatrix}\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ QA + C^*C & -A^* \end{pmatrix}\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ A^*Q + QA + C^*C & -A^* \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \end{aligned}$$

Also

$$\begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}\begin{pmatrix} B \\ C^*D \end{pmatrix} = \begin{pmatrix} B \\ C^*D + QB \end{pmatrix}$$

and

$$(D^*C \quad -B^*)\begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}(D^*C + B^*Q \quad -B^*)$$

We have therefore

$$I = \left(\begin{array}{cc|c} A & 0 & B \\ 0 & -A^* & C^*D + QB \\ \hline D^*C + B^*Q & -B^* & I \end{array}\right)$$

This implies $C^*D + QB = 0$ or

$$Q^{\#}QB = -Q^{\#}C^*D$$

where $Q^{\#}$ is the Moore-Penrose pseudoinverse of $Q$. Multiplying (9) by $Q^{\#}$ on both sides we get

$$\begin{aligned} 0 &= Q^{\#}A^*QQ^{\#} + Q^{\#}QAQ^{\#} + Q^{\#}C^*CQ^{\#} \\ &= Q^{\#}A^* + AQ^{\#} + Q^{\#}C^*CQ^{\#} - Q^{\#}A^*(I - QQ^{\#}) \\ &\quad + (I - Q^{\#}Q)AQ^{\#} \\ &= Q^{\#}A^* + AQ^{\#} + Q^{\#}C^*CQ^{\#} \end{aligned}$$

since

$$\mathrm{Im}\,(I - QQ^{\#}) \perp \mathrm{Im}\,QQ^{\#}$$

implies

$$(I - Q^{\#}Q)AQ^{\#} = 0$$

and

$$\mathrm{Ker}(I - QQ^{\#}) \perp \mathrm{Ker}\,QQ^{\#}$$

entails $Q^{\#}A^*(I - QQ^{\#}) = 0$. We set now $Y = Q^{\#}$, so $Y$ is a non-negative definite solution to the the Riccati equation (6). That it is maximal follows from simple rank considerations on the solution to the Lyapunov equation. Finally, since $B = Y^{\#}YB + (I - Y^{\#}Y)B$, and since it is well known that $\mathrm{Ker}\ C(sI - A)^{-1} = \mathrm{Ker}\ Y^{\#}YA = \mathrm{Im}\ (I - YY^{\#})$, we conclude that

$$C(sI - A)^{-1}(I - YY^{\#})B = 0$$

We can therefore always choose $B$ so that (5) is satisfied.

Conversely, if $R$ is given by the realization (4), with the extra conditions satisfied, then it is straightforward to check that $R$ is indeed rigid.

(2) The proof is similar, or can be obtained by duality considerations. □

The next proposition studies the embedding of rigid functions in inner ones. This is a special case of Darlington synthesis.

**Proposition 2:**

(1) *Let $\hat{R}$ be a $p \times m_0$ column rigid function, that is $\hat{R}^*\hat{R} = I$. Then there exists a $p \times (p - m_0)$ column rigid function $\tilde{R}$ such that*

(a) $R = (\hat{R}\ \tilde{R})$ *is inner.*

(b) *We have the equality of McMillan degrees*

$$\delta(R) = \delta(\hat{R}) \tag{10}$$

$\tilde{R}$ *is uniquely determined up to a right constant unitary factor.*

(2) *Let $\hat{R}$ be a $m_0 \times p$ row rigid function, that is $\hat{R}\hat{R}^* = I$. Then there exists a $(p - m_0) \times p$ row rigid function $\tilde{R}$ such that*

(a) $R = \begin{pmatrix} \hat{R} \\ \tilde{R} \end{pmatrix}$ *is inner.*

(b) *We have the equality of McMillan degrees*

$$\delta(R) = \delta(\hat{R}) \tag{11}$$

$\tilde{R}$ *is uniquely determined up to a left constant unitary factor.*

**Proof:**

(1) Choose any $D'$ such that $(D\ D')$ is unitary. Such a $D'$ is unique up to a right unitary factor of size $p \times (p - m_0)$. Set

$$R = \left(\begin{array}{c|cc} A & B & B' \\ \hline C & D & D' \end{array}\right)$$

For $R$ to be inner it is necessary and sufficient that

$$(B \quad B') = -YC^*(D \quad D')$$

where $Y$ is the non-negative definite solution of the Riccati equation (6). So we make the choice

$$B' = -YC^*D'$$

and we get an inner embedding. It is obvious that $\delta(R) = \delta(\hat{R})$.

(2) The proof is similar or can be obtained from the first part by duality considerations. □

We will refer to the extensions obtained in the previous proposition as **minimal inner embeddings**.

### 2.2. *Rectangular spectral factors*

Let $W$ be a $p \times m$ stable rational function. Then, from the factorization (1), (2) and (3) we know that there exist functions $W_-$, $W_+$, $\bar{W}_-$, $\bar{W}_+$ (we will call them *extremal spectral factors*) which are, respectively outer, maximum-phase, conjugate maximum-phase and conjugate outer, and rigid functions $\hat{Q}'$, $\hat{Q}''$, $\hat{\bar{Q}}'$, $\hat{\bar{Q}}''$ such that

$$W = W_-\hat{Q}' = W_+(\hat{Q}'')^* = \bar{W}_-\hat{\bar{Q}}'K = \bar{W}_+(\hat{\bar{Q}}'')^*K$$

(we refer to Fuhrmann and Gombani 1998 for details).

The term *spectral factor* comes from the fact that the absolute value on the boundary $\Phi(i\omega)$ of all these functions is the same, i.e.

$$\Phi := WW^* = W_- W_-^* = W_+ W_+^* = \bar{W}_- \bar{W}_-^* = \bar{W}_+ \bar{W}_+^*$$

Therefore, we will often refer to the functions $W$, $W_-$ etc. as spectral factors.

It can also be shown (by a straightforward application of Proposition 2 to the above factorizations) that there exist inner functions $Q_-$, $Q_+$, $K_-$, $K_+$ of size $m_0 \times m_0$ such that

$$W_+ = W_- Q_+ = \bar{W}_- K_- Q_+ = \bar{W}_+ K_+$$

i.e. Figure 1 commutes.

While the dimension of $W$ is $p \times m$ (we will suppose it's fixed from now on), it is clear from the definitions of outer and maximum-phase factors that the extremal factors $W_-$, $W_+$, $\bar{W}_-$, $\bar{W}_+$ have all dimension $p \times m_0$ and, as we said, the inner functions $Q_-$, $Q_+$, $K_-$, $K_+$ have all size $m_0 \times m_0$. In the following analysis, though, it will be crucial to have all factors of dimension $p \times m$ and the inner functions of dimension $m \times m$. To this end we define the *extended spectral factors* as

$$W_-^e = [W_-,\ 0] \qquad W_+^e = [W_+,\ 0]$$
$$\bar{W}_-^e = [\bar{W}_-,\ 0] \qquad \bar{W}_+^e = [\bar{W}_+,\ 0]$$

where the 0 matrix is obviously of dimension $p \times (m - m_0)$. Similarly, from the inner matrices $Q_-$, $Q_+$, $K_-$, $K_+$, we can obtain the extended inner matrices

$$Q_-^e = \begin{bmatrix} Q_- & 0 \\ 0 & I_{m-m_0} \end{bmatrix} \qquad Q_+^e = \begin{bmatrix} Q_+ & 0 \\ 0 & I_{m-m_0} \end{bmatrix}$$
$$K_-^e = \begin{bmatrix} K_- & 0 \\ 0 & I_{m-m_0} \end{bmatrix} \qquad K_+^e = \begin{bmatrix} K_+ & 0 \\ 0 & I_{m-m_0} \end{bmatrix}$$

**Proposition 3:** *Let $W$ be a stable rational $p \times m$ spectral factor and let $W_-^e$, $W_+^e$, $\bar{W}_-^e$, $\bar{W}_+^e$.*

(1) *There exist essentially unique inner functions $Q'$, $Q''$, of minimal McMillan degree, for which*

$$\left.\begin{aligned} \bar{W} &= \bar{W}_-^e \bar{Q}' \\ \bar{W}_+^e &= \bar{W} \bar{Q}'' \end{aligned}\right\} \tag{12}$$

(2) *Let $\bar{W}_-^e$ and $\bar{W}_+^e$ be the extended antistable spectral factors. Given any minimal antistable spectral factor $\bar{W}$, there exist essentially unique inner functions $\bar{Q}'$, $\bar{Q}''$ for which*

$$\left.\begin{aligned} \bar{W} &= \bar{W}_-^e \bar{Q}' \\ \bar{W}_+^e &= \bar{W} \bar{Q}'' \end{aligned}\right\} \tag{13}$$

(3) *(DSS factorization) Let $W_+$, $W_+$, $\bar{W}_-^e$ and $\bar{W}_+^e$ be the extended extremal spectral factors and let $W$ be any stable factor. Then there exist essentially unique inner functions $K_-^e$, $K_+^e$ and $K$ such that*

$$W_-^e = \bar{W}_-^e K_-^e$$
$$W_+^e = \bar{W}_+^e K_+^e$$
$$W = \bar{W} K$$

*where $\bar{W}$ is the antistable DSS factor of $W$.*

(4) *With the above notation, we have the equalities*

$$K_-^e Q_+^e = K_-^e Q' Q'' = \bar{Q}' K Q'' = \bar{Q}' \bar{Q}'' K_+^e = Q_-^e K_+^e$$
$$Q_+^e (K_+^e)^* = Q' Q'' (K_+^e)^* = Q' K^* \bar{Q}'' = (K_-^e)^* \bar{Q}' \bar{Q}''$$
$$= (K_-^e)^* Q_+^e$$

*In other words, figure 2 is commutative.*

(5) *There exists an $(m - m_0) \times (m - m_0)$ inner function $R$ for which the following relations hold*

$$\begin{tikzcd}
W_- \arrow[r, "Q_+"] & W_+ \\
\bar{W}_- \arrow[u, "K_-"] \arrow[r, "Q_-"'] & \bar{W}_+ \arrow[u, "K"']
\end{tikzcd}$$

Figure 1.

$$\begin{tikzcd}
 & W \arrow[dr, "Q''"] & \\
W_-^e \arrow[ur, "Q'"] \arrow[rr, "Q_+"] & & W_+^e \\
\bar{W}_-^e \arrow[u, "K_-^e"] \arrow[rr, "Q_-"] \arrow[dr, "\bar{Q}'"'] & & \bar{W}_+^e \arrow[u, "K_+^e"'] \\
 & \bar{W} \arrow[uuu, "K"] \arrow[ur, "\bar{Q}''"'] &
\end{tikzcd}$$

Figure 2.

$$Q'Q'' = \begin{bmatrix} Q_+ & 0 \\ 0 & R \end{bmatrix}$$

$$\bar{Q}'\bar{Q}'' = \begin{bmatrix} Q_- & 0 \\ 0 & R \end{bmatrix}$$

For the proof see Fuhrmann and Gombani (1998).

A complete characterization of all such factorizations is available in terms of non-negative definite solutions of a homogeneous Riccati equation or in terms of invariant subspaces of a linear transformation. For more on this, see Willems (1971), Finesso and Picci (1982), Picci and Pinzoni (1994) and Fuhrmann (1995).

It is clear that stable, internal spectral factors are in a bijective correspondence with the set of all left inner factors of $Q_+^e$.

### 2.3. *Lyapunov equation and Hardy space metrics*

As we said in the introduction, the aim of this work is to provide a connection between co-invariant and controlled invariant subspaces by means of state space formulas. Since in the Hardy space setting we have natural notions of orthogonality, and we freely use orthogonal projections, we need a mechanism that allows the same operations to be done in the state space context. In particular, we need to find a metric in the state space that is equivalent to the $H_+^2$ metric, when restricted to an invariant subspace. The key to this is provided by the solution of a Lyapunov equation associated with a realization. We begin with the following lemma which characterizes the solution of a Lyapunov equation as a Gram matrix.

**Proposition 4:**

(1) *Let $(A, C)$ be an observable pair, with $A$ stable. Let $Q = (q_{ij})$ be the unique, positive definite, solution of the Lyapunov equation*

$$A^*Q + QA + C^*C = 0 \tag{14}$$

*Let $e_1, \dots, e_n$ be the standard column unit vectors and set*

$$v_i = C(sI - A)^{-1}e_i$$

*Then we have*

$$q_{ij} = (v_j, v_i)_{H_+^2} \tag{15}$$

(2) *Let $(A, B)$ be a reachable pair, with $A$ stable. Let $P = (p_{ij})$ be the unique, positive definite, solution of the Lyapunov equation*

$$AP + PA^* + BB^* = 0 \tag{16}$$

*Let $e_1, \dots, e_n$ be the standard row unit vectors and set*

$$v_i = e_i(sI - A)^{-1}B$$

*Then we have*

$$p_{ij} = (v_i,\ v_j)_{H_+^2} \tag{17}$$

**Proof:**

(1) The solution of the Lyapunov equation (14) is known to be given by

$$Q = \int_0^\infty \mathrm{e}^{A^* t}\, C^*C\, \mathrm{e}^{At}\, \mathrm{d}t$$

and hence

$$q_{ij} = \int_0^\infty \mathrm{e}_i\, \mathrm{e}^{A^* t}\, C^*C\ \mathrm{e}^{At}\, \mathrm{e}_j\, \mathrm{d}t$$
$$= (\hat{v}_j, \hat{v}_i)_{L^2(0,\infty)}$$

where $\hat{v}_i = C\,\mathrm{e}^{At}\,e_j \in L^2(0,\infty)$. Since the Fourier–Plancherel transform maps, unitarily, $L^2(0,\infty)$ onto $H_+^2$, and maps $C\,\mathrm{e}^{At}\,e_i$ to $C(sI - A)^{-1}e_i$, we get (15).

(2) The proof is similar. □

We now need a well known result (see Fuhrmann and Ober 1993, Fuhrmann 1995) about state space formulas for the Douglas–Shapiro–Shields factorization of a rational stable function.

**Lemma 1:** *Let*

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

*be a minimal realization of $W \in H_+^\infty$ and let $W = \bar{W}K$ be the right coprime DSS factorization of $W$ over $H_+^\infty$. Then*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

*where $P$ satisfies the Lyapunov equation $AP + PA^* + BB^* = 0$ and*

$$\bar{W} = \left(\begin{array}{c|c} -A^* & P^{-1}B \\ \hline DB^* + CP & D \end{array}\right)$$

*Conversely, let*

$$\bar{W} = \left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right)$$

*be a minimal realization of $\bar{W}$. Then $W$ has minimal realization*

$$W = \left( \begin{array}{c|c} -\overline{A}^* & \overline{PB} \\ \hline -\overline{DB}^* + \overline{CP}^{-1} & \overline{D} \end{array} \right)$$

*where* $\bar{P}$ *satisfies the Lyapunov equation* $\overline{AP} + \overline{PA}^* - \overline{BB}^* = 0$.

**Proof:** The computations are similar to those of Proposition 1. It is well known that the degree of $K$ is equal to the degree of the stable part of $W$, which coincides with the degree of $W$, since $W$ is stable. We show that with the above definition of $K$, $WK^*$ is in $H_-^\infty$. In fact, from the realization of $K$ we obtain

$$K^* = \left( \begin{array}{c|c} -A^* & P^{-1}B \\ \hline B^* & I \end{array} \right)$$

Then we have

$$WK^* = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \times \left( \begin{array}{c|c} -A^* & P^{-1}B \\ \hline B^* & I \end{array} \right)$$

$$= \left( \begin{array}{cc|c} -A^* & 0 & P^{-1}B \\ BB^* & A & B \\ \hline DB^* & C & D \end{array} \right)$$

Applying the similarity transformation

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}$$

we compute

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} -A^* & 0 \\ BB^* & A \end{pmatrix} \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$$

$$= \begin{pmatrix} -A^* & 0 \\ AP + PA^* + BB^* & A \end{pmatrix} = \begin{pmatrix} -A^* & 0 \\ 0 & A \end{pmatrix}$$

Also

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} P^{-1}B \\ B \end{pmatrix} = \begin{pmatrix} P^{-1}B \\ 0 \end{pmatrix}$$

and

$$(DB^* \quad C) \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} = (DB^* + CP \quad C)$$

We have therefore

$$\overline{W} = WK^* = \left( \begin{array}{cc|c} -A^* & 0 & P^{-1}B \\ 0 & A & 0 \\ \hline DB^* + CP & C & D \end{array} \right)$$

$$= \left( \begin{array}{c|c} -A^* & P^{-1}B \\ \hline DB^* + CP & D \end{array} \right)$$

i.e. $\overline{W} \in H_-^\infty$ and has the representation we wanted. Since the degree is minimal, it is a right coprime DSS factorization. As for the expression of $W$ in terms of a realization of $\overline{W}$, it follows from a dual argument. □

The next proposition gives a state space representation of finite dimensional, backward invariant subspaces. This is an analogue of a polynomial result of Hautus and Heymann (1979) and Wimmer 1979.

**Lemma 2:** *Let $S$ be a $m \times m$ rational matrix inner function and let*

$$S = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{18}$$

*be a non-necessarily minimal realization of dimension $n$. Then*

(1) *A representation of $H_r(S)$ is given by*

$$H_r(S) = \{\xi^*(sI - A)^{-1}B | \xi \in \mathbb{C}^n\} \tag{19}$$

(2) *A representation of $H_c(S)$ is given by*

$$H_c(S) = \{C(sI - A)^{-1}\xi | \xi \in \mathbb{C}^n\} \tag{20}$$

**Proof:** We give a proof for (1). The proof for (2) follows by duality.

Suppose first the realization of $S$ is minimal. Let then $g \in H_+^2$ and let $\Xi := [\xi_1, \xi_2, \ldots, \xi_n]$ where $\{\xi_i;\ i = 1, \ldots, n\}$ is a basis in $\mathbb{C}^n$. Set $X = \Xi^*(sI - A)^{-1}B$. Then the inner product of the $X$ and $gS$ is

$$\langle X, gS \rangle = \int_I \Xi^*(sI - A)^{-1}BS(s)^*g^*(s)\,\mathrm{d}s$$

Applying Lemma 1 to

$$X = \left( \begin{array}{c|c} A & B \\ \hline \Xi & 0 \end{array} \right) \quad \text{and} \quad S = \left( \begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array} \right)$$

we obtain

$$\langle X, gS \rangle = \int_I \Xi^*P(sI + A^*)^{-1}P^{-1}Bg^*(s)\,\mathrm{d}s = 0$$

since the function is analytic in the negative half-plane $\mathbb{C}^-$ and is the product of two strictly proper functions. Conversely, if $g \in H_+^2$ and $\langle \xi^*(sI - A)^{-1}B, g \rangle = 0$ for all $\xi \in \mathbb{C}^n$, then $\xi^*(sI - A)^{-1}Bg^*$ is analytic in $C^-$ or, equivalently, $g^*$ is divisible by the DSS factor of $(s1 - A)^{-1}B$ over $H_+^\infty$. But this factor is precisely $S^*$, and therefore the conclusion under the restrictive assumption.

Suppose now that the realization of $S$ is not minimal, and let $T$ be an invertible transformation such that

$$TAT^{-1} = \begin{pmatrix} \hat{A} & Z \\ 0 & \tilde{A} \end{pmatrix} \qquad TB = \begin{pmatrix} \hat{B} \\ 0 \end{pmatrix}$$

with $(\hat{A}, \hat{B})$ controllable. Then clearly

$$\{\xi^*(sI - A)^{-1}B;\ \xi \in \mathbb{C}^n\} = \{\xi^* T^{-1}(sI - \hat{A})^{-1}\hat{B};\ \xi \in \mathbb{C}^n\}$$
$$= \{\xi'^*(sI - \hat{A})^{-1}\hat{B};\ \xi' \in \mathbb{C}^n\}$$

and

$$S = \left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline -\hat{B}^*\hat{P}^{-1} & D \end{array}\right)$$

is obviously a minimal realization, so that we fall in the previous case. □

We prove now a slight extension of a result in Fuhrmann and Gombani (2000). Here we remove the coprimeness restriction. Given two $m \times m$ inner functions $S_1$ and $S_2$, we let $S$ be a common left inner multiple. Thus we have $S = \bar{S}_2 S_1 = \bar{S}_1 S_2$. The inner function $S$ is only defined up to a left inner factor. Now the coinvariant subspace $H_r(S)$ has two natural, orthogonal direct sum representations, namely

$$H_r(S) = H_r(\bar{S}_2 S_1) = H_r(\bar{S}_2)S_1 \oplus H_r(S_1)$$
$$= H_r(\bar{S}_1 S_2) = H_r(\bar{S}_1)S_2 \oplus H_r(S_2)$$

This means that computing orthogonal projections, like $P_{H_r(\bar{S}_2)S_1}|H_r(S_2)$ or $P_{H_r(S_2)}|H_r(S_1)$, can be done in the state space $H_r(\bar{S}_2 S_1)$.

**Lemma 3:** *Let*

$$S_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline C_i & I \end{array}\right) \qquad i = 1, 2$$

*be minimal realizations of two inner functions, and let*

$$A_e := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B_e := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C_e := (C_1 \quad C_2)$$

(1) *Let $S_L$ be a common left multiple of $S_1$ and $S_2$, and let $\bar{S}_1 := S_L S_2^*$. Then*

$$P_{H_r(\bar{S}_1)S_2}(sI - A_1)^{-1}B_1 = (sI - A_1)^{-1}\hat{B}_1 S_2(s) \quad (21)$$

*and*

$$P_{H_r(S_2)}(sI - A_1)^{-1}B_1 = P_{12}P_{22}^{-1}(sI - A_2)^{-1}B_2 \quad (22)$$

*where $\hat{B}_1 := B_1 - P_{12}P_{22}^{-1}B_2$ and*

$$P_e = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix}$$

*is the solution to the Lyapunov equation*

$$A_e P_e + P_e A_e^* + B_e B_e^* = 0 \quad (23)$$

*If $S_L$ is the least common left multiple of $S_1$ and $S_2$ then*

$$\bar{S}_1(s) = I - \hat{B}_1^* \hat{P}_{11}^{\#}(sI - A_1)^{-1}\hat{B}_1 \quad (24)$$

*where*

$$\hat{P}_{11} = P_{11} - P_{12}P_{22}^{-1}P_{12}^*$$

(2) *Define now $S_R$ to be a common right multiple of $S_1$ and $S_2$, and let $\bar{S}_2 := S_1^* S_R$. Then*

$$P_{S_1 H_c(\bar{S}_2)}C_2(sI - A_2)^{-1} = S_1(s)\hat{C}_2(sI - A_2)^{-1} \quad (25)$$

*and*

$$P_{H_c(S_1)}C_2(sI - A_2)^{-1} = C_1(sI - A_1)^{-1}Q_{11}^{-1}Q_{12} \quad (26)$$

*where $\hat{C}_2 := C_2 - C_1 Q_{11}^{-1}Q_{12}$ and*

$$Q_e = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix}$$

*is the solution to the Lyapunov equation*

$$A_e^* Q_e + Q_e A_e + C_e^* C_e = 0$$

*If $S_R$ is the least common right multiple of $S_1$ and $S_2$ then*

$$\bar{S}_2(s) = I - \hat{C}_2^*(sI - A_2)^{-1}\hat{Q}_{22}^{\#}\hat{C}_2 \quad (27)$$

*where*

$$\hat{Q}_{22} = Q_{22} - Q_{12}^* Q_{11}^{-1}Q_{12}$$

**Proof:** We only prove (1), since the proof of (2) is similar and can be obtained by duality considerations. First, suppose $S_L = \tilde{S}_L \hat{S}_L$, where $\hat{S}_L$ is the least left common multiple of $S_1$ and $S_2$; then $H_r(S_1) \vee H_r(S_2) = H_r(\hat{S}_L)$ (see Fuhrmann 1981b), and from the factorization $S_L = \tilde{S}_L(\hat{S}_L S_2^*)S_2$, we get $\bar{S}_1 = S_L S_2^* = \tilde{S}_L(\hat{S}_L S_2^*)$ which yields the decomposition

$$H_r(\bar{S}_1)S_2 = H(\hat{S}_L S_2^*)S_2 \oplus H(\tilde{S}_L)\hat{S}_L$$

and thus

$$P_{H_r(\bar{S}_1)S_2}|H_r(S_1) = P_{H_r(\hat{S}_L S_2^*)S_2 \oplus H_r(\tilde{S}_L)\hat{S}_L}|H_r(S_1)$$
$$= P_{H_r(\hat{S}_L S_2^*)S_2}|H_r(S_1)$$

since $S_1$ divides $S_L$. So, we can assume that $S_L$ is the least left common multiple of $S_1$ and $S_2$, i.e. $H_r(S_L) = H_r(S_1) \vee H_r(S_2)$.

In view of Lemma 2, $H_r(S_L)$ is spanned by the rows of $(sI - A_e)^{-1}\ B_e$. Observe that the diagonal blocks of equation (23) are $A_i P_{ii} + P_{ii}A_i^* + B_i B_i^* = 0$, for $i = 1, 2$, and therefore $C_i = -B_i^* P_{ii}^{-1}$, since the realizations of the $S_i$ are minimal. Applying to the extended state space $H_r(S_L)$ the similarity transformation

$$T = \begin{pmatrix} I & -P_{12}P_{22}^{-1} \\ 0 & I \end{pmatrix}$$

we see that it block diagonalizes $P_e$, in fact, with $\hat{B}_1 = B_1 - P_{12}P_{22}^{-1}B_2$, we have

$$\hat{P} := TP_eT^* = \begin{pmatrix} P_{11} - P_{12}P_{22}^{-1}P_{12}^* & 0 \\ 0 & P_{22} \end{pmatrix}$$

$$\hat{A} := TA_eT^{-1} = \begin{pmatrix} A_1 & -P_{12}P_{22}^{-1}A_2 + A_1P_{12}P_{22}^{-1} \\ 0 & A_2 \end{pmatrix}$$

$$\hat{B} := TB_e = \begin{pmatrix} \hat{B}_1 \\ B_2 \end{pmatrix}$$

We claim that

$$-P_{12}P_{22}^{-1}A_2 + A_1P_{12}P_{22}^{-1} = \hat{B}_1C_2 \tag{28}$$

In fact, the upper right corner of $\hat{A}\hat{P} + \hat{P}\hat{A}^* + \hat{B}\hat{B}^* = 0$ yields

$$(-P_{12}P_{22}^{-1}A_2 + A_1P_{12}P_{22}^{-1})P_{22} = -\hat{B}_1B_2^* = \hat{B}_1C_2P_{22}$$

since $C_2 = -B_2^*P_{22}^{-1}$. As the realization of $S_2$ is minimal, $P_{22}$ is invertible and (28) is proved. From (28) we get

$$P_{12}P_{22}^{-1}(sI - A_2) - (sI - A_1)P_{12}P_{22}^{-1} = \hat{B}_1C_2$$

and hence also the following partial fraction decomposition

$$\begin{aligned}(sI - A_1)^{-1}&P_{12}P_{22}^{-1} - P_{12}P_{22}^{-1}(sI - A_2)^{-1} \\ &= (sI - A_1)^{-1}\hat{B}_1C_2(sI - A_2)^{-1}\end{aligned}$$

We compute now

$$\begin{aligned}(sI - A_1)^{-1}B_1 &= (I \quad 0)(sI - A_e)^{-1}B_e \\ &= (I \quad 0)\begin{pmatrix} I & P_{12}P_{22}^{-1} \\ 0 & I \end{pmatrix}(sI - \hat{A})^{-1}\hat{B} \\ &= (I \quad P_{12}P_{22}^{-1}) \\ &\quad\times \begin{pmatrix} (sI - A_1)^{-1} & (sI - A_1)^{-1}\hat{B}_1C_2(sI - A_2)^{-1} \\ 0 & (sI - A_2)^{-1} \end{pmatrix} \\ &\quad\times \begin{pmatrix} \hat{B}_1 \\ B_2 \end{pmatrix} \\ &= (sI - A_1)^{-1}\hat{B}_1 + (sI - A_1)^{-1}\hat{B}_1C_2(sI - A_2)^{-1}B_2 \\ &\quad + P_{12}P_{22}^{-1}(sI - A_2)^{-1}B_2 \\ &= (sI - A_1)^{-1}\hat{B}_1S_2(s) + P_{12}P_{22}^{-1}(sI - A_2)^{-1}B_2\end{aligned}$$

Since

$$e_i(sI - A_2)^{-1}B_2 \in H_r(S_2)$$

and

$$e_j(sI - A_1)^{-1}\hat{B}_1S_2 \perp H_r(S_2)$$

we get (21) and (22), as wanted.

If $S_L$ is the least common left multiple of $S_1$ and $S_2$, the rows of $(sI - A_1)^{-1}\hat{B}_1S_2$ span $H(S_1)S_2$, and thus $\bar{S}_1 = I - \hat{B}_1^*\hat{P}_{11}^{\#}(sI - A_1)^{-1}\hat{B}_1$ where $\hat{P}_{11} = P_{11} - P_{12}P_{22}^{-1}P_{12}$, in view of Proposition 1 and Lemma 2, and this completes the proof. □

## 3. Geometric control

In this section we derive the connection between coinvariant subspaces in $H_+^2$ and special classes of controlled invariant subspaces in $\mathbb{C}^n$ (Theorem 1). It is then shown how output nulling subspaces relate to spectral factorizations (Theorem 2).

We recall a few definitions from geometric control theory (see Wonham 1979). A subspace $\mathcal{V} \subset \mathbb{C}^n$ is said to be a **controlled invariant subspace** if there exists a feedback matrix $F$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$. Any such feedback matrix $F$ is called a **friend** of $\mathcal{V}$. We say that the controlled invariant subspace $\mathcal{V}$ is **inner antistabilizable** if there exists a feedback matrix $F$ that is a friend of $\mathcal{V}$ such that $(A + BF)_{|\mathcal{V}}$ is antistable. Similarly we say that $\mathcal{V}$ is **inner stabilizable** if there exists a feedback $\bar{F}$ that is a friend of $\mathcal{V}$) such that $(A + B\bar{F})_{|\mathcal{V}}$ is stable. A subspace $\mathcal{V}$ is **output nulling** if $(A + BF)\mathcal{V} \subset \mathcal{V} \subset \text{Ker } (C + DF)$ for some feedback matrix $F$.

Note that, given a realization of a rational matrix function, for inner (anti)stabilizability we only use the pole information, whereas for the definition of output nulling, we use the full state space information.

Let $X \subset \mathbb{C}^n$. By the $\langle A|X\rangle$ we denote the subspace span $\{A^kX;\, 0 \le k\}$.

We want, as a first step, to provide a Hilbert space characterization of inner stabilizable and antistabilizable subspaces. In order to do that, we need to translate vectors of $\mathbb{C}^n$ into object of $H_+^2$. Such an approach is based on the shift and translation realization theory, using co-invariant subspaces of Hardy spaces as developed in Fuhrmann (1981b). To explain this, assume we have a stable reachable pair $(A, B)$ in $\mathbb{C}^n$. We consider the coprime factorization

$$(sI - A)^{-1}B = H(s)D(s)^{-1} \tag{29}$$

Necessarily, the polynomial matrix $D(s)$ is stable. Moreover, the rows of $H(s)$ are a basis for the row polynomial model $X_D^r$. Equivalently, the rows of $H(s)D(s)^{-1} = (sI - A)^{-1}B$ are a basis for the row rational model $X_r^D$. Rewriting the coprime factorization in the form $BD(s) = (sI - A)H(s)$, we have a natural isomorphic map from $X_D^c$ given by $f \mapsto \pi_D Bf$. On the other hand, the coprime factorization (29) provides a natural map from $\mathbb{C}_r^n$, i.e. the space $\mathbb{C}^n$ considered as a row space, onto $X_r^D$ given by $\xi \mapsto \xi(sI - A)^{-1}B$. Now the row rational model $X_r^D$ is, because of the stability of $D(s)$, actually a co-invariant subspace. In fact, if $\bar{D}$ is the antistable solution of the polynomial spectral factorization problem

$$\bar{D}^*(s)\bar{D}(s) = D^*(s)D(s) \tag{30}$$

Here $D^*(s) = D(-\bar{s})^*$, so $K(s) = \bar{D}(s)D(s)^{-1}$ is an inner function that satisfies $H_r(K) = X_r^D$ (see Fuhrmann 1981b). This provides the connection to Hardy spaces. In the state space $H_r(K)$ we define a pair $(\mathbf{A}_K, \mathbf{B}_K)$ by

$$\left.\begin{aligned} (\mathbf{A}_K f)(s) &= -sf(s) + [\lim_{s\to\infty} sf(s)] \cdot K(s) \\ \mathbf{B}_K &= \xi^*(I - K(s)) \end{aligned}\right\} \tag{31}$$

Since our starting point is the pair $(A, B)$ acting in $\mathbb{C}^n$, we need an appropriate map from $\mathbb{C}^n$ to $H_r(K)$. Since we map a column vector into a row vector function, such a map involves a conjugation and as a result it will be an antilinear map. Our choice is the following one. Let

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

be a minimal realization of a given inner function $K$, with $P$ the positive definite solution of the Lyapunov equation

$$AP + PA^* + BB^* = 0 \tag{32}$$

We define the map

$$\left.\begin{aligned} &I_{A,B} : \mathbb{C}^n \to H_r(K) \\ &I_{A,B}\xi = \xi^*P^{-1}(sI - A)^{-1}B \end{aligned}\right\} \tag{33}$$

For any subspace $\mathcal{V}$ in $\mathbb{C}^n$, we can then define its image in $H_+^2$ as the subspace

$$X_{\mathcal{V}} := I_{A,B}\mathcal{V}$$

Our next step is to relate the pair $(A, B)$ to the pair $(\mathbf{A}_K, \mathbf{B}_K)$.

**Proposition 5:** *Let* $(A, B)$ *be a stable, controllable pair in* $\mathbb{C}^n$. *Let*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

*where* $P$ *is the unique, positive definite solution of the Lyapunov equation* (32). *Let* $(\mathbf{A}_K,\ \mathbf{B}_K)$ *be defined by* (31). *The*

(1) *The map* $I_{A,B} : \mathbb{C}^n \to H_r(K)$ *is an antilinear isomorphism.*

(2) *Figure 3 is commutative.*

(3) *The image, under* $I_{A,B}$, *of an inner (anti)stabilizable subspace for* $(A, B)$ *is an inner (anti)stabilizable subspace for* $(\mathbf{A}_K,\ \mathbf{B}_k)$.

$$\begin{tikzcd} & \mathbb{C}^m \arrow[dl, "B"'] \arrow[dr, "\mathbf{B}_K"] & \\ \mathbb{C}^n \arrow[rr, "I_{A,B}"] \arrow[d, "A"'] & & H_r(K) \arrow[d, "\mathbf{A}_K"] \\ \mathbb{C}^n \arrow[rr, "I_{A,B}"] & & H_r(K) \end{tikzcd}$$

Figure 3.

**Proof:**

(1) Clearly, $I_{A,B}$ is by definition an antilinear map. That it is an isomorphism follows from the representation (19).

(2) Let $\eta \in \mathbb{C}^m$, then $B\eta \in \mathbb{C}^m$. We compute

$$\begin{aligned} I_{A,B}B\eta &= \eta^*B^*P^{-1}(sI - A)^{-1}B \\ &= \eta^*(I - K(s)) = \mathbf{B}_K\eta \end{aligned}$$

The Lyapunov equation (32) can be rewritten as $P^{-1}A + A^*P^{-1} + P^{-1}BB^*P^{-1} = 0$. Hence

$$P^{-1}(A + BB^*P^{-1}) = -A^*P^{-1} \tag{34}$$

With this, letting $f = \xi^*P^{-1}(sI - A)^{-1}B$, we can compute

$$\begin{aligned} \mathbf{A}_K I_{A,B}\xi &= (\mathbf{A}_K f)(s) = \mathbf{A}_K(\xi^*P^{-1}(sI - A)^{-1}B) \\ &= -sf(s) + [\lim_{s\to\infty} sf(s)] \cdot K(s) \\ &= -\xi^*P^{-1}s(sI - A)^{-1}B \\ &\quad + [\xi^* \lim_{s\to\infty} s(sI + A^*)^{-1}P^{-1}B]K(s) \\ &\qquad \text{(Lemma 1)} \\ &= -\xi^*P^{-1}s(sI - A)^{-1}B \\ &\quad + \xi^*P^{-1}B[I - B^*P^{-1}(sI - A)^{-1}B] \\ &= \xi^*P^{-1}[-sI + (sI - A) - BB^*P^{-1}](sI - A)^{-1}B \\ &= \xi^*P^{-1}[-A - BB^*P^{-1}](sI - A)^{-1}B \\ &= \xi^*A^*P^{-1}(sI - A)^{-1}B \\ &= I_{A,B}(A\xi) \end{aligned}$$

(3) Clearly, even though the map $I_{A,B}$ is antilinear, the property of controlled invariance is preserved under it. So it remains to show that if $\mathcal{V}$ is an inner (anti)stabilizable subspace with respect to $A$, $B$, so is $I_{A,B}\mathcal{V}$ with respect to $\mathbf{A}_K$, $\mathbf{B}_K$. To see this, let $L\colon \mathbb{C}^n \to \mathbb{C}^m$ be a friend of $\mathcal{V}$. Using previous computations, we have, for $\xi \in \mathcal{V}$

$$I_{A,B}(A+BL)\xi = (\mathbf{A}_K + \mathbf{B}_K L) I_{A,B}\xi$$

i.e. $I_{A,B}(A+BL)|\mathcal{V} = (\mathbf{A}_K + \mathbf{B}_K L) I_{A,B}\mathcal{V}$. Since $I_{A,B}\lambda\xi = \bar{\lambda} I_{A,B}\xi$, we have

$$\lambda \in \sigma(A+BL|\mathcal{V}) \Leftrightarrow \bar{\lambda} \in \sigma(\mathbf{A}_K + \mathbf{B}_K L | I_{A,B}\mathcal{V})$$

□

Now that we have the proper tools, we can start relating controlled invariant subspaces to coinvariant subspaces of $H^2$. The rest of the section is devoted to the proof of three results: in Theorem 1 we establish a correspondence between a general controlled invariant subspace for a given pair $(A, B)$ and a co-invariant subspace in $H^2$. In Lemma 4 and Theorem 2 we characterize, for a given stable function

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

the stabilizable output nulling and supremal output nulling subspaces in the Hardy space setting.

**Theorem 1:** *Let $(A, B)$ be a stable, controllable pair, and $\mathcal{V}$ a controlled invariant subspace of $\mathbb{C}^n$. Let $P$ be the unique, solution to the Lyapunov equation*

$$AP + PA^* + BB^* = 0 \tag{35}$$

*Set*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

*to be the inner matrix associated to $(A, B)$.*

(1) *The subspace $\mathcal{V}_+ \subset \mathbb{C}^n$ is an inner antistabilizable subspace if and only if there exists an inner function $Q'$ such that*

$$X_{\mathcal{V}_+} := I_{A,B}\mathcal{V}_+ = P_{H_r(K)} H_r(Q') \tag{36}$$

*If*

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}^*_{Q'}\mathcal{P}^{-1}_{Q'} & I \end{array}\right)$$

*and $\mathcal{P}_{Q'}$ is the unique solution of the Lyapunov equation*

$$\mathcal{A}_{Q'}\mathcal{P}_{Q'} + \mathcal{P}_{Q'}\mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} = 0 \tag{37}$$

*then*

$$\mathcal{V}_+ = \operatorname{Im} P_{KQ'}$$

*where $P_{KQ'}$ is the unique solution of the equation*

$$AP_{KQ'} + P_{KQ'}\mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} = 0$$

(2) *A subspace $\mathcal{V}_+ \subset H_r(K)$ is an inner antistabilizable subspace, for the pair $(\mathbf{A}_K, \mathbf{C}_K)$ defined in (31), if and only if there exists an inner function $Q'$ such that $\mathcal{V}_+ = P_{H_r(K)} H_r(Q')$.*

(3) *The subspace $\mathcal{V}_- \subset \mathbb{C}^n$ is an inner stabilizable subspace if and only if there exists an inner function $\bar{Q}''$ such that*

$$X_{\mathcal{V}_-} := I_{A,B}\mathcal{V}_- = P_{H_r(K)} \bar{H}_r(\bar{Q}''^*)K \tag{38}$$

*If*

$$\bar{Q}'' = \left(\begin{array}{c|c} \mathcal{A}_{\bar{Q}''} & \mathcal{B}_{\bar{Q}''} \\ \hline -\mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''} & I \end{array}\right)$$

*then*

$$\mathcal{V}_- = \operatorname{Im} PQ_{K\bar{Q}''}$$

*where $Q_{K\bar{Q}''}$ is the solution to*

$$A^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''} + P^{-1}B\mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''} = 0$$

**Proof:**

(1) We first show that if $X_{\mathcal{V}_+} = P_{H_r(K)} H_r(Q')$, then $\mathcal{V}_+ = I^{-1}_{A,B} X_{\mathcal{V}+}$ is inner antistabilizable. Let

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}^*_{Q'}\mathcal{P}^{-1}_{Q'} & I \end{array}\right)$$

be a minimal realization, say of degree $r'$, of the $m \times m$ inner function $Q'$. Let

$$A_e = \begin{pmatrix} \mathcal{A}_{Q'} & 0 \\ 0 & A \end{pmatrix}, \qquad B_e = \begin{pmatrix} \mathcal{B}_{Q'} \\ B \end{pmatrix}$$

and let

$$P_e = \begin{pmatrix} \mathcal{P} & P_{Q'K} \\ P_{KQ'} & P \end{pmatrix}$$

be the solution to the Lyapunov equation

$$A_eP_e + P_eA^*_e + B_eB^*_e = 0 \tag{39}$$

We divide the proof into several steps.

(a) We claim that $\mathcal{V}_+ := \operatorname{Im} P^*_{Q'K} = \operatorname{Im} P_{KQ'}$. In fact, in view of Lemma 3, we can write

$$\begin{aligned}\mathcal{V}_+ &= I_{A,B}^{-1} P_{H_r(K)} H_r(Q') \\ &= I_{A,B}^{-1} P_{H_r(K)} \{\zeta^*(sI - \mathcal{A}_{Q'})^{-1} \mathcal{B}_{Q'} | \zeta \in \mathbb{C}^{r'}\} \quad \text{Lemma 2} \\ &= I_{A,B}^{-1} \{\zeta^* P_{Q'K} P^{-1} (sI - A)^{-1} B | \zeta \in \mathbb{C}^{r'}\} \quad \text{Lemma 3} \\ &= \{P_{Q'K}^* \zeta | \zeta \in \mathbb{C}^{r'}\} = \text{Im } P_{Q'K}^*\end{aligned}$$

(b) Define $A_1 := P_{Q'K}^{\#} \mathcal{A}_{Q'} P_{Q'K}$, where $P_{Q'K}^{\#}$ is the Moore–Penrose pseudoinverse of $P_{Q'K}$. Then

$$\begin{aligned}A_1^* \mathcal{V}_+ &= P_{Q'K}^* \mathcal{A}_{Q'}^* (P_{Q'K}^{\#})^* \text{Im } P_{Q'K}^* \subset \text{Im } P_{Q'K}^* \\ &= \mathcal{V}_+\end{aligned}$$

and thus $\mathcal{V}_+$ is invariant for $A_1^*$.

(c) We have

$$A^*_{1|\mathcal{V}_+} = -(A + B\mathcal{B}^*_{Q'} P^{\#}_{KQ'})_{|\mathcal{V}_+}$$

In fact, the element in the lower left corner of (39) yields

$$AP_{KQ'} + P_{KQ'} \mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} = 0$$

or

$$AP_{KQ'} P^{\#}_{KQ'} + P_{KQ'} \mathcal{A}^*_{Q'} P^{\#}_{KQ'} + B\mathcal{B}^*_{Q'} P^{\#}_{KQ'} = 0$$

Thus, since the restriction of $P_{KQ'} P^{\#}_{KQ'}$ to $\mathcal{V}_+$ is the identity, we can write

$$P_{KQ'} \mathcal{A}^*_{Q'} P^{\#}_{KQ'|\mathcal{V}_+} = -(A + B\mathcal{B}^*_{Q'} P^{\#}_{KQ'})_{|\mathcal{V}_+}$$

that is, $\mathcal{V}_+$ is controlled invariant.

(d) Finally, if $\bar{\lambda}$ is an eigenvalue of $P_{KQ'} \mathcal{A}^*_{Q'} P^{\#}_{KQ'}$ with an eigenvector $\xi$ then it is also an eigenvalue of $\mathcal{A}^*_{Q'}$, with an eigenvector $P^{\#}_{KQ'} \xi$. Thus, as $\mathcal{A}_{Q'}$ is stable, this shows that $(A + B\mathcal{B}^*_{Q'} P^{\#}_{KQ'})_{|\mathcal{V}+}$ is antistable.

Conversely, assume now that $\mathcal{V}_+$ is an inner antistabilizable subspace. Then there exists a feedback matrix $F$ such that $(A + BF)\mathcal{V}_+ \subset \mathcal{V}_+$ and $(A + BF)_{|\mathcal{V}+}$ is antistable. Let $P_{KQ'}$ to be any full column-rank matrix having image $\mathcal{V}_+$. Since $P_{KQ'} P^{\#}_{KQ'}$, is a projection onto $\mathcal{V}_+$

$$(A + BF)_{|\mathcal{V}_+} = P_{KQ'} P^{\#}_{KQ'} (A + BF) P_{KQ'} P^{\#}_{KQ'}$$

Therefore we can define

$$\mathcal{A}^*_{Q'} := -P^{\#}_{KQ'} (A + BF) P_{KQ'}$$

and

$$\mathcal{B}_{Q'} = P^*_{KQ'} F^*$$

A simple computation yields

$$\begin{aligned}&AP_{KQ'} + P_{KQ'} \mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} \\ &= AP_{KQ'} - P_{KQ'} P^{\#}_{KQ'} (A + BF) P_{KQ'} + BFP_{KQ'} \\ &= AP_{KQ'} - (A + BF) P_{KQ'} + BFP_{KQ'} = 0\end{aligned} \tag{40}$$

We claim that $(\mathcal{A}_{Q'}, \mathcal{B}_{Q'})$ is controllable. In fact, if $\mathcal{P}_{Q'}$ is the controllability gramian of $\mathcal{A}Q'$, $\mathcal{B}Q'$), the matrix

$$P_e = \begin{pmatrix} \mathcal{P}_{Q'} & P_{Q'K} \\ P_{KQ'} & P \end{pmatrix}$$

solves (39) by construction, because the diagonal blocks are the controllability gramians and the equation for the off-diagonal block is precisely (40). Since $A_e$ is stable, $P_e$ is non-negative definite. But this forces the rank of $P_{Q'}$ to be not less than the rank of $P_{Q'K}$. Since $P_{Q'K}$ has full row rank and the number of rows equals the dimension of $\mathcal{P}_{Q'}$, we conclude that $\mathcal{P}_{Q'}$ has full rank and thus $(\mathcal{A}_{Q'}, \mathcal{B}_{Q'})$ is controllable.

In conclusion, if we define

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}^*_{Q'} \mathcal{P}^{-1}_{Q'} & I \end{array}\right)$$

equation (40) becomes the lower left block of the Lyapunov equation (22) associated to the projection of $H_r(Q')$ onto $H_r(K)$ (see Lemma 3).

(2) Follows from the previous part and Proposition 5.

(3) For the proof of this part we could develop the above argument in $H^2_-$, obtaining dual formulas. Nevertheless, we would have to translate the relation so obtained to $H^2_+$. So we find it more constructive to derive the relations directly in $H^2_+$, also in view of the connection with geometric control.

(a) Let

$$\bar{Q}'' = \left(\begin{array}{c|c} \mathcal{A}_{\bar{Q}''} & \mathcal{B}_{\bar{Q}''} \\ \hline -\mathcal{B}^*_{\bar{Q}''} \mathcal{P}^{-1}_{\bar{Q}''} & I \end{array}\right)$$

be a minimal realization of the $m \times m$ inner function $\bar{Q}''$ of degree $r''$. Let

$$\bar{A}_e = \begin{pmatrix} \mathcal{A}_{\bar{Q}''} & 0 \\ 0 & A \end{pmatrix}, \quad \bar{B}_e = \begin{pmatrix} \mathcal{B}_{\bar{Q}''} \\ B \end{pmatrix},$$

$$\bar{C}_e = -(\mathcal{B}_{\bar{Q}''} \mathcal{P}^{-1}_{\bar{Q}''} \quad B\bar{P}^{-1})$$

As above, let

$$\bar{P}_e = \begin{pmatrix} \mathcal{P}_{\bar{Q}''} & \bar{P}_{\bar{Q}''K} \\ \bar{P}_{K\bar{Q}''} & \bar{P} \end{pmatrix}$$

be the solution to the Lyapunov equation

$$\bar{A}_e\bar{P}_e + \bar{P}_e\bar{A}_e^* + \bar{B}_e\bar{B}_e^* = 0 \tag{41}$$

and let

$$\bar{Q}_e = \begin{pmatrix} \mathcal{Q}_{\bar{Q}''} & Q_{\bar{Q}''K} \\ Q_{K\bar{Q}''} & \bar{Q} \end{pmatrix}$$

be the solution to the Lyapunov equation

$$\bar{A}_e^*\bar{Q}_e + \bar{Q}_e\bar{A}_e + \bar{C}_e^*\bar{C}_e = 0 \tag{42}$$

Then

$$\mathcal{P}_{\bar{Q}''} = \mathcal{Q}_{\bar{Q}''}^{-1}$$

and

$$P = \bar{P} = \bar{Q}^{-1}$$

The relations between $\mathcal{P}_{\bar{Q}''}$ and $\mathcal{Q}_{\bar{Q}''}$ follow from the fact that $\bar{Q}''$ is inner. The equality between $P$ and $\bar{P}$ follows by inspection, for they solve the same Lyapunov equation.

(b) We show now that the following equalities hold

$$\begin{aligned} P_{H_r(K)}(sI - \mathcal{A}_{\bar{Q}''})^{-1}\mathcal{B}_{\bar{Q}''}\bar{Q}''^*K &= -\mathcal{P}_{\bar{Q}''}Q_{\bar{Q}''K}P(sI + A^*)^{-1}PBK \\ &= \mathcal{P}_{\bar{Q}''}Q_{\bar{Q}''K}(sI - A)^{-1}B \end{aligned} \tag{43}$$

We recall that, since we are working with row vectors, in our notation it is

$$(P_X y)S = P_{XS}yS \tag{44}$$

for $X \subset H_+^2$, $y \in H_+^2$ and $S$ inner. Then we have the equalities

$$\begin{aligned} &P_{H_r(K)}(sI - \mathcal{A}_{\bar{Q}''})^{-1}\mathcal{B}_{\bar{Q}''}\bar{Q}''^*K \\ &= [P_{H_r(K)K^*}(sI - \mathcal{A}_{\bar{Q}''})^{-1}\mathcal{B}_{\bar{Q}''}\bar{Q}''^*]K && \text{from (44)} \\ &= [P_{H_r(K^*)}\mathcal{P}_{\bar{Q}''}(sI + \mathcal{A}_{\bar{Q}''}^*)^{-1}\mathcal{P}_{\bar{Q}''}^{-1}\mathcal{B}_{\bar{Q}''}]K && \text{Corollary1} \\ &= -[P_{H_c(K)}\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}(sI - \mathcal{A}_{\bar{Q}''})^{-1}\mathcal{P}_{\bar{Q}''}]^*K \\ &= -(B^*P^{-1}(sI - A)^{-1}Q^{-1}Q_{K\bar{Q}''}\mathcal{P}_{\bar{Q}''})^*K && \text{Lemma 3} \\ &= \mathcal{P}_{\bar{Q}''}Q_{\bar{Q}''K}P(sI + A^*)^{-1}P^{-1}BK \\ &= \mathcal{P}_{\bar{Q}''}Q_{\bar{Q}''K}(sI - A)^{-1}B && \text{Corollary 1} \end{aligned}$$

and (43) is proved.

(c) We claim now that $\mathcal{V}_- = \text{Im } PQ_{K\bar{Q}''}$. In fact, we can write

$$\begin{aligned} \mathcal{V} &= I_{A,B}^{-1}P_{H_r(K)}(sI - \mathcal{A}_{\bar{Q}''})^{-1}\mathcal{B}_{\bar{Q}''}\bar{Q}''^*K \\ &= I_{A,B}^{-1}P_{H_r(K)}\{\zeta^*(sI - \mathcal{A}_{\bar{Q}''})^{-1}\mathcal{B}_{\bar{Q}''}\bar{Q}''^*K;\ \zeta \in \mathbb{C}^{r''}\} && \text{Lemma 2} \\ &= I_{A,B}^{-1}\{\zeta^*\mathcal{P}_{\bar{Q}''}Q_{\bar{Q}''K}(sI - A)^{-1}B;\ \zeta \in \mathbb{C}^{r''}\} && \text{from (43)} \\ &= \{PQ_{\bar{Q}''K}^*\mathcal{P}_{\bar{Q}''}\zeta;\ \zeta \in \mathbb{C}^{r''}\} = P \text{ Im } Q_{\bar{Q}''K}^* \end{aligned}$$

(d) Define $\bar{A}_1 := PQ_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#}P^{-1}$. Then, again

$$\bar{A}_1\mathcal{V} = PQ_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''}Q_{K\bar{Q}''}^{\#}P^{-1}P \text{ Im } Q_{K\bar{Q}''} \subset P \text{ Im } Q_{K\bar{Q}''} = \mathcal{V}_-$$

(e) We have

$$\bar{A}_1 = (A + B(B^* - \mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}Q_{K\bar{Q}''}^{\#})P^{-1})_{|\mathcal{V}_-}$$

To see this, observe that the equation for the (2, 1)-block of (42) yields

$$A^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''} + P^{-1}B\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1} = 0$$

or, since $A^* = -P^{-1}AP - P^{-1}BB^*$

$$-P^{-1}APQ_{K\bar{Q}''} - P^{-1}BB^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''} + P^{-1}B\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1} = 0$$

Multiplying by $Q_{K\bar{Q}''}^{\#}$ on the right and by $P$ on the left

$$\begin{aligned} PQ_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''}Q_{K\bar{Q}''}^{\#} = APQ_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#} + BB^*Q_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#} \\ - B\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}Q_{K\bar{Q}''}^{\#} = 0 \end{aligned} \tag{45}$$

Now $\xi \in \mathcal{V}$ if and only if $\xi = PQ_{K\bar{Q}''}\zeta$ and since $Q_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#}$ is a projection, it acts as the identity on its image, and so $\xi = PQ_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#}P^{-1}\xi$ for all $\xi \in \mathcal{V}_-$. Thus we can write

$$\begin{aligned} &PQ_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''}Q_{K\bar{Q}''}^{\#}P_{|\mathcal{V}_-}^{-1} \\ &\quad = A_{|\mathcal{V}_-} + B(B^* - \mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}Q_{K\bar{Q}''}^{\#})P_{|\mathcal{V}_-}^{-1} \end{aligned}$$

that is, $\mathcal{V}_-$ is $(A, B)$-invariant

Conversely, assume now that $\mathcal{V}_-$ is a stabilizable, controlled invariant subspace. Then there exists a feedback matrix $\bar{F}$ such that $(A + B\bar{F})\mathcal{V}_- \subset \mathcal{V}$ and $(A + B\bar{F})_{|\mathcal{V}_-}$ is stable. Let $Q_{K\bar{Q}''}$ to be any full column-rank matrix having image $P^{-1}\mathcal{V}_-$. Since $Q_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#}$ is a projection onto $P^{-1}\mathcal{V}_-$, and obviously $P^{-1}(A + B\bar{F})\mathcal{V}_- \subset P^{-1}\mathcal{V}_-$

$$Q_{K\bar{Q}''}Q^{\#}_{K\bar{Q}''}P^{-1}(A+B\bar{F})PQ_{K\bar{Q}''}Q^{\#}_{K\bar{Q}''_{|P^{-1}\mathcal{V}}}$$

$$= P^{-1}(A+B\bar{F})P_{|P^{-1}\mathcal{V}_-}$$

Thus, setting $\mathcal{A}_{\bar{Q}''} := Q^{\#}_{K\bar{Q}''}P^{-1}(A+B\bar{F})PQ_{K\bar{Q}''}$ and $\bar{\mathcal{C}}_{\bar{Q}''} = \bar{F}PQ_{K\bar{Q}''} - B^*Q_{K\bar{Q}''}$, we define

$$\bar{Q}'' - \left(\begin{array}{c|c} \mathcal{A}_{\bar{Q}''} & -\mathcal{Q}^{-1}\bar{\mathcal{C}}^*_{\bar{Q}''} \\ \hline \mathcal{C}_{\bar{Q}''} & I \end{array}\right)$$

and the controllability of $(\mathcal{A}_{\bar{Q}''},\ \mathcal{Q}^{-1}\bar{\mathcal{C}}^*_{\bar{Q}''})$ is shown as above. By construction, $X_{\mathcal{V}_-} = P_{H_r(K)}H_r(\bar{Q}'')\bar{Q}''^*K = P_{H_r(K)}\bar{H}_r(\bar{Q}''^*)K$. $\square$

A remark is in order concerning the representation (38), especially in connection with the analysis of spectral factors carried out in Fuhrmann and Gombani (1998). The corresponding representation there, though with the state space taken as $H_r(K)Q''$, is $P_{H_r(K)Q''}H_r(\bar{Q}'')K^e_+$ where $K^e_+$ is the inner function that satisfies $KQ'' = \bar{Q}''K^e_+$. We note that $P_{H_r(K)}f = [P_{H_r(K)Q''}fQ'']Q''^*$. Therefore we have

$$\begin{aligned} P_{H_r(K)}H_r(\bar{Q}''^*)K &= [P_{H_r(K)Q''}H_r(\bar{Q}'')KQ'']Q''^* \\ &= [P_{H_r(K)Q''}H_r(\bar{Q}''^*)\bar{Q}''K^e_+]Q''^* \\ &= [P_{H_r(K)Q''}H_r(\bar{Q}'')K^e_+]Q''^* \end{aligned}$$

This shows the equivalence of the two representations.

Using duality in the coinvariant subspace $H_r(K)$, one can easily derive characterizations for outer (anti)-detectable subspaces with respect to the pair $(\mathbf{A}_K,\ \mathbf{B}_K)$ given by (31).

The approach taken here is complementary to that of Fuhrmann (1998a). In that paper one starts with a functional characterization of outer (anti)detectable subspaces in the polynomial model setting. This one is easy to pull back to the state space $H_r(K)$ and from there to $\mathbb{C}^n$. The present characterizations for inner (anti)stabilizable subspaces are then obtained by duality.

The discrepancy between the approaches to the two cases is not new. In Fuhrmann and Willems (1980) and Fuhrmann (1981a) characterizations, in the context of functional models, were given for controlled and conditioned invariant subspaces, with respect to the reachable and observable pairs respectively, arising from coprime factorizations. These characterizations, in terms of intersections and projections, are precursors of the characterizations obtained here. It turned out that characterization of conditioned invariant subspaces is technically much easier within the polynomial model context than the characterization of controlled invariant subspaces, although conceptually the last one poses less difficulties.

The characterizations obtained in Theorem 1 used only the pole information, whereas zeros played no role at all. In the next lemma we extend the scope of our investigation. We begin with a transfer function $W$ and use two inner functions that act as a measure of its antistable and stable zeros.

**Lemma 4:** *Let*

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

*be a minimal realization of the stable spectral factor $W$, and let $K$ be the associated inner function, i.e.*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

*where $P$ solves $AP + PA^* + BB^* = 0$.*

(1) *Let $\mathcal{V}_+$ be a controlled invariant subspace, and let*

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}^*_{Q'}\mathcal{P}^{-1}_{Q'} & I \end{array}\right)$$

*be an inner function such that $X_{\mathcal{V}_+} = \mathbf{P}_{H_r(K)}H_r(Q')$. Then $W(Q')^*$ is stable if and only if*

$$\mathcal{V}_+ \subset \operatorname{Ker}\,(C + D\mathcal{B}^*_{Q'}P^{\#}_{KQ'}) \tag{46}$$

*where $P_{KQ'}$ satisfies the equation*

$$AP_{KQ'} + P_{KQ'}\mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} = 0$$

(2) *Let $\mathcal{V}_-$ be a controlled invariant subspace and let*

$$\bar{Q}'' = \left(\begin{array}{c|c} \mathcal{A}_{\bar{Q}''} & \mathcal{B}_{\bar{Q}''} \\ \hline -\mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''} & I \end{array}\right)$$

*be a minimal realization of the $m \times m$ inner functions $\bar{Q}''$ such that $X_{\mathcal{V}_-} = \mathbf{P}_{H_r(K)}\bar{H}_r(\bar{Q}''^*)K$. Then $\overline{WQ}'' \in H^\infty_-$ if and only if*

$$\mathcal{V}_- \subset \operatorname{Ker}\,(C - D(\mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''}Q^{\#}_{K\bar{Q}''}P^{-1} - B^*P^{-1})) \tag{47}$$

*where $Q_{K\bar{Q}''}$ satisfies*

$$A^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''} + P^{-1}B\mathcal{B}^*_{\bar{Q}''}\bar{\mathcal{P}}^{-1}_{\bar{Q}''} = 0$$

**Proof:**

(1) Assume first that $W(Q')^*$ is stable. This is equivalent to the stability of $Wu^*$ for any $u \in H_r(Q')$. Let $u = \zeta^*(sI - \mathcal{A}_{Q'})^{-1}\mathcal{B}_{Q'}$ be an arbitrary element of $H_r(Q)$. Then $Wu^*$ is analytic in the positive closed right half-plane $C^+$ and so

$$\int_{\Gamma} W(s)u^*(s)\,\mathrm{d}s = 0$$

for any closed, positively oriented, curve $\Gamma$ contained in $C^+$; in particular

$$\frac{1}{2\pi i}\int_{\Gamma_1} Du^*(s)\,\mathrm{d}s + \frac{1}{2\pi i}\int_{\Gamma_2} C(sI - A)^{-1}Bu^*(s)\,\mathrm{d}s = 0 \tag{48}$$

for any pair of closed, positively oriented, curves $\Gamma_1$ and $\Gamma_2$ containing the poles of $u^*$. Now it is easy to see that

$$\frac{1}{2\pi i}\int_{\Gamma_1} Du^*(s)\,\mathrm{d}s = \frac{1}{2\pi i}\int_{\Gamma_1} D\mathcal{B}^*_{Q'}(-sI - \mathcal{A}^*_{Q'})^{-1}\zeta\,\mathrm{d}s$$
$$= -D\mathcal{B}^*_{Q'}\zeta \tag{49}$$

since $(1/2\pi i)\int_{\Gamma_1}(si + \mathcal{A}^*_{Q'})^{-1}\zeta\ \mathrm{d}s = I$. The second integral is easily computed observing that, since the two functions are strictly proper, we can integrate along the imaginary axis, with a change of sign, since we are reversing the direction of integration, but we obtain, in this manner the inner product matrix of the basis $(-sI - A^*)^{-1}B$ and $(-sI - \mathcal{A}^*_{Q'})^{-1}\mathcal{B}_{Q'}$. We can therefore substitute for $(-sI - \mathcal{A}^*_{Q'})^{-1}\mathcal{B}_{Q'}$ its projection onto $H_r(K)$. Thus we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_2} C(sI - A)^{-1}Bu^*(s)\,\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{I} C(sI - A)^{-1}B\mathcal{B}^*_{Q'}(-sI - \mathcal{A}^*_{Q'})^{-1}\zeta\,\mathrm{d}s$$
$$= -\frac{1}{2\pi i}\int_{I} C(sI - A)^{-1}BB^*(-sI - A^*)^{-1}P^{-1}P_{KQ'}\zeta\,\mathrm{d}s$$
$$= -CP_{KQ'}\zeta \tag{50}$$

Putting (48), (49) and (50) together we obtain $(D\mathcal{B}^*_{Q'} + CP_{KQ'})\zeta = 0$ and remembering that $\mathcal{V}_+ = \mathrm{Im}\ P_{KQ'}$, we can write

$$(D\mathcal{B}^*_{Q'}P^{\#}_{KQ'} + C)\xi = 0 \qquad \forall \xi \in \mathcal{V}_+ \tag{51}$$

To prove the converse, suppose that $\mathcal{V}_+$ is a controlled invariant subspace, and that

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}^*_{Q'}\mathcal{P}^{-1}_{Q'} & I \end{array}\right)$$

is an inner function associated to $\mathcal{V}_+$ such that (46) is satisfied. Then, by reversing the above argument, we obtain $\int_{\Gamma} W(s)u^*(s)\,\mathrm{d}s = 0$ for each $u \in H_r(Q')$. Since $H_r(Q')$ is coinvariant, this implies that $W \in H^2_+Q'$, as wanted.

(2) Assume first that $\overline{WQ}''$ is analytic in the closed left half-plane $\mathbb{C}^-$. This means that $\bar{W}u$ is analytic in $\mathbb{C}^-$ for every $u \in H_c(\bar{Q}'')$. So, let $u = \mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''}(sI - \mathcal{A}_{\bar{Q}''})^{-1}\zeta$. Then

$$\int_{\bar{\Gamma}} \bar{W}(s)u(s)\,\mathrm{d}s = 0$$

for any positively oriented, closed curve $\bar{\Gamma}$ contained in $\mathbb{C}^-$. In particular, remembering that, in view of Lemma 1

$$\bar{W} = \left(\begin{array}{c|c} -A^* & P^{-1}B \\ \hline DB^* + CP & D \end{array}\right)$$

the above formula becomes

$$\frac{1}{2\pi i}\int_{\bar{\Gamma}_1} Du(s)\,\mathrm{d}s$$
$$+\frac{1}{2\pi i}\int_{\bar{\Gamma}_2} (DB^* + CP)(sI + A^*)^{-1}P^{-1}Bu(s)\,\mathrm{d}s = 0 \tag{52}$$

for any pair of positively oriented, closed curves $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ containing the poles of $u$. The first integral is computed as above

$$\frac{1}{2\pi i}\int_{\bar{\Gamma}_i} D\mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''}u(s)\,\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{\bar{\Gamma}_2} D\mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''}(sI - \mathcal{A}_{\bar{Q}''})^{-1}\zeta\,\mathrm{d}s = D\mathcal{B}^*_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''}\zeta \tag{53}$$

The second integral is again computed observing that we can replace the path of integration by the imaginary axis. This time we obtain the inner product matrix of the basis $B^*P^{-1}(sI - A^*)^{-1}$ and $\mathcal{B}_{\bar{Q}''}\mathcal{P}^{-1}_{\bar{Q}''}(sI - \mathcal{A}^*_{\bar{Q}''})^{-1}$. To compute this, we apply Lemma 3 with $S_1 = K$ and $S_2 = \bar{Q}''$. We set

$$A_e = \begin{pmatrix} A & 0 \\ 0 & \mathcal{A}_{\bar{Q}''} \end{pmatrix}, \quad C_e = (-B^*P^{-1} \quad -\mathcal{B}^*_{\bar{Q}''}P^{-1}_{\bar{Q}''}),$$
$$Q_e = \begin{pmatrix} Q & Q_{K\bar{Q}''} \\ Q^*_{K\bar{Q}''} & Q_{\bar{Q}''} \end{pmatrix}$$

We note that the 1,1 term of the Lyapunov equation $A^*_eQ_e + Q_eA_e + C^*_eC_e = 0$ becomes $A^*Q + QA + P^{-1}BB^*P^{-1} = 0$, which has the unique solution $Q = P^{-1}$. Similarly, the 2,1 term is

$$A^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''} + P^{-1}B\mathcal{B}^*_{\bar{Q}''}P^{-1}_{\bar{Q}''} = 0$$

We can therefore, in view of Lemma 3, substitute for $\mathcal{B}_{\bar{Q}''}\mathcal{P}_{\bar{Q}''}^{-1}(sI-\mathcal{A}_{\bar{Q}''}^*)^{-1}$ its projection onto $H_c(K)$. Thus we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\bar{\Gamma}_2}(DB^*+CP)(sI+A^*)^{-1}P^{-1}Bu(s)\,\mathrm{d}s\\
&=\frac{1}{2\pi i}\int_I(DB^*+CP)(sI+A^*)^{-1}\\
&\quad\times P^{-1}B\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}(sI-\mathcal{A}_{\bar{Q}''})^{-1}\zeta\,\mathrm{d}s\\
&=(DB^*+CP)\frac{1}{2\pi i}\int_I(sI+A^*)^{-1}P^{-1}BB^*P^{-1}(sI-A)^{-1}\\
&\quad\times Q^{-1}Q_{K\bar{Q}''}\zeta\,\mathrm{d}s\\
&=(DB^*+CP)\frac{1}{2\pi i}\int_I(sI+A^*)^{-1}P^{-1}BB^*P^{-1}(sI-A)^{-1}\\
&\quad\times PQ_{K\bar{Q}''}\zeta\,\mathrm{d}s\\
&=-(DB^*+CP)Q_{K\bar{Q}''}\zeta \qquad (54)
\end{aligned}$$

In the computation we used the partial fraction decomposition

$$(sI+A^*)^{-1}P^{-1}BB^*P^{-1}(sI-A)^{-1}=(sI+A^*)^{-1}P^{-1}-P^{-1}(sI-A)^{-1}$$

In conclusion, putting (52), (53) and (54) together, we get

$$D\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}\zeta-(DB^*+CP)_{K\bar{Q}''}\zeta=0$$

Since, by Theorem 1, for any $\xi\in\mathcal{V}_-$ we can find a $\zeta$ such that $\xi=PQ_{K\bar{Q}''}\zeta$, we have eventually

$$(C-D(\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}Q_{K\bar{Q}''}^{\#}-B^*)P^{-1})\xi=0\quad\forall\xi\in\mathcal{V}_-$$

To prove the converse, we assume, as above, that $\mathcal{V}_-$ is a controlled invariant subspace, and that

$$\bar{Q}''=\left(\begin{array}{c|c}\mathcal{A}_{\bar{Q}''}&\mathcal{B}_{\bar{Q}''}\\\hline-\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}&I\end{array}\right)$$

is an inner function associated to $\mathcal{V}_-$ such that (47) is satisfied. Again, by reversing the above argument, we obtain $\int_{\bar{\Gamma}}\bar{W}(s)u(s)\,\mathrm{d}s=0$ for each $u\in H_c(\bar{Q}'')$. Since $H_c(\bar{Q}'')$ is coinvariant in $H_-^2$, this implies that $\bar{W}\bar{Q}''\in H_-^2$, as wanted. □

**Theorem 2:** *Let*

$$W=\left(\begin{array}{c|c}A&B\\\hline C&D\end{array}\right)$$

*and let $Q'$ and $Q''$ be as in Proposition 3, i.e.*

$$W=W_-Q'=W_+Q''$$

*then the maximal inner antistabilizable, output nulling subspace is*

$$\mathcal{V}_+^*:=I_{A,B}^{-1}[\mathbf{P}_{H_r(K)}H_r(Q')]$$

*and the maximal, inner stabilizable, output nulling subspace is*

$$\mathcal{V}_-^*:=I_{A,B}^{-1}[\mathbf{P}_{H_r(K)}H_r(\bar{Q}'')\bar{Q}''^*K]$$

**Proof:** In view of Theorem 1, $\mathcal{V}_+^*$ is an antistabilizable controlled invariant subspace, and in view of Lemma 4, it is an output nulling subspace. Suppose $W=W_1Q_1$ and denote by $\mathcal{V}_1$ the controlled invariant, output nulling subspace associated to $Q_1$. Since $Q'$ is the inner factor of $W$, $Q_1|_RQ'$, and therefore it is easy to see that $\mathcal{V}_1\subset\mathcal{V}_+^*$. The proof of the other statement is similar. □

**Corollary 1:** *Let*

$$W=\left(\begin{array}{c|c}A&B\\\hline C&D\end{array}\right)$$

*be given and assume D has full column rank. Let*

$$\mathcal{V}_+=I_{A,B}^{-1}\mathbf{P}_{H_r(K)}H_r(Q')$$

*and*

$$\mathcal{V}_-=I_{A,B}^{-1}\mathbf{P}_{H_r(K)}\bar{H}_r(\bar{Q}''^*)K$$

*be output nulling subspaces with*

$$Q'=\left(\begin{array}{c|c}\mathcal{A}_{Q'}&\mathcal{B}_{Q'}\\\hline-\mathcal{B}_{Q'}^*\mathcal{P}_{Q'}^{-1}&I\end{array}\right)$$

*and*

$$\bar{Q}''=\left(\begin{array}{c|c}\mathcal{A}_{\bar{Q}''}&\mathcal{B}_{\bar{Q}''}\\\hline-\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1}&I\end{array}\right)$$

*Then*

$$(A+B\mathcal{B}_{Q'}^*P_{KQ'}^{\#})_{|\mathcal{V}_+}=(A-BD^{\#}C)_{|\mathcal{V}_+}$$

*and*

$$(A+B(B^*-\mathcal{B}_{\bar{Q}''}^*\mathcal{P}\bar{Q}''^{\,-1}Q_{K\bar{Q}''}^{\#})P^{-1})_{|\mathcal{V}_-}=(A-BD^{\#}C)_{|\mathcal{V}_-}$$

**Proof:** Since $D$ has full column rank, it has a left inverse $D^{\#}$ such that $D^{\#}D=I$. Therefore from (51) we get

$$D^{\#}C_{|\mathcal{V}_+}=-\mathcal{B}_{Q'}^*P_{KQ'}^{\#}{}_{|\mathcal{V}_+}$$

and eventually

$$(A+B\mathcal{B}_{Q'}^*P_{KQ'}^{\#})_{|\mathcal{V}_+}=(A-BD^{\#}C)_{|\mathcal{V}_+}$$

Similarly we obtain the result for $\mathcal{V}_-$. □

**Remark:** The above corollary holds in particular for the supremal, inner (anti)stabilizable, output nulling subspaces $\mathcal{V}_+^*$ and $\mathcal{V}_-^*$. In the case when $W$ is invertible, $\mathcal{V}_+^* \vee \mathcal{V}*_- = \mathbb{C}^n$ and therefore we find that the zeros of $W$ are given by the spectrum of $A - BD^{-1}C$, as might be expected.

The above corollary also indicates that the output nulling condition determines uniquely the subspaces $\mathcal{V}_+^*$ and $\mathcal{V}_-^*$. It is natural to ask the converse question, namely, is $W$ uniquely determined by a given output nulling subspaces. The next section is devoted to the investigation of this question.

We would like to comment, at this point, on the substantial difference between our approach and the one followed by Lindquist *et al*. (1995). In that paper, the *left* zeros of a spectral factor $W$ are considered; moreover the density $\Phi = WW^*$ is, unlike here, assumed to be full rank. In our notation, this corresponds to factorizing first $W$ as

$$W = W_- Q' = W_- Q_1' Q_2' \tag{55}$$

where $W_-$ and $Q$ are the outer and rigid factors of $W$ and $Q_1'Q_2'$ is in turn the inner–outer factorization of $Q'$; then we can define $W_{0-} := W_- Q_1'$ and take the inner–outer factorization of

$$W_{0-} = \tilde{Q}_1' \tilde{W}_-$$

where now $\tilde{Q}_1'$ represents the antistable left zeros of $W$. Under the full rank assumption on $\Phi$, $W_{0-}$, $Q_1'$ and $\tilde{Q}_1'$ are square and therefore it can be shown that $\det Q_1' = \det \tilde{Q}_1'$ and thus we can talk about antistable invariant zeros *tout court*; a similar construction can be carried out for the stable zeros. The main drawback of this approach is that the left zeros are only the invariant ones, and so the whole structure of the non-invariant zeros and of the controllability subspace of $W$, which are both related to $Q_2'$, is lost. Moreover, the full rank assumption on $\Phi$ in this approach is essential. We feel that, in general, considering right zeros leads to a more complete and thorough analysis of the partial ordering of coinvariant subspaces related to different spectral factors (see Fuhrmann and Gombani 2000). We refer the reader to Fuhrmann and Gombani (1998) for a more detailed analysis of the factorizations of the form (55) and of the resulting state space decomposition into internal and external parts, as well as for its connection with the *tightest internal bound* of Lindquist and Picci (1991). In spite of all these differences, we would like to acknowledge once more the seminal influence of the work of Lindquist, Michaletzky and Picci on the present paper.

## 4. Controllability subspaces

We proceed now to characterize controllability subspaces and controllabihty output nulling subspaces in terms of inner functions. More in detail, in Theorem 3 we show that any controllability subspace for $(A, B)$ can be represented in terms of inner functions $Q'$ and $\bar{Q}''$. This derivation is quite straightforward, but it is lengthy and requires the introduction of some new definitions (minimal proper reductions, see Definition 1) to avoid pathological situations. But, as in the case of controlled invariant subspaces, this representation is not unique. Nevertheless, in view of Proposition 3, for any factor $W$ we can write $W = W_-^e Q' = \bar{W}_+^e (\bar{Q}'')^* K$. It will be shown in Theorem 5 that, in this case, $Q'$ and $\bar{Q}''$ yield the same controllability subspace. The shortcoming of this construction is that it depends on $W$. The question which raises naturally then (and that we try to answer with Theorem 4) is whether there exist conditions on the functions $Q'$ and $\bar{Q}''$ (without knowing if they come from a factorization process of a factor $W$ as the above) such that $Q'$ and $\bar{Q}''$ yield the same controllability subspace. In Theorem 5 we consider the characterization of output nulling controllability subspaces.

Given a reachable pair $(A, B)$, we say that a subspace $\mathcal{R} \subset \mathbb{C}^n$ is called a **controllability subspace** if it is a controlled invariant subspace and there exist a feedback matrix $F$ and a matrix $G_2$ such that

$$\mathcal{R} = \langle A + BF | \mathrm{Im}\ BG_2 \rangle$$

Clearly, it is not restrictive to assume that $G_2$ is an orthogonal projection in $\mathbb{C}^m$. Then $G_1$ will denote the projection onto the orthogonal complement. A subspace $\mathcal{R} \subset \mathcal{V}$ is a **supremal controllability subspace** in $\mathcal{V}$ if it is not properly contained in any other controllability subspace of $\mathcal{V}$. It is well known, see Wonham (1991), that for any subspace $\mathcal{V}$ this space is uniquely determined. Let therefore $\mathcal{R}_\mathcal{V}$ be the supremal controllability subspace of $\mathcal{V}$, and set $\mathcal{V}^0 := \mathcal{V}/\mathcal{R}_\mathcal{V}$. It is well known, again see Wonham (1991), that $\mathcal{V}$ is (anti)stabilizable if and only if $(A + BF)_{|\mathcal{V}^0}$ is (anti)stable. As above, given a controllable pair $(A, B)$ and a controllability subspace $\mathcal{R}$ for the pair, we can define its image $X_\mathcal{R} := I_{A,B}\mathcal{R}$.

Since a controllability subspace is, at the same time, inner stabilizable and inner antistabilizable, it is quite natural, in view of the results in the above section, to seek a characterization in terms of inner functions. It turns out that this is the right idea. In particular, if $K$ and $Q'$ were right coprime inner functions, and $G_2$ were a given matrix, we could define the skew-prime factorization of the least common left inner multiple of $K$ and $Q'$

$$K \vee_L Q' = \bar{Q}' K = K_- Q'$$

Then we will see that if the space $\mathcal{R} = I_{A,B}^{-1} \mathbf{P}_{H_r(K)} H_r(Q')$ is a controllability subspace (with respect to $G_2$) then, for instance, $H_r(K_-) \subset \operatorname{Ker} G_2$ (in fact we are going to show a more general result).

Nevertheless, as the following simple example shows, it might happen that we do not have right coprimeness of the inner functions $K$ and $Q'$. Suppose

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \quad \text{and} \quad Q' = \begin{bmatrix} I & 0 \\ 0 & K_2 \end{bmatrix}$$

with $K_1$, $K_2$ inner functions. Then $K$ and $Q'$ are not coprime at all. Nevertheless, the space

$$\mathbf{P}_{H_r(K)} H_r(Q') = H_r(Q')$$

is a controllability subspace with respect to

$$G_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

In fact it is trivially controlled invariant and, letting

$$K = \left( \begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline x & x & x & x \end{array} \right)$$

be a minimal realization of $K$, we clearly have

$$\mathcal{R} = \operatorname{Im}\left\langle \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix} \middle| \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix} \right\rangle = I_{A,B}^{-1} \mathbf{P}_{H_r(K)} H_r(Q')$$

Observe that, letting $K_- := KQ'^*$

$$H_r(K_-) = H_r(KQ'^*) G_2 = H_r\left( \begin{bmatrix} K_1 & 0 \\ 0 & I \end{bmatrix} \right) G_2 = 0$$

We will now develop a formal derivation of the above idea which also accommodates the example shown above. It should be kept in mind though, for an easier understanding of the construction, that if we were assuming that $K \wedge_R Q' = I$ and $K \wedge_L \bar{Q}'' = I$, then $K_-$ and $K_+$ would be given by the skew prime factorizations $K_- Q' = \bar{Q}' K$ and $\bar{Q}'' K_+ = K Q''$. In fact, in this case, as we already mentioned above, we will show, see Lemma 6, that $\mathcal{R}$ is a controllability subspace with respect to $G_2$ if

$$I_{A,B} \mathcal{R} = \mathbf{P}_{H_r(K)} H_r(Q') \cap \mathbf{P}_{H_r(K)} \bar{H}_r(\bar{Q}''^*) K$$

and

$$H_r(K_-) \subset \operatorname{Ker} G_2 \qquad H_c(K_+) \subset \operatorname{Ker} G_2 \tag{56}$$

The shortcoming of using coprime inner functions is, as explained above, that not all controllability subspaces can be represented in this way. Nevertheless, we are going to show that the basic idea goes through even if, for instance, $Q'$ and $K$ are not right coprime. But in this case we need to say what we mean by $K_-$ if we still want to use (56) in order to characterize controllability subspaces. This is why we need the following definition.

**Definition 1:** Let $K$, $Q'$ and $\bar{Q}''$ be inner functions and $G_2$ a constant projection matrix on $\mathbb{C}^m$.

(1) Define

$$S_L := Q' \vee_L K$$

to be the least common left inner multiple of $K$ and $Q'$. We say that $Q'$ **reduces** $K$ **on the right with respect to** $G_2$ if, letting $K_- := S_L Q'^*$, implies

$$H_r(K_-) \subset \operatorname{Ker} G_2$$

The reduction is **proper** if $P_{H_r(K)} | H_r(Q')$ is injective. It is **minimal** if $H_c(Q') \cap G_1 H_+^2 = 0$.

(2) Similarly, define

$$S_R := \bar{Q}'' \vee_R K$$

to be the least right inner multiple of $K$ and $\bar{Q}''$. We say that $\bar{Q}''$ **reduces** $K$ **on the left with respect to** $G_2$ if, letting $K_+ := \bar{Q}''^* S_R$, implies

$$H_c(K_+) \subset \operatorname{Ker} G_2$$

The reduction is **proper** if $P_{H_r(K)} | H_r(\bar{Q}''^*) K$ is injective. It is **minimal** if $H_r(\bar{Q}'') \cap H_+^2 G_1 = 0$.

Some comments on these definition. Reduction simply seems to be the right notion to characterize controllability subspaces, as the next lemma shows. Properness is to avoid a pathological situation: in the scalar case, to say that $P_{H_r(K)} | H_r(Q')$ is not injective, means that the degree of $Q'$ is greater than the degree of $K$ and thus a transfer function $W$ having the same poles a $K$ and the same zeros as $Q'$ is not proper. The situation is slightly more complicated in the multivariable case (it might happen that $^\circ Q' \leq {}^\circ K$ but still might not be proper); the way to avoid this is by imposing properness of the reduction. Minimality has to do with the fact that $H(K_-)$ is in the kernel $G_2$ and thus everything orthogonal to that kernel will not be univoquely determined: if, for example, $Q'$ reduces $K$ with respect to $G_2$ and $Q_0$ is such that $H_r(Q_0) \subset \operatorname{Im} G_2$, then also $Q'_1 := Q_0 Q'$ will reduce $K$. We simply want to avoid this lack of uniqueness.

**Lemma 5:** *Let*

$$Q' = \left( \begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}_{Q'}^* \mathcal{P}_{Q'}^{-1} & I \end{array} \right), \quad \bar{Q}'' = \left( \begin{array}{c|c} \mathcal{A}_{\bar{Q}''} & \mathcal{B}_{\bar{Q}''} \\ \hline -\mathcal{B}_{\bar{Q}''}^* \mathcal{P}_{\bar{Q}''}^{-1} & I \end{array} \right)$$

*and*

$$K = \left( \begin{array}{c|c} A & B \\ \hline -B^* P^{-1} & I \end{array} \right)$$

*be inner functions. Then:*

(1) *$Q'$ reduces $K$ on the right with respect to $G_2$ if and only if*

$$BG_2 = P_{KQ'}\mathcal{P}_{Q'}^{-1}\mathcal{B}_{Q'}G_2 \tag{57}$$

*where $P_{KQ'}$ satisfies $AP_{KQ'} + P_{KQ'}\mathcal{A}_{Q'}^* + B\mathcal{B}_{Q'}^* = 0$*

(2) *$\bar{Q}''$ reduces $K$ on the left with respect to $G_2$ if and only if*

$$G_2B^*P^{-1} = G_2\mathcal{B}_{\bar{Q}''}^*\bar{Q}_{K\bar{Q}''}^* \tag{58}$$

*where $Q_{KQ''}$ satisfies*

$$A^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}\mathcal{A}_{\bar{Q}''} + P^{-1}B\mathcal{B}_{\bar{Q}''}^*\mathcal{P}_{\bar{Q}''}^{-1} = 0.$$

**Proof:** In view of Lemma 3

$$K = I - \hat{B}^*\hat{P}^{\#}(sI - A)^{-1}\hat{B}$$

where

$$\hat{B} = B - P_{KQ'}\mathcal{P}_{Q'}^{-1}\mathcal{B}_{Q'}$$

and

$$\hat{P} = P - P_{KQ'}\mathcal{P}_{Q'}^{-1}P_{KQ'}^*$$

Therefore $H_r(K_-)G_2 = \operatorname{span}\{(sI - A)^{-1}\hat{B}G_2\}$, which is zero if and only if $\hat{B}G_2 = 0$.

Similarly, $K_+ = I - \hat{C}(sI - A)^{-1}\hat{Q}^{\#}\hat{C}$ where $\hat{C} = C - \bar{\mathcal{C}}\mathcal{Q}^{-1}Q_{\bar{Q}''K} = B^*P^{-1} - \mathcal{B}_{\bar{Q}''}^*Q_{K\bar{Q}''}^*$. Therefore $G_2H_r(K_+) = \operatorname{span}\{G_2\hat{C}(sI - A)^{-1}\}$, which is zero if and only if $G_2B^*P^{-1} = G_2\mathcal{B}_{\bar{Q}''}^*\bar{Q}_{K\bar{Q}''}^*$. □

The following is the main result (the equivalent of Theorem 1) for the characterization of controllability subspaces.

**Theorem 3:** *Let*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

*and let $Z \subset X$ and a projection matrix $G_2$ be given. The following are equivalent:*

(1) *There exists a matrix $F$ such that $Z = I_{A,B}\mathcal{R}$ where $\mathcal{R} = \langle A + BF|BG_2\rangle$, i.e. $\mathcal{R}$ is a controllability subspace.*

(2) *$Z = \mathbf{P}_{H_r(K)}H_r(Q')$ and $Q'$ reduces $K$ on the right with respect to $G_2$ and the reduction is minimal and proper.*

(3) *$Z = \mathbf{P}_{H_r(K)}H_r(\bar{Q}''^*)K$ and $\bar{Q}''$ reduces $K$ on the left with respect to $G_2$ and the reduction is minimal and proper.*

**Proof:** (1) ⇒ (2) First, observe that, if $\mathcal{R} = \langle A + BF|BG_2\rangle$ is a controllability subspace, then it has also the representation

$$\mathcal{R} = \langle A + BG_1F|BG_2\rangle$$

where $G_1 = I - G_2$. That is, $G_1F$ is a friend of $\mathcal{R}$. This is because $\operatorname{Im} BG_2 \subset \mathcal{R}$. Let now $P_{KQ'}$ be a full column-rank matrix such that $\operatorname{Im} P_{KQ'} = \mathcal{R}$.

Set $\mathcal{A}_0 := P_{KQ'}^{\#}(A + BG_1F)P_{KQ'}$, $\mathcal{B}_0 := P_{KQ'}^{\#}BG_2$, $\mathcal{F} := G_1FP_{KQ'}$. Observe that $\mathcal{B}_0\mathcal{F} = 0$. Then, since by constriction $(\mathcal{A}_0, \mathcal{B}_0)$ is controllable and $(\mathcal{F}, \mathcal{A}_0)$ is detectable, the Riccati equation

$$\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^* + \mathcal{B}_0\mathcal{B}_0^* - \mathcal{X}\mathcal{F}^*\mathcal{F}\mathcal{X} = 0 \tag{59}$$

has a unique positive definite solution $\mathcal{X}_+$. Therefore, adding and subtracting and changing sign

$$\begin{aligned}&(-\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\mathcal{X}_+^{-1})\mathcal{X}_+ + \mathcal{X}_+((-\mathcal{B}_0\mathcal{B}_0^*\mathcal{X}_+^{-1})^* - \mathcal{A}_0^*)\\ &\quad + \mathcal{B}_0\mathcal{B}_0^* + \mathcal{X}_+\mathcal{F}^*\mathcal{F}\mathcal{X}_+ = 0\end{aligned}$$

which means that the matrix $-\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\mathcal{X}_+^{-1}$ is stable and that the pair $(-\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\mathcal{X}_+^{-1}, \mathcal{B} + \mathcal{X}_+\mathcal{F}^*)$ is controllable with gramian $\mathcal{X}_+$. That is, setting $\mathcal{P} := \mathcal{X}_+^{-1}$, the pair

$$(-\mathcal{A}_0^* - \mathcal{P}\mathcal{B}_0\mathcal{B}_0^*,\ \mathcal{P}\mathcal{B}_0 + \mathcal{F}^*)$$

is controllable with gramian $\mathcal{P}$.

Set

$$\begin{aligned}\mathcal{A}^* &:= -\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\mathcal{P}\\ &= -P_{KQ'}^{\#}(A + BG_1F + BG_2B^*(P_{KQ'}^*)^{\#}\mathcal{P}P_{KQ'}^{\#})P_{KQ'}\\ \mathcal{B}^* &:= \mathcal{B}_0^*\mathcal{P} + \mathcal{F} = G_2B^*(P_{KQ'}^{\#})^*\mathcal{P} + G_1FP_{KQ'}\end{aligned}$$

Then

$$\begin{aligned}&AP_{KQ'} + P_{KQ'}\mathcal{A}^* + B\mathcal{B}^*\\ &\quad = AP_{KQ'} + P_{KQ'}P_{KQ'}^{\#}(-A - BG_1F\\ &\qquad - BG_2B^*(P_{KQ'}^{\#})^*\mathcal{P}P_{KQ'}^{\#})P_{KQ'}\\ &\qquad + BG_2B^*(P_{KQ'}^{\#})^*\mathcal{P} + BG_1FP_{KQ'} = 0\end{aligned}$$

Therefore, if we set

$$Q' := \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\mathcal{B}^*\mathcal{P}^{-1} & I \end{array}\right)$$

in view of Lemma 4, we can write $\mathcal{R} = I_{A,B}^{-1}\mathbf{P}_{H_r(K)}H_r(Q')$. Note that, since $\operatorname{Im} BG_2 \subset \mathcal{R}$, it is $P_{KQ'}P_{KQ'}^{\#}BG_2 = BG_2$ and therefore the equality

$$\mathcal{P}^{-1}\mathcal{B}G_2 = P_{KQ'}^{\#}BG_2$$

holds; in view of Lemma 5, $Q'$ reduces $K$ on the right.

The reduction is proper by construction ($\deg Q' = \dim \mathcal{R}$).

It is also minimal. In fact, since

$$H_r(Q') = \operatorname{span}\{\xi(sI - \mathcal{A})^{-1}\mathcal{B}|\xi \in \mathbb{C}_r^n\}$$

and $(\mathcal{A}^*, \mathcal{P}^{-1}\mathcal{B}G_2)$ is controllable

$$\mathbf{P}_{G_2H^2}H_c(Q') = \mathrm{span}\{G_2\mathcal{B}^*\mathcal{P}^{-1}(sI-\mathcal{A})^{-1}\}$$

has dimension equal to $\deg Q'$. But this implies $H_c(Q') \cap G_1H^2 = 0$.

(2) $\Rightarrow$ (1) Let now

$$\left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right) \quad \text{and} \quad Q' := \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\mathcal{B}^*\mathcal{P}^{-1} & I \end{array}\right)$$

be given. We already know, from Theorem 1, that $\mathcal{R} = I_{A,B}^{-1}\mathbf{P}_{H_r(K)}H_r(Q')$ is an antistabilizable controlled invariant subspace and therefore $\mathcal{R} = \mathrm{Im}\ P_{KQ'}$ where $P_{KQ'}$ solves

$$AP_{KQ'} + P_{KQ'}\mathcal{A}^* + B\mathcal{B}^* = 0 \tag{60}$$

Since $Q'$ reduces $K$ properly, $P_{KQ'}$ has full column rank, which implies that $(\mathcal{A}, \mathcal{B})$ is controllable and thus so is $(\mathcal{A}^*, \mathcal{P}^{-1}\mathcal{B})$. Since the reduction is minimal, $\mathbf{P}_{G_2H^2}H_c(Q') = \mathrm{span}\ \{G_2\mathcal{B}^*\mathcal{P}^{-1}(sI-\mathcal{A})^{-1}\}$ has dimension equal to degree of $Q'$. Therefore, also $(\mathcal{A}^*, \mathcal{P}^{-1}\mathcal{B}G_2)$ is controllable, i.e.

$$\mathcal{R} = P_{KQ'}\langle\mathcal{A}^*|\mathcal{P}^{-1}\mathcal{B}G_2\rangle = \langle P_{KQ'}\mathcal{A}^*P_{KQ'}^{\#}|P_{KQ'}\mathcal{P}^{-1}\mathcal{B}G_2\rangle \tag{61}$$

But from (60) we get

$$(A + B\mathcal{B}^*P_{KQ'}^{\#})_{|\mathcal{R}} = -P_{KQ'}\mathcal{A}^*P_{KQ'}^{\#}$$

and since $Q'$ reduces $K$, Lemma 5 yields $BG_2 = P_{KQ'}\mathcal{P}^{-1}\mathcal{B}G_2$. Therefore substitution in (61) yields

$$\mathcal{R} = \langle A + B\mathcal{B}^*P_{KQ'}^{\#}|BG_2\rangle$$

(1) $\Rightarrow$ (3) Again

$$\mathcal{R} = \langle A + BG_1F|BG_2\rangle$$

Let now $Q_{K\bar{Q}''}$ be a full column-rank matrix such that $\mathrm{Im}\ PQ_{K\bar{Q}''} = \mathcal{R}$.

Set

$$\mathcal{A}_0 := Q_{K\bar{Q}''}^{\#}(A^* + P^{-1}BG_1F)Q_{K\bar{Q}''},$$
$$\mathcal{B}_0 := Q_{K\bar{Q}''}^{\#}P^{-1}BG_2, \qquad \bar{\mathcal{F}} := G_1FQ_{K\bar{Q}''}.$$

Observe that $\mathcal{B}_0\bar{\mathcal{F}} = 0$. Then, since by construction $(\mathcal{A}_0, \mathcal{B}_0)$ is controllable and $(\bar{\mathcal{F}}, \mathcal{A}_0)$ is detectable, the Riccati equation

$$\mathcal{A}_0\bar{\mathcal{Y}} + \bar{\mathcal{Y}}\mathcal{A}_0^* + \mathcal{B}_0\mathcal{B}_0^* - \overline{\mathcal{Y}\mathcal{F}}^*\overline{\mathcal{F}\mathcal{Y}} = 0 \tag{62}$$

has a unique positive definite solution $\bar{\mathcal{Y}}_+$. Therefore, adding and subtracting and changing sign

$$(-\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\bar{\mathcal{Y}}_+^{-1})\bar{\mathcal{Y}}_+ + \bar{\mathcal{Y}}_+((-\mathcal{B}_0\mathcal{B}_0^*\bar{\mathcal{Y}}_+^{-1})^* - \mathcal{A}_0^*)$$
$$+ \mathcal{B}_0\mathcal{B}_0^* + \bar{\mathcal{Y}}_+\bar{\mathcal{F}}^*\overline{\mathcal{F}\mathcal{Y}}_+ = 0$$

which means that the matrix $-\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\bar{\mathcal{Y}}_+^{-1}$ is stable and that the pair $(-\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\bar{\mathcal{Y}}_+^{-1}, \mathcal{B}_0 + \bar{\mathcal{Y}}_+\bar{\mathcal{F}}^*)$ is controllable with gramian $\bar{\mathcal{Y}}_+$. That is, setting $\mathcal{Q} := \bar{\mathcal{Y}}_+^{-1}$, the pair

$$(-\mathcal{A}_0^* - \mathcal{Q}\mathcal{B}_0\mathcal{B}_0^*,\ \mathcal{Q}\mathcal{B}_0 + \bar{\mathcal{F}}^*)$$

is controllable with gramian $\mathcal{Q}$.

Set

$$\mathcal{A} := -\mathcal{A}_0 - \mathcal{B}_0\mathcal{B}_0^*\mathcal{Q} = -Q_{K\bar{Q}''}^{\#}(A^* + P^{-1}BG_1F$$
$$+ P^{-1}BG_2B^*P^{-1}(\bar{Q}_{K\bar{Q}''}^*)^{\#}\mathcal{Q}Q_{K\bar{Q}''}^{\#})Q_{K\bar{Q}''}$$
$$\mathcal{B}^* := \mathcal{B}_0^* + \bar{\mathcal{F}}\mathcal{Q}^{-1} = G_2B^*P^{-1}Q_{K\bar{Q}''}^{\#} + G_1FQ_{K\bar{Q}''}\mathcal{Q}^{-1}$$

Then

$$A^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}\mathcal{A} + P^{-1}B\mathcal{B}^*\mathcal{Q}$$
$$= A^*Q_{K\bar{Q}''} + Q_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#}(-A^* - P^{-1}BG_1F$$
$$- P^{-1}BG_2B^*P^{-1}(Q_{K\bar{Q}''}^{\#})^*\mathcal{Q}Q_{K\bar{Q}''}^{\#})Q_{K\bar{Q}''}$$
$$+ P^{-1}BG_2B^*P^{-1}Q_{K\bar{Q}''}^{\#}\mathcal{Q} + P^{-1}BG_1FQ_{K\bar{Q}''} = 0$$

Therefore, if we set

$$\bar{Q}'' := \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\mathcal{B}^*\mathcal{Q} & I \end{array}\right)$$

in view of Theorem 1, we can write $\mathcal{R} = I_{A,B}^{-1}\mathbf{P}_{H_r(K)}\bar{H}_r(\bar{Q}''^*)K$. Again, since $\mathrm{Im}\ BG_2 \subset \mathcal{R}$, it is $Q_{K\bar{Q}''}Q_{K\bar{Q}''}^{\#}BG_2 = BG_2$ and the equality

$$\mathcal{Q}\mathcal{B}G_2 = Q_{K\bar{Q}''}^{\#}P^{-1}BG_2$$

ensures us, in view of Lemma 5, that $\bar{Q}''$ reduces $K$ on the left.

The reduction is proper by construction since $\deg \bar{Q}'' = \dim \mathcal{R}$.

It is minimal: in fact, since

$$H_c(\bar{Q}'') = \mathrm{span}\ \{\mathcal{B}^*\mathcal{Q}(sI-\mathcal{A})^{-1}\}$$

and $(\mathcal{A}^*, \mathcal{Q}\mathcal{B}G_2)$ is controllable

$$\mathbf{P}_{G_2H^2}H_c(Q') = \mathrm{span}\ \{G_2\mathcal{B}^*\mathcal{Q}(sI-\mathcal{A})^{-1}\}$$

has dimension equal to $\deg \bar{Q}''$. This however implies $H_r(\bar{Q}'') \cap H^2G_1 = 0$.

(3) $\Rightarrow$ (1) The proof is similar to the second step (2) $\Rightarrow$ (1) and it is omitted. $\square$

The functions $Q'$ and $\bar{Q}''$ are, in general, not determined uniquely by $\mathcal{R}$ and $G_2$. Nevertheless, if we impose a mild condition, we get a uniqueness result.

Given an arbitrary subspace $\mathcal{R}$ of $\mathbb{C}^n$ and a matrix $B$, we say that $G_2$ is **maximal** for $\mathcal{R}$ w.r.t. $B$ if $\mathrm{Ker}\ BG_1 = \mathrm{Ker}\ G_1$ and $\mathrm{Im}\ BG_1 \cap \mathcal{R} = 0$. If $\mathcal{R}$ is controlled invariant, $B$ is already specified implicitly, and we will simply say that $G_2$ is maximal for $\mathcal{R}$. Similarly, given $K$ and $Q'$, we can define, in view of Theorem 3, a unique controllability subspace $\mathcal{R}_{Q'}$. We will say that $G_2$ is maximal for $Q'$ if it is maximal for $\mathcal{R}_{Q'}$. Note that not all controlled invariant subspaces have a maximal $G_2$. Nevertheless, in the application we have in mind, namely spectral factors, this condition is satisfied.

**Lemma 6:** *Let $\mathcal{R}$ be a controlled invariant subspace for $A$, $B$; suppose $G_2$ is maximal for $\mathcal{R}$ with respect to $B$ and $F_1$ and $F_2$ are friends of $\mathcal{R}$ such that $F_i = G_1 F_i$: then $F_{1|\mathcal{R}} = F_{2|\mathcal{R}}$.*

**Proof:** If $F_1$ and $F_2$ are two different feedbacks friends of $\mathcal{R}$, then for any $x \in \mathcal{R}$,

$$\begin{aligned} y &:= (A + BG_1F_1)x - (A + BG_1F_2)x \\ &= BG_1(F_1 - F_2)x \in \mathcal{R} \end{aligned}$$

But since $\operatorname{Im} BG_1 \cap \mathcal{R} = 0$, $y = 0$. Since $\operatorname{Ker} BG_1 = \operatorname{Ker} G_1$, this implies $(F_1 - F_2)_{|\mathcal{R}} = 0$.

**Lemma 7:** *Let $\mathcal{R}$ be a controllability subspace for $A$, $B$; suppose $G_2$ is maximal for $\mathcal{R}$ with respect to $B$. Then there exist unique $Q'$ and $\bar{Q}''$ which are properly and minimally reducing*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

*and such that*

$$I_{A,B}\mathcal{R} = \mathbf{P}_{H_r(K)}H_r(Q') = \mathbf{P}_{H_r(K)}\bar{H}_r(\bar{Q}''^*)K$$

**Proof:** Let $\mathcal{R} = I_{A,B}^{-1}\mathbf{P}_{H_r(K)}H_r(Q')$ be a controllability subspace and let, as usual

$$Q' = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\mathcal{B}^*\mathcal{P}^{-1} & I \end{array}\right)$$

with $Q'$ properly and minimally reducing for $K$ w.r.t. $G_2$. Since $\mathcal{R}$ is controlled invariant, the usual Lyapunov equation

$$AP_{KQ'} + P_{KQ'}\mathcal{A}^* + B\mathcal{B}^* = 0 \tag{63}$$

holds, with $\operatorname{Im} P_{KQ'} = \mathcal{R}$. Moreover, by a change of coordinates, we can assume that it is $P_{KQ'}^*P_{KQ'} = I$. Then (63) yields

$$\mathcal{A}^* = -P_{KQ'}^*(A + B\mathcal{B}^*P_{KQ'}^*)P_{KQ'} \tag{64}$$

The condition that $Q'$ reduces $K$ is, in view of Lemma 5

$$\mathcal{B}G_2 = \mathcal{P}P_{KQ'}BG_2 \tag{65}$$

where $\mathcal{P}$ satisfies

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0 \tag{66}$$

Substituting (64) and (65) into (66), we obtain

$$\begin{aligned} 0 &= -P_{KQ'}^*(A^* + P_{KQ'}\mathcal{B}B^*)P_{KQ'}\mathcal{P} \\ &\quad - \mathcal{P}P_{KQ'}^*(A + B\mathcal{B}^*P_{KQ'}^*)P_{KQ'} + \mathcal{B}G_1\mathcal{B}^* + \mathcal{B}G_2\mathcal{B}^* \\ &= -P_{KQ'}^*(A^* + P_{KQ'}\mathcal{B}G_1B^*)P_{KQ'}\mathcal{P} \\ &\quad - \mathcal{P}P_{KQ'}^*(A + BG_1\mathcal{B}^*P_{KQ'}^*)P_{KQ'} + \mathcal{B}G_1\mathcal{B}^* \\ &\quad - \mathcal{B}G_2B^*P_{KQ'}\mathcal{P} - \mathcal{P}P_{KQ'}^*BG_2\mathcal{B}^* + \mathcal{B}G_2\mathcal{B}^* \\ &= -P_{KQ'}^*(A^* + P_{KQ'}\mathcal{B}G_1B^*)P_{KQ'}\mathcal{P} \\ &\quad - \mathcal{P}P_{KQ'}^*(A + BG_1\mathcal{B}^*P_{KQ'}^*)P_{KQ'} + \mathcal{B}G_1\mathcal{B}^* - \mathcal{B}G_2\mathcal{B}^* \\ &= -P_{KQ'}^*(A^* + P_{KQ}\mathcal{B}G_1B^*)P_{KQ'}\mathcal{P} \\ &\quad - \mathcal{P}P_{KQ'}^*(A + BG_1\mathcal{B}^*P_{KQ'}^*)P_{KQ'} + \mathcal{B}G_1\mathcal{B}^* \\ &\quad - \mathcal{P}P_{KQ'}^*BG_2B^*P_{KQ'}\mathcal{P} \end{aligned} \tag{67}$$

Note that, in view of maximality of $G_2$, Lemma 7 ensures that the matrices $\mathcal{B}$, $G_1$, $\mathcal{B}^*$ and $\mathcal{B}G_1B^*$ are uniquely determined by $\mathcal{R}$. Therefore, in the above equation we have fixed $A$, $B$, $P_{KQ'}$ and $\mathcal{B}G_1\mathcal{B}^*$. The only variable is $\mathcal{P}$ and we get again a Riccati equation. Now the pair $(P_{KQ'}^*(A^* + P_{KQ'}\mathcal{B}G_1B^*)P_{KQ'},\ \mathcal{B}G_2)$ is controllable, since it is obtained by a change of basis from the pair $(P_{KQ'}^*(A + BG_1\mathcal{B}^*P_{KQ'}^*)P_{KQ'},\ \mathcal{P}^{-1}\mathcal{B}G_2)$, which is controllable by construction. Detectability of $(P_{KQ'}^*(A^* + P_{KQ'}\mathcal{B}G_1B^*)P_{KQ'},\ G_2B^*P_{KQ'})$ follows from the fact that $((A + BG_1\mathcal{B}P_{KQ'})_{|\mathcal{R}},\ BG_2)$ is stabilizable. Therefore equation (67) has a unique positive definite solution $\mathcal{P}$. Since $\mathcal{B}$ and then $\mathcal{A}$ are uniquely determined by $\mathcal{P}$ the proof is achieved. □

To characterize controllability subspaces we will need the following technical result.

**Lemma 8:** *Let $K$, $Q'$ be given inner functions and $G_2$ a constant projection matrix; suppose that there exist inner functions $Q''$, $\bar{Q}'$, $\bar{Q}''$, $K_-$, $K_+$, $Q_-$, $Q_+$, $R$ such that*

- $Q'Q'' = Q_+G_1 + RG_2$
- $\bar{Q}'\bar{Q}'' = G_1Q_- + G_2R$
- $K_-Q_+ = Q_-K_+$
- $\bar{Q}'\bar{Q}''K_+ = \bar{Q}'KQ'' = K_-Q'Q''$

*Then*

(1) $\mathbf{P}_{H_r(K_+)|H_r(Q_+)}$ *is injective if and only if* $\mathbf{P}_{H_c(K_-)}|H_c(Q_-)$ *is injective.*

(2) *If* $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ *is injective, then also*
 (a) $\mathbf{P}_{H_r(K)}|H_r(Q')$
 (b) $\mathbf{P}_{H_r(K)}|H_r(\bar{Q}'')K^*$.
 *are injective.*

(3) *If* $\mathbf{P}_{H_r(K)}|H_r(Q')$ *is injective and* $\deg \bar{Q}'' = \deg R$, *then also* $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ *is injective.*

**Proof:**

(1) This follows from Theorem 3 in Fuhrmann and Gombani (1998) which studies Toeplitz operators with all-pass symbols. But we give also a direct proof. There is no loss of generality in assuming that

$$G_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array}\right)$$

Suppose that the projection $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ is injective; we show that $\mathbf{P}_{H_r(K_-^*)}|H_r(Q_-^*)$ is injective, from which the injectivity of $\mathbf{P}_{H_c(K_-)}|H_c(Q_-)$ follows easily.

We have that $[\mathbf{P}_{H_r(K_-^*)}|H_r(Q_-^*)]K_-Q_+ = \mathbf{P}_{H_r(K_-)Q_+}|H_r(Q_-)K_+$. Now, if $v \in H_r(Q_-)K_+$, we can decompose it uniquely as $v = \hat{v} + \tilde{v}$, where $\hat{v} \in H_r(Q_+)$ and $\tilde{v} \in H_r(K_-)Q_+$ and therefore $\mathbf{P}_{H_r(K_-)Q_+} v = \tilde{v}$. Suppose now that $v \neq 0$ but $\tilde{v} = 0$. This means that $v \in H_r(Q_+) \cap H_r(Q_-)K_+$. But this in turns leads to a contradiction since, at the same time, $\mathbf{P}_{H_r(K_+)} v \neq 0$ since $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ is injective, and $\mathbf{P}_{H_r(K_+)} v = 0$, since $H_r(Q_-)K_+$ is orthogonal to $H_r(K_+)$. Thus $\tilde{v} = 0$ implies $v = 0$ and the application is injective. The opposite implication is shown similarly.

(2) (a) Observe first that $\mathbf{P}_{H_r(K)}|H_r(Q')$ is injective if and only if $\mathbf{P}_{H_r(K)Q''}|H_r(Q')Q''$. Since $H_r(Q'')$ is orthogonal to $H_r(K)Q''$ and $H_r(Q')Q''$, $\mathbf{P}_{H_r(K)Q''}|H_r(Q')Q''$ is injective if and only if

$$\mathbf{P}_{H_r(KQ'')}|H_r(Q'Q'') \tag{68}$$

is injective, or equivalently if the same holds for $\mathbf{P}_{H_r(\bar{Q}''K_+)}|H_r(Q_+R)$. Note that $Q_+$ and $R$ commute.

Suppose now there exists a vector $v \in H_r(Q_+R)$ such that $v \perp H_r(\bar{Q}''K_+)$. Then

$$\begin{aligned} 0 = \mathbf{P}_{H_r(\bar{Q}''K_+)} v &= \mathbf{P}_{H_r(\bar{Q}'')K_+} v + \mathbf{P}_{H_r(K_+)} v \\ &= \mathbf{P}_{H_r(\bar{Q}'')K_+}\mathbf{P}_{H_r(Q_+)} v + \mathbf{P}_{H_r(\bar{Q}'')K_+}\mathbf{P}_{H_r(R)} v \\ &\quad + \mathbf{P}_{H_r(K_+)}\mathbf{P}_{H_r(Q_+)} v + \mathbf{P}_{H_r(K_+)}\mathbf{P}_{H_r(R)} v \end{aligned} \tag{69}$$

Now, since $H_r(K_+)$ is orthogonal to $H_r(R)$, we have

$$\mathbf{P}_{H_r(K_+)} H_r(R) = 0$$

Since the range of the first and second summands of (69) are orthogonal to $H_r(K_+)$, (69) is satisfied if and only if $\mathbf{P}_{H_r(K_+)}\mathbf{P}_{H_r(Q_+)} v = 0$. So, if $v \neq 0$, $\mathbf{P}_{H_r(K)}|H_r(Q')$ is not injective and this implies that $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ is not injective either. This contradicts the assumption and therefore the statement is proved.

(b) Again, note that, since $KQ'' = \bar{Q}''K_+$, injectivity of $\mathbf{P}_{H_r(K)}|H_r(\bar{Q}''^*)K$ is equivalent to that of $\mathbf{P}_{H_r(K)Q''}|H_r(\bar{Q}'')K_+$. Then the reasoning is similar to that of 1. Since $H_r(\bar{Q}')KQ''$ is orthogonal to $H_r(K)Q''$ and to $H_r(\bar{Q}'')K_+$, $\mathbf{P}_{H_r(K)Q''}|H_r(\bar{Q}'')K_+$ is injective if and only if $\mathbf{P}_{H_r(\bar{Q}'K)Q''}|H_r(\bar{Q}'\bar{Q}'')K_+$ is injective, or equivalently if the same holds for $\mathbf{P}_{H_r(K_-Q')Q''}|H_r(Q_-R)K_+$. Again we can decompose

$$H_r(K_-Q')Q'' = H_r(K_-)Q_-R \oplus H_r(Q')Q''$$

(for $Q_-$ and $R$ commute) and

$$H_r(Q_-R)K_+ = H_r(Q_-)K_+ \oplus H(R)$$

As above, we can suppose that there exists a vector $v \in H_r(K_-R)K_+$ such that $v \perp H_r(K_-Q')Q''$. Then

$$\begin{aligned} 0 = \mathbf{P}_{H_r(K_-Q')Q''} v &= \mathbf{P}_{H_r(K_-)Q'Q''} v + \mathbf{P}_{H_r(Q')Q''} v \\ &= \mathbf{P}_{H_r(K_-)Q'Q''}\mathbf{P}_{H_r(Q_-)K_+} v + \mathbf{P}_{H_r(K_-)Q'Q''}\mathbf{P}_{H_r(R)} v \\ &\quad + \mathbf{P}_{H_r(Q')Q''}\mathbf{P}_{H_r(Q_-)K_+} v + \mathbf{P}_{H_r(Q')Q''}\mathbf{P}_{H_r(R)} v \end{aligned} \tag{70}$$

Now, since $H_r(K_+)Q'Q'' = H_r(K_+)Q_+$ is orthogonal to $H_r(R)$, we have

$$\mathbf{P}_{H(K_+)Q'Q''} H(R) v = 0$$

Since the range of the first and second summands of (70) are orthogonal to $H_r(K_+)Q_+$, (70) is satisfied if and only if

$$\mathbf{P}_{H_r(K_+)Q_+}\mathbf{P}_{H_r(Q_-)K_+} v = 0$$

Multiplication by $K_+^*Q_-^*R^*$ and conjugation yield that (70) is satisfied if and only if $\mathbf{P}_{H_c(K_-)}\mathbf{P}_{H_c(Q_-)} v_1 = 0$ for $v_1 = Q_-RK_+v^*$. So, if $v \neq 0$, $\mathbf{P}_{H_r(K)}|H_r(Q')$ is not injective and this implies that $\mathbf{P}_{H_r(K_-)}|H_r(Q_-)$ is not injective either: this again contradicts the assumption and therefore the second statement is proved.

(3) Suppose $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ is not injective. That is, there exists an element $v \in H(Q_+)$ which is orthogonal to $H_r(K_+)$. Now, we know from (68) that injectivity of $\mathbf{P}_{H_r(K)}|H_r(Q')$ is equivalent to that of $\mathbf{P}_{H_r(\bar{Q}''K_+)}|H_r(RQ_+)$. But the space $Z := H_r(R) \vee \langle v \rangle$ is orthogonal to $H(K_+)$ and thus

$$\mathbf{P}_{H_r(\bar{Q}''K_+)}|Z = \mathbf{P}_{H_r(\bar{Q}'')K_+}|Z$$

and since the dimension of $Z$ is strictly bigger than that of $H_r(\bar{Q}'')K_+$, the map cannot be one to one and we get the conclusion. $\square$

The previous lemma is put into use in the next result.

**Lemma 9:** *Suppose*

$$Q' = \left(\begin{array}{c|c} A_{Q'} & B_{Q'} \\ \hline -B^*_{Q'}P^{-1}_{Q'} & I \end{array}\right)$$

*reduces $K$ on the right w.r.t. to $G_2$ and the reduction is proper and minimal. Then, there exist inner functions $Q''$, $\bar{Q}'$, $\bar{Q}''$, $K_-$, $K_+$, $Q_-$, $Q_+$, $R$ satisfying the hypothesis of Lemma 8 and such that $\bar{Q}''$ is minimally and properly reducing $K$ w.r.t. to $G_2$ on the left. If $G_2$ is maximal, all these functions are (essentially) unique.*

**Proof:** Since $G_2H_r(Q')$ and $G_1H_r(Q')$ are coinvariant subspaces, there exist unique (normalized) inner functions $Q_+$ and $R$ such that

$$H_r(Q_+) = H_r(Q')G_1 \qquad H_r(R) = H_r(Q')G_2 \tag{71}$$

moreover, since $H_r(Q') \subset H_r(Q_+) \oplus H_r(R)$, we can define the *completion* of $Q'$ w.r.t. $G_2$ as the inner functions $Q'' := Q'^*RQ_+$ (recall that $Q_+$ and $R$ commute).

Now we can define $K_-$ and $\bar{Q}'$ as

$$K_- := (K \vee_L Q')Q'^* \qquad \bar{Q}' := (K \vee_L Q')K^*$$

Define next $\bar{Q}''$ as the right completion of $\bar{Q}'$ with respect to $G_2$: that is, we set $Q_-$ and $\bar{R}$ to be the inner functions such that

$$H(Q_-) = G_1H_c(\bar{Q}') \qquad H(\bar{R}) = G_2H_c(\bar{Q}') \tag{72}$$

we can again define $\bar{Q}'' := \bar{Q}'^*Q_-\bar{R}$ and $K_+ := \bar{Q}''^*(\bar{Q}'' \vee K)$, so that we have the relation

$$K_-Q'Q'' = \bar{Q}'\bar{Q}''K_+ \tag{73}$$

We claim that $R = \bar{R}$. In fact, since $G_2K_- = G_2$

$$\begin{aligned} H_c(R) &= G_2H_c(Q') = G_2H_c(K_-Q') \\ &= G_2(H_c(K_-) \vee H_c(\bar{Q}')) = G_2H_c(\bar{Q}') = H_c(\bar{R}) \end{aligned}$$

This in turn implies that $H_c(K_+) \subset \operatorname{Ker} G_2$ (multiply (73) on both side by $G_2$ ) and so $K_-Q_+ = Q_-K_+$. Thus $\deg\ Q_- \le \deg\ \bar{Q}'$; but $\deg\ K_+ \le \deg K = \deg K_-$ and thus $\deg\ Q_- = \deg\ Q_+ = \deg\ R = \deg\ \bar{R}$ which entails $H(\bar{Q}') \cap H^2G_1 = 0$.

It is immediate to verify that the inner functions thus defined satisfy the conditions of Lemma 8. Moreover, by assumption, $Q'$ reduces $K$ w.r.t. $G_2$ minimally and properly on the right and, by construction, $\bar{Q}''$ reduces $K$ on the left and the reduction is minimal. Since $Q'$ reduces $K$ properly and $\deg\ \bar{Q}'' = \deg\ R$, in view of Lemma 8 the reduction on the left of $K$ by $Q''$ is also proper. If $G_2$ is maximal, $\bar{Q}''$ which is properly and minimally reducing $K$ on the left w.r.t. $G_2$ is unique, in view of Lemma 7. $\square$

The above lemma basically says that, if $Q'$ is properly and minimally reducing $K$ with respect to $G_2$, then it uniquely determines a controlled invariant subspace together with a $\bar{Q}''$ which is also minimally and properly reducing.

Since in the end we are interested in the connection of output nulling controllability subspaces with a stable proper rational function, we would like to drop the minimality assumption on the reduction; so, it might be tempting to deduce that, given $K$, $G_2$ and $Q'$, $\bar{Q}''$ which are proper but not minimal and satisfy the usual coprimeness conditions with $K$, a controlled invariant subspace is uniquely determined. Unfortunately this is not true, for the following reason: suppose $Q'$ reduces minimally and properly $K$, so that $K_-Q' = \bar{Q}'K$. Let now $\bar{Q}'_1$ be skew-prime with $K_-$ and such that $H(Q') \subset \ker G_2$. Let $\bar{K}_-$ be such that

$$\bar{K}_-Q'_1 = \bar{Q}'_1K_-$$

then, since $H(\bar{K}_-) \subset \ker G_2$, also $Q'_1Q'$ reduces $K$. The reduction, by definition, will not be minimal, but if the degree of $Q$ is less than the degree of $K$ it is generically possible to choose $Q'_1$ so that the reduction of $K$ by $Q'_1Q'$ is proper and the degree of $K$ equals that of $Q'_1Q'$. But this operation can be performed independently on $\bar{Q}''\bar{Q}''_1$, obtaining again a proper but non minimal reduction and again we can assume that $K$ and $\bar{Q}''\bar{Q}'_1$ have the same degree. Let $Q''Q''_1$ be such that $\bar{Q}''\bar{Q}''_1K_+ = KQ''Q'_1$ and the reduction is proper. Now the degree of $G_1Q'_1Q'Q''Q''_1$ will be equal to $\deg Q'_1 + \deg\ Q' + \deg\ Q''_1$ and thus strictly greater than the degree of $K_+$. Thus $\mathbf{P}_{H(K_+)}H(Q'_1Q'Q''Q'_1)$ cannot be injective. In other words, the functions $Q'_1Q'$ and $Q''Q''_1$ cannot be associated with a stable proper rational function. But there is a small technical definition which allows us to circumvent this problem.

**Definition 2:** Let $Q'$ and $\bar{Q}''$ be reducing for $K$ w.r.t. $G_2$, with $Q'$ and $K$ left coprime and $\bar{Q}''$ and $K$ right coprime. We say that $Q'$ and $\bar{Q}''$ are simultaneously proper if, defining

$$\begin{aligned} K_+ &:= \bar{Q}''^*(K \vee \bar{Q}'') \\ Q'' &:= (\bar{Q}'' \vee K)K^*_+ \\ Q_+ &:= Q'Q''G_1 + G_2 \end{aligned}$$

we have that $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ is injective.

**Theorem 4:** *Suppose that $Q'$ and $\bar{Q}''$ are reducing for $K$ with respect to $G_2$, and that the coprimeness relations*

$$Q' \wedge_R K = I \qquad \bar{Q}'' \wedge_L K = I$$

*hold. Suppose moreover that $Q'$ and $Q''$ are simultaneously proper. If $G_2$ is maximal, then the space*

$$\mathcal{R} := I_{A,B}^{-1}(\mathbf{P}_{H_r(K)}H_r(Q') \cap \mathbf{P}_{H_r(K)}H_r(\bar{Q}''*)K)$$

*is a controllability subspace with respect to $G_2$*

**Proof:** Observe first that, since $Q'$ and $\bar{Q}''$ are reducing, $K_-$ and $K_+$ are in Ker $G_2$. Next, define $Q_i'$ as the inner function associated to the coinvariant subspace $H_c(Q') \cap G_1H_+^2$. Then setting

$$Q_e' := Q_i'^*Q'$$

we have by construction that $H(Q_e) \cap G_1H_+^2 = 0$. Moreover, since, in view of maximality of $G_2$, it is $Q_iG_2 = G_2$, we claim that $BG_2 = P_{KQ'}\mathcal{P}_e^{-1}\mathcal{B}_eG_2$, where

$$Q_e = \left(\begin{array}{c|c} \mathcal{A}_e & \mathcal{B}_e \\ \hline -\mathcal{B}_e^*\mathcal{P}_e^{-1} & I \end{array}\right) \quad \text{and} \quad Q_i = \left(\begin{array}{c|c} \mathcal{A}_i & \mathcal{B}_i \\ \hline -\mathcal{B}_i^*\mathcal{P}_i^{-1} & I \end{array}\right)$$

In fact, we can write

$$Q' = Q_i'Q_e' = \left(\begin{array}{c|c} \mathcal{A}_i & \mathcal{B}_i \\ \hline -\mathcal{B}_i^*\mathcal{P}_i^{-1} & I \end{array}\right)\left(\begin{array}{c|c} \mathcal{A}_e & \mathcal{B}_e \\ \hline -\mathcal{B}_e^*\mathcal{P}_e^{-1} & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} \mathcal{A}_e & 0 & \mathcal{B}_e \\ -\mathcal{B}_e^*\mathcal{B}_i\mathcal{P}_i^{-1} & \mathcal{A}_i & \mathcal{B}_i \\ \hline -\mathcal{B}_e^*\mathcal{P}_e^{-1} & -\mathcal{B}_i^*\mathcal{P}_i^{-1} & I \end{array}\right)$$

So, since $Q_i'G_2 = G_2$, it is $G_2\mathcal{B}_i^*\mathcal{P}_i^* = 0$ or equivalently

$$\mathcal{P}^{-1}\mathcal{B}G_2 = \begin{bmatrix} \mathcal{P}_e^{-1}\mathcal{B}_eG_2 \\ 0 \end{bmatrix}$$

From the usual relation

$$AP_{KQ'} + P_{KQ'}\mathcal{A}^* + B\mathcal{B}_{Q'}^* = 0$$

applied to the cascade $Q_i'Q_e'$ we get

$$A[P_{KQ'}^e,\ P_{KQ'}^i] + [P_{KQ'}^e,\ P_{KQ'}^i]\begin{bmatrix} \mathcal{A}_e^* & -\mathcal{P}_i^{-1}\mathcal{B}_i\mathcal{B}_e^* \\ 0 & \mathcal{A}_i^* \end{bmatrix} + B[\mathcal{B}_e^*,\ \mathcal{B}_i^*] = 0$$

Therefore $P_{KQ'}$ satisfies

$$AP_{KQ'}^e + P_{KQ'}^e\mathcal{A}_e^* + B\mathcal{B}_e^* = 0$$

i.e. $P_{KQ'}^e = P_{KQ_e'}$. In conclusion

$$P_{KQ'}\mathcal{P}^{-1}\mathcal{B}G_2 = [P_{KQ'}^e,\ P_{KQ'}^i]\begin{bmatrix} \mathcal{P}_e^{-1}\mathcal{B}_eG_2 \\ 0 \end{bmatrix} = P_{KQ_e'}\mathcal{P}_e^{-1}\mathcal{B}_eG_2$$

as claimed. In view of Lemma 5, $Q_e'$ is reducing. It is obviously properly reducing and it is minimal by construction. Therefore $\mathcal{R} = I_{A,B}^{-1}\mathbf{P}_{H_r(K)}H_r(Q_e')$ is a controllability subspace. An analogous argument shows that $\mathcal{R} = I_{A,B}^{-1}\mathbf{P}_{H_r(K)}H_r(\bar{Q}_e''*)$ (with the obvious dual definitions). In conclusion, since

$$\mathbf{P}_{H_r(K)}H_r(Q') = \mathbf{P}_{H_r(K)}[H_r(Q_e') \oplus H_r(Q_i')Q_e']$$

and

$$\mathbf{P}_{H_r(K)}H_r(\bar{Q}''*)K = \mathbf{P}_{H_r(K)}[H_r(\bar{Q}''*) \oplus H_r(\bar{Q}_i''*)\bar{Q}_e''*]K$$

we have

$$\mathcal{R} \subset I_{A,B}^{-1}(\mathbf{P}_{H_r(K)}H_r(Q') \cap \mathbf{P}_{H_r(K)}H_r(\bar{Q}''*)K)$$

To see the reverse inclusion, we claim that

$$\dim[\mathbf{P}_{H_r(K)}H_r(Q') \cap \mathbf{P}_{H_r(K)}H_r(\bar{Q}''*)K] \leq \deg Q_2''$$

We recall first that if $M$, $N$ are subspaces of a Hilbert space $L$, $(M \cap N)^\perp = M^\perp \vee N^\perp$, and $(M \cap N)_M^\perp = \mathbf{P}_MN^\perp$, where the subscript $M$ in the notation $(M \cap N)_M^\perp$ indicates that the orthogonal complement has to be taken in $M$.

In view of the above projection formulas, we can write

$$\begin{aligned}[\mathbf{P}_{H_r(K)}H_r(Q')]Q'' &= \mathbf{P}_{H_r(K)Q''}H_2(Q')Q'' \\ &= (H_r(K)Q'' \cap H_+^2Q'Q'')_{H_n(K)Q''}^\perp\end{aligned}$$

and

$$\begin{aligned}[\mathbf{P}_{H_r(K)}H_r(\bar{Q}''*)K]Q'' &= \mathbf{P}_{H_r(K)Q''}H_r(\bar{Q}'')K_+ \\ &= (\mathbf{P}_{H_r(K^*)}H_r((\bar{Q}'')^*))KQ'' \\ &= (H_r(K^*) \cap H_-^2[\bar{Q}'']^*)^\perp KQ'' \\ &= (H_r(K)Q'' \cap H_-^2K_+)_{H_r(K)Q''}^\perp\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbf{P}_{H_r(K)}H_r(Q')Q'' \cap \mathbf{P}_{H_r(K)}H_r(\bar{Q}''*)KQ'' \\ &= (H_r(K)Q'' \cap H_+^2Q'Q'')_{H_r(K)Q''}^\perp \cap (H_r(K)Q'' \\ &\quad \cap H_-^2K_+)_{H_r(K)Q''}^\perp \\ &= [(H_r(K)Q'' \cap H_+^2Q_+R) \\ &\quad \cup (H_r(K)Q'' \vee H_-^2K_+)]_{H_r(K)Q''}^\perp \\ &\subset [H_r(K)Q'' \cap (H_+^2Q_+R \cup H_-^2K_+)]_{H_r(K)Q''}^\perp \\ &= [H_r(K)Q'' \cap [L^2G_1 \vee H_+^2R]]_{H(K)Q''}^\perp \\ &\subset [H_r(K)Q'' \cap L^2G_1]_{H_r(K)Q''}^\perp\end{aligned}$$

where we have used the fact that the injectivity of $\mathbf{P}_{H_r(K_+)}|H_r(Q_+)$ implies that $H_+^2Q_+G_1 \vee H_-^2K_+G_1 = L^2G_1$ (see Theorem 4.1 in Fuhrmann and Gombani 1998). Therefore

$$\begin{aligned}&\dim(\mathbf{P}_{H_r(K)}H_r(Q') \cap \mathbf{P}_{H_r(K)}H_r(\bar{Q}''*)K) \\ &\leq n - \dim(H_r(K)Q'' \cap L^2G_1)\end{aligned}$$

But, denoting by $n$ the degree of $K$, and recalling that if $M \subset L$, then $\dim[(M \oplus N) \cap L] = \dim M \oplus \dim (N \cap L)$

$$\begin{aligned}\dim (H_r(K)Q'' \cap L^2G_1) &\geq \dim [(H_r(K)Q'' \oplus H_r(Q'')) \\ &\quad \cap L^2G_1] - \dim H_r(Q'') \\ &= \dim [(H(\bar{Q}'')K_+ \oplus H(K_+)) \\ &\quad \cap L^2G_1] - \dim H_r(\bar{Q}'') \\ &= \dim [H_r(\bar{Q}'') \cap L^2G_1] \\ &\quad + n - \dim H_r(\bar{Q}'') \\ &= \dim [[H_r(\bar{Q}_2'')\bar{Q}_1'' \oplus H_r(\bar{Q}_1'')] \\ &\quad \cap L^2G_1] + n - \dim H_r(\bar{Q}'') \\ &= \dim [H_r(\bar{Q}_2'') \cap L^2G_1] \\ &\quad + n - \dim H_r(\bar{Q}'') \\ &\quad + \dim H_r(\bar{Q}_1'') \\ &\geq n - \dim H_r(\bar{Q}_2'')\end{aligned}$$

since $\dim\ H_r(\bar{Q}_2'') = \dim\ H_r(\bar{Q}'') - H_r(\bar{Q}_1'')$. Thus

$$\begin{aligned}&\dim (\mathbf{P}_{H_r(K)}H_r(Q')Q'' \cap \mathbf{P}_{H_r(K)}H(\bar{Q}''^*)K) \\ &\quad \leq \dim\ H_r(\bar{Q}_2'') = \dim H_r(Q_2'')\end{aligned}$$

as wanted. □

**Theorem 5:** *Let $W$ be a proper stable rational function (typically a spectral factor) and let*

$$W = (W_- \quad 0)Q, \qquad (W_+ \quad 0)Q''^*$$

*be the outer–inner and maximum-phase inner factorizations of $W$. Then the supremal controllability subspace $\mathcal{R}^*$ which is output nulling for $W$ is*

$$\mathcal{R}^* := I_{A,B}^{-1}(\mathbf{P}_{H_r(K)}H_r(Q') \cap \mathbf{P}_{H_r(K)}\bar{H}_r(\bar{Q}''^*)K) \qquad (74)$$

*where $\bar{Q}''$ is defined as*

$$\bar{Q}'' = K^*[K \vee_R Q'']$$

*Thus*

$$\mathcal{R}^* = \mathrm{Im}\ P_{KQ'} \cap \mathrm{Im}\ P_{K\bar{Q}''}$$

**Proof:** In view of Theorem 2, $\mathcal{R}^*$ is contained in the right-hand side of (74); thus we only need to show that this right-hand side is itself a controllability output nulling subspace. But since $W$ is a proper rational function, $Q'$ and $\bar{Q}''$ are simultaneously proper in view of Lemma 8; in view of Lemma 4, this space is output nulling; Theorem 4 eventually implies that it is a controllability subspace. □

The phase function of a spectral density is defined by means of its maximum and minimum-phase spectral factor as the function

$$T = K_+Q_+^* = Q_-^*K_-$$

where $K_-$ and $K_+$ are the DSS factors of the minimum-phase and maximum-phase factors $W_-$ and $W_+$ respectively; $Q_+$ and $Q_-$ are the maximal inner divisors of $W_+$ (i.e. $W_+ = W_-Q_+$) and $\bar{W}_+$ (see figure 2). The phase function of an arbitrary spectral factor $W$ is the phase function of the associated density $\Phi = WW^*$. In practice, a phase function is any all-pass function which can be expressed as $T = K_+Q_+^*$ with $K_+$ and $Q_+$ inner and such that $\mathbf{P}_{H(K_+)}H(Q_+)$ is injective. A natural question which can be asked is which spectral factors have the same phase function. We can give a simple answer to this problem.

**Theorem 6:** *Suppose on all-pass function $T = K_+Q_+^*$ is given with $K_+$, $Q_+$ inner; let*

$$K_+ = \left(\begin{array}{c|c} A_+ & B_+ \\ \hline -B_+^*P_+^{-1} & I \end{array}\right) \quad \textit{and} \quad Q_+ = \left(\begin{array}{c|c} \mathcal{A}_{Q_+} & \mathcal{B}_{Q_+} \\ \hline -\mathcal{B}_{Q_+}^*\mathcal{P}_{Q_+}^{-1} & I \end{array}\right)$$

*then a maximum phase factor $W_+$ has phase function $T$ if and only if*

$$W_+ = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

*with $A$, $B$ as in the realization of $K_+$ and*

$$D^{-L}C = \mathcal{B}_{Q_+}^*P_{K_+Q_+}^{\#} \qquad (75)$$

*where $P_{K_+Q_+}$ satisfies*

$$AP_{K_+Q_+} + P_{K_+Q_+}\mathcal{A}_{Q_+}^* + B\mathcal{B}_{Q_+}^* = 0$$

**Proof:** The function $K_+$ is the DSS factor of $W_+$ and therefore $A$ and $B$ can be chosen to be equal to $A_+$ and $B_+$. Condition (75) then follows from Lemma 4 after noting that all zeros of $W_+$ are, by construction, unstable and that $D$ is left invertible since the factor is maximum phase. □

Observe that, in the above theorem, no assumption on coprimeness of $T_+$ and $Q_+$ is made, so that the spectral factor might not be minimal (choose for instance $Q_+ = K_+$). To get minimal factors we will need to assume that $K_+$ and $Q_+$ are right coprime.

## 5. Unstable transfer functions

So far, we have only considered stable transfer functions. As we said in the beginning, this is mainly due to expository reasons, since the main ideas are already present in the Hardy space setting. We proceed now to extend the previous results to the unstable case.

We need to introduce some further notation, since $W$ will be now unstable. The idea is to keep the previous notation, with an overline, for the **antistable** factors and inner functions related to them; but we will indicate by

**underlined** letters the **stable** factors and the relative inner functions.

So, for example, let $W$ be an unstable spectral factor. We denote its right DSS factorization over $H_+^\infty$ by $W = \underline{\underline{W}}\underline{K}^*$ and its right factorization over $H_-^\infty$ by $W = \overline{W}\overline{K}$. Thus $\underline{W} = \overline{W}\overline{K}\underline{K}$. The product $\overline{K}\underline{K}$ will be denoted by $\hat{K}$. Note that, with this convention, $\bar{W} \in H_-^\infty$ and $\underline{W} \in H_+^\infty$, but $\overline{K}$, $\underline{K} \in H_+^\infty$.

Before continuing, we need to introduce a few technical lemmas which connect factors of inner functions with invariant subspaces and solutions to the Riccati equation. It is well known that, given an inner $K$, there exists a bijective correspondence between inner divisors of $K$ and non-negative definite solutions of a homogeneous Riccati equation. The following lemma is taken from Fuhrmann (1995).

**Lemma 10:** *Let*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right)$$

*be a minimal realization of the inner function $K$. There exists a bijective correspondence between right inner divisors $K_\alpha$ of $K$ and non-negative definite solutions $X_\alpha$ of the control homogeneous Riccati equation (CHRE)*

$$A^*X_\alpha + X_\alpha A + X_\alpha BB^*X_\alpha = 0 \qquad (76)$$

*given by the relation*

$$K_\alpha = \left(\begin{array}{c|c} A & B \\ \hline -B^*X_\alpha & I \end{array}\right) \qquad (77)$$

*Similarly, there exists a bijective correspondence between left inner divisors $K_\beta$ of $K$ and solutions $Y_\beta$ to the filtering homogeneous Riccati equation (FHRE)*

$$AY_\beta + Y_\beta A^* + Y_\beta P^{-1}BB^*P^{-1}Y_\beta = 0 \qquad (78)$$

*given by the relation*

$$K_\beta = \left(\begin{array}{c|c} A & Y_\beta B \\ \hline -B^*P^{-1} & I \end{array}\right) \qquad (79)$$

We now use the above result to extend Lemma 3 to projection onto a subspace of a coinvariant subspace.

**Lemma 11:** *Let $K = K_\beta K_\alpha$ be inner, with*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right), \quad K_\alpha = \left(\begin{array}{c|c} A_\alpha & B_\alpha \\ \hline -B_\alpha^*P_\beta^{-1} & I \end{array}\right)$$

*and*

$$K_\beta = \left(\begin{array}{c|c} A_\beta & B_\beta \\ \hline -B_\beta^*P_\beta^{-1} & I \end{array}\right)$$

(1) *Let*

$$Q' = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\mathcal{B}^*\mathcal{P}^{-1} & I \end{array}\right)$$

*Suppose $P_{Q'K}$ is the solution to*

$$\mathcal{A}P_{Q'K} + P_{Q'K}A^* + \mathcal{B}B^* = 0$$

*Then*

$$\mathbf{P}_{H_r(K_\alpha)}H_r(Q') = P_{Q'K}X_\alpha(sI - A)^{-1}B$$

*where $X_\alpha$ is the solution to the homogeneous Riccati equation*

$$A^*X_\alpha + X_\alpha A + X_\alpha BB^*X_\alpha = 0$$

*corresponding to the left factor $K_\alpha$ as in (77). Similarly*

$$\mathbf{P}_{H_r(K_\beta)K_\alpha}H(Q') = P_{Q'K}X_\beta(sI - A)^{-1}B$$

*where $X_\beta := P^{-1} - X_\alpha$.*

(2) *Similarly, let*

$$Q'' = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\mathcal{B}^*\mathcal{P}^{-1} & I \end{array}\right)$$

*be inner. Suppose $Q_{KQ''}$ is the solution to*

$$AQ_{KQ''} + Q_{KQ''}\mathcal{A}^* + P^{-1}B\mathcal{B}^*\mathcal{P}^{-1} = 0$$

*Then*

$$\mathbf{P}_{H_c(K_\alpha)}H(Q'') = P^{-1}B^*(sI - A)^{-1}Y_\beta Q_{KQ''}$$

*where $Y_\beta$ is the solution to the homogeneous Riccati equation*

$$AY_\beta + Y_\beta A^* + Y_\beta P^{-1}BB^*P^{-1}Y_\beta = 0$$

*corresponding to the left factor $K_\beta$ as in (79). Similarly*

$$\mathbf{P}_{H_c(K_\beta K_\alpha)}H(Q'') = P^{-1}B^*(sI - A)^{-1}Y_\alpha Q_{KQ''}$$

*where $Y_\alpha := P - Y_\beta$.*

*Moreover, it is*

$$X_\alpha P = P^{-1}Y_\alpha \qquad X_\beta P = P^{-1}Y_\beta \qquad (80)$$

**Proof:** Again we can assume that $K$ has realization the cascade of $K_\beta$ and $K_\alpha$, i.e.

$$K = \left[\begin{array}{c|c} A & B \\ \hline -B^*P^{-1} & I \end{array}\right]$$

with

$$A = \begin{pmatrix} A_\alpha & 0 \\ -B_\beta B_\alpha^* P_\alpha^{-1} & A_\beta \end{pmatrix} \quad B = \begin{bmatrix} B_\alpha \\ B_\beta \end{bmatrix} \quad P = \begin{bmatrix} P_\alpha & 0 \\ 0 & P_\beta \end{bmatrix}$$

Then, in view of Lemma 3,

$$\begin{aligned}
\mathbf{P}_{H(K)}\xi^*(sI-\mathcal{A})^{-1}\mathcal{B} &= \xi^* P_{Q'K}P^{-1}(sI-A)^{-1}B \\
&= \xi^* P_{Q'K}\begin{bmatrix} P_\alpha^{-1} & 0 \\ 0 & 0 \end{bmatrix}(sI-A)^{-1}B \\
&\quad + \xi^* P_{Q'K}\begin{bmatrix} 0 & 0 \\ 0 & P_\beta^{-1} \end{bmatrix}(sI-A)^{-1}B \\
&= \xi^* P_{Q'K_\alpha}P_\alpha^{-1}(sI-A_\alpha)^{-1}B_\alpha \\
&\quad + \xi^* P_{Q'K}\begin{bmatrix} 0 \\ P_\beta^{-1} \end{bmatrix}[-(sI-A_\beta)^{-1} \\
&\quad \times B_\beta B_\alpha^* P_\alpha^{-1}(sI-A_\alpha)^{-1}B_\alpha + (sI-A_\beta)^{-1}B_\beta] \\
&= \xi^* P_{Q'K_\alpha}P_\alpha^{-1}(sI-A_\alpha)^{-1}B_\alpha \\
&\quad + \xi^* P_{Q'K}\begin{bmatrix} 0 \\ P_\beta^{-1} \end{bmatrix}(sI-A_\beta)^{-1} \\
&\quad \times B_\beta[-B_\alpha^* P_\alpha^{-1}(sI-A_\alpha)^{-1}B_\alpha + I] \\
&= \mathbf{P}_{H_r(K_\alpha)}\xi^*(sI-\mathcal{A})^{-1}\mathcal{B} + \mathbf{P}_{H_r(K_\beta)K_\alpha}\xi^*(sI-\mathcal{A})^{-1}\mathcal{B}
\end{aligned}$$

where the first term in the last line derives again from Lemma 3 and the second term follows from orthogonality of the decomposition. So

$$\mathbf{P}_{H_r(K_\alpha)}\xi^*(sI-\mathcal{A})^{-1}\mathcal{B} = \xi^* P_{Q'K}\begin{bmatrix} P_\alpha^{-1} & 0 \\ 0 & 0 \end{bmatrix}(sI-A)^{-1}B$$

and

$$\mathbf{P}_{H_r(K_\beta)K_\alpha}\xi^*(sI-\mathcal{A})^{-1}\mathcal{B} = \xi^* P_{Q'K}\begin{bmatrix} 0 & 0 \\ 0 & P_\beta^{-1} \end{bmatrix}(sI-A)^{-1}B$$

Observe now that

$$\begin{bmatrix} P_\alpha^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is the solution to the homogeneous Riccati equation corresponding to $H_r(\bar{K})$. Thus the conclusion.

The dual statement is proved analogously. Equalities (80) follow from the fact that $X_\beta = P^{-1} - X_\alpha$, $Y_\alpha = P - Y_\beta$ and in the given basis

$$Y_\beta = \begin{bmatrix} 0 & 0 \\ 0 & P_\beta \end{bmatrix}$$

The proof is then by inspection. □

In the previous sections we often used the uniqueness of the solution to a Lyapunov equation associated with the controllable pair of a minimal realization

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

of $W$. But the extension to unstable $W$ may lead to a lack of this uniqueness if some of poles of $W$ and $W^*$ coincide. We will therefore assume that in the sequel that this never happens, i.e. $W$ has *unmixing poles*. Similarly we will assume that the matrix $A$ has *unmixing spectrum*, i.e. $\sigma(A)\cap\sigma(-A^*)=\emptyset$.

**Lemma 12:** *Let $(A,B)$ be a controllable pair and suppose moreover that $A$ has unmixing spectrum; let $X_+$ and $X_-$ be the maximal non-negative definite solutions to*

$$A^*X + XA + XBB^*X = 0$$

*and*

$$A^*X + XA - XBB^*X = 0$$

*Then, setting $\hat{A} := A - BB^*X$, the solution $\hat{P}$ to*

$$\hat{A}\hat{P} + \hat{P}\hat{A}^* + BB^* = 0$$

*is $\hat{P} = (X_+ + X_-)$.*

**Proof:** We can write

$$\begin{aligned}
&\hat{P}^{-1}\hat{A}^* + \hat{A}\hat{P}^{-1} + \hat{P}^{-1}BB^*\hat{P}^{-1} \\
&= (A^* - X_-BB^*)(X_+ + X_-) + (X_+ + X_-)(A - BB^*X_-) \\
&\quad + (X_+ + X_-)BB^*(X_+ + X_-) \\
&= A^*X_+ + A^*X_- - X_-BB^*X_+ - X_-BB^*X_- \\
&\quad + X_+A + X_-A - X_+BB^*X_- - X_-BB^*X_- \\
&\quad + X_+BB^*X_+ + X_+BB^*X_- + X_-BB^*X_+ + X_-BB^*X_- \\
&= A^*X_+ + X_+A + X_+BB^*X_+ \\
&\quad + A^*X_- + X_-A - X_-BB^*X_- \\
&= 0
\end{aligned}$$

as wanted. □

We now go back to unstable factors. Let $\underline{K}$, $\overline{K}$ be inner functions. Define the subspace $H_r(\overline{K}, \underline{K}^*)$ of $L^2$ by

$$H_r(\overline{K},\underline{K}^*) := H_r(\underline{K}^*) \oplus H_r(\bar{K}) \tag{81}$$

Then we have the following version of Lemma 2.

**Lemma 13:** *Let $\underline{K}$, $\overline{K}$ be $m\times m$ rational matrix inner functions and let*

$$\underline{K} = \left(\begin{array}{c|c} A_{\underline{K}} & B_{\underline{K}} \\ \hline -B^*_{\underline{K}}P_{\underline{K}}^{-1} & I \end{array}\right), \quad \overline{K} = \left(\begin{array}{c|c} A_{\overline{K}} & B_{\overline{K}} \\ \hline -B^*_{\overline{K}}P_{\overline{K}}^{-1} & I \end{array}\right) \tag{82}$$

*be minimal realizations of dimensions $n_{\underline{K}}$, $n_{\overline{K}}$, respectively, and such that $\sigma(A_K)\cap\sigma(-A_{\overline{K}}^*)=\emptyset$. Then, with $n = n_{\underline{K}} + n_{\overline{K}}$, we have that:*

(1) *a representation of $H_r(\bar{K}, \underline{K}^*)$ is given by*

$$H_r(\overline{K}, \underline{K}^*) = \left\{ \xi^* \left[ sI - \begin{pmatrix} -P_{\underline{K}} A^*_{\underline{K}} P^{-1}_{\underline{K}} & 0 \\ 0 & A_{\overline{K}} \end{pmatrix} \right]^{-1} \times \begin{pmatrix} B_{\underline{K}} \\ B_{\overline{K}} \end{pmatrix} | \xi \in \mathbb{C}^n \right\}. \qquad (83)$$

(2) *a representation of* $H_c(\overline{K}, \underline{K}^*)$ *is given by*

$$H_c(\overline{K}, \underline{K}^*) = \left\{ (B^*_{\underline{K}} P^{-1}_{\underline{K}} \quad - B^*_{\overline{K}} P^{-1}_{\overline{K}}) \times \left( sI - \begin{pmatrix} -P_{\underline{K}} A^*_{\underline{K}} P^{-1}_{\underline{K}} & 0 \\ 0 & A_{\overline{K}} \end{pmatrix} \right)^{-1} \xi | \xi \in \mathbb{C}^n \right\} \qquad (84)$$

**Proof:** Remembering that if

$$\underline{K} = \left( \begin{array}{c|c} A_{\underline{K}} & B_{\underline{K}} \\ \hline -B^*_{\underline{K}} P^{-1}_{K} & I \end{array} \right)$$

then

$$\underline{K}^* = \left( \begin{array}{c|c} -A^*_{\underline{K}} & P^{-1}_{\underline{K}} B_{\underline{K}} \\ \hline B^*_{\underline{K}} & I \end{array} \right)$$

the result for (1) immediately follows from Lemma 2 applied to $H_r(\bar{K})$ and $H_r(\overline{K}^*)$. The proof for (2) follows by duality. □

The space $H(\overline{K}, \underline{K})$ has a very simple representation.

**Lemma 14:** *Let* $(A, B)$ *be a controllable pair where* $A$ *has unmixing spectrum. Then there exist inner functions* $\overline{K}$ *and* $\underline{K}$ *such that*

$$\{\xi^*(sI - A)^{-1} B;\ \xi \in \mathbb{C}^n\} = H(\overline{K}, \underline{K})$$

**Proof:** We can clearly block diagonalise $A$ so that

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$$

with $A_+$ antistable and $A_-$ stable. If

$$B = \begin{pmatrix} B_+ \\ B_- \end{pmatrix}$$

is a conformal partition of $B$, then setting

$$\overline{K} = \left( \begin{array}{c|c} A_- & B_- \\ \hline -B^*{}_- P^{-1}_- & I \end{array} \right) \quad \text{and} \quad \overline{K} = \left( \begin{array}{c|c} A_- & B_- \\ \hline -B^*_- P^{-1}_- & I \end{array} \right)$$

we get the result.

We need now the following representation result, see Theorem 5.1 in Fuhrmann (1995).

**Lemma 15:** *Let* $(A, B)$ *be a controllable pair where* $A$ *has unmixing spectrum and let* $\overline{K}$, $\underline{K}$ *be defined as in Lemma 14; set* $\hat{K} := \overline{K}\underline{K}$. *Then there exists a realization*

$$\hat{K} = \left( \begin{array}{c|c} A_{\hat{K}} & B_{\hat{K}} \\ \hline -B^*_{\hat{K}} P^{-1}_{\hat{K}} & I \end{array} \right)$$

*such that*

$$A_{\hat{K}} = A - BB^* X_{\underline{K}} \qquad B_{\hat{K}} = B \qquad (85)$$

*where* $X_{\underline{K}}$ *is the maximal solution to*

$$A^*X + XA - XBB^*X = 0$$

*moreover*

$$(sI - A)^{-1} B = (sI - A_{\hat{K}})^{-1} B_{\hat{K}} \underline{K}^* \qquad (86)$$

**Proof:** Let

$$\underline{K} = \left( \begin{array}{c|c} A_{\underline{K}} & B_{\underline{K}} \\ \hline -B^*_{\underline{K}} P^{-1}_{\underline{K}} & I \end{array} \right) \quad \text{and} \quad \overline{K} = \left( \begin{array}{c|c} A_{\overline{K}} & B_{\overline{K}} \\ \hline -B^*_{\overline{K}} P^{-1}_{\overline{K}} & I \end{array} \right)$$

Thus we can always assume that, after a change of coordinates in $H_r(\overline{K}, \underline{K}^*)$, the realization is the cascade of the two inner functions, i.e.

$$A_{\hat{K}} = \begin{bmatrix} A_{\underline{K}} & 0 \\ -B_{\overline{K}} B^*_{\underline{K}} P^{-1}_{\underline{K}} & A_{\overline{K}} \end{bmatrix} \qquad B_{\hat{K}} = \begin{bmatrix} B_{\underline{K}} \\ B_{\overline{K}} \end{bmatrix}$$

Then, in view of the fact that $A_{\underline{K}} P_{\underline{K}} + P_{\underline{K}} A^*_{\underline{K}} + B_{\underline{K}} B^*_{\underline{K}} = 0$, it is easily seen that

$$\underline{K}^* = \left( \begin{array}{c|c} -A^*_{\underline{K}} & P^{-1}_{\underline{K}} B_{\underline{K}} \\ \hline B^*_{\underline{K}} & I \end{array} \right) = \left( \begin{array}{c|c} A_{\underline{K}} + B_{\underline{K}} B^*_{\underline{K}} P^{-1}_{\underline{K}} & B_{\underline{K}} \\ \hline B^*_{\underline{K}} P^{-1}_{\underline{K}} & I \end{array} \right)$$

Finally, the Riccati solution corresponding to $\underline{K}$ is easily seen to be

$$X_{\hat{K}} = \begin{bmatrix} P^{-1}_{\underline{K}} & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore

$$BB^*X = \begin{bmatrix} B_{\underline{K}} B^*_{\underline{K}} P^{-1}_{\underline{K}} & 0 \\ B_{\overline{K}} B^*_{\underline{K}} P^{-1}_{\underline{K}} & 0 \end{bmatrix}$$

and this shows (85). To see (86) we can write the following chain of equalities for the basis of $H_r(\overline{K}, \underline{K}^*)$

$$\left(\begin{array}{c|c} A_{\hat K} & B_{\hat K}\\ \hline I & 0\end{array}\right)\left(\begin{array}{c|c} A + B_{\underline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & B_{\underline{K}} \\ \hline -B^*_{\underline{K}}P^{-1}_{\underline{K}} & I\end{array}\right)$$

$$= \left(\begin{array}{ccc|c} A_{\underline{K}} + B_{\underline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & 0 & 0 & B_{\underline{K}} \\ B_{\underline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & A_{\underline{K}} & 0 & B_{\underline{K}} \\ B_{\overline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & -B_{\overline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & A_{\overline{K}} & B_{\overline{K}} \\ \hline 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} A_{\underline{K}} + B_{\underline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & 0 & 0 & B_{\underline{K}} \\ 0 & A_{\underline{K}} & 0 & 0 \\ 0 & -B_{\overline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & A_{\overline{K}} & B_{\overline{K}} \\ \hline I & I & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} A_{\underline{K}} + B_{\underline{K}}B^*_{\underline{K}}P^{-1}_{\underline{K}} & 0 & B_{\underline{K}} \\ 0 & A_{\overline{K}} & B_{\overline{K}} \\ \hline I & 0 & 0 \\ 0 & I & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -P_{\underline{K}}A^*_{\underline{K}}P^{-1}_{\underline{K}} & 0 & B_{\underline{K}} \\ 0 & A_{\overline{K}} & B_{\overline{K}} \\ \hline I & 0 & 0 \\ 0 & I & 0 \end{array}\right)$$

$$= \begin{bmatrix} P_{\underline{K}} & 0 \\ 0 & I \end{bmatrix}\left(sI - \begin{bmatrix} -A^*_{\underline{K}} & 0 \\ 0 & A_{\overline{K}} \end{bmatrix}\right)^{-1}\begin{bmatrix} P^{-1}_{\underline{K}}B_{\underline{K}} \\ B_{\overline{K}} \end{bmatrix}$$

$$= (sI - A)^{-1}B$$

which is a basis for $H_r(\overline{K}, \underline{K}^*)$, as wanted. The dual statement is proved analogously. □

In the case of stable $W$ we had defined inner functions $Q'$, $Q''$, $\bar Q'$, $\bar Q''$ to characterize the extremal $W$. Since now $W$ is no longer stable, we will need to introduce, as we did for $K$, the inner functions $Q'$ and $Q''$.

**Lemma 16:** *Let*

$$\hat K = \overline{K}\underline{K} = \left(\begin{array}{c|c} A_{\hat K} & B_{\hat K} \\ \hline -B^*_{\hat K}P^{-1}_{\hat K} & I \end{array}\right)$$

*be an inner function where $A_{\hat K}$ has unmixing spectrum.*

- *Let $\underline{Q}'$ be inner and suppose it is right coprime with $\underline{K}$. Denote the skew-prime factors by $Q'$ and $\underline{K}_-$, i.e.*

$$Q'\underline{K} = \underline{K}_- Q'$$

*Then, if $X_{\underline{K}}$ is the solution to (76) there exists a realization*

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}^*_{Q'}\mathcal{P}^{-1}_{Q'} & I \end{array}\right)$$

*with*

$$\mathcal{A}_{Q'} = \mathcal{A}_{\underline{Q}'} \qquad \mathcal{B}_Q = \mathcal{B}_{\underline{Q}'} - P_{\underline{Q}'\hat K}X_{\underline{K}}B_{\hat K}$$

*where $P_{\underline{Q}'\hat K}$ is the solution to*

$$A_{\hat K}P_{\hat K\underline{Q}'} + P_{\hat K\underline{Q}'}\mathcal{A}^*_{\underline{Q}'} + B\mathcal{B}^*_{\underline{Q}'} = 0$$

- *Let $\bar Q''$ be inner and suppose it is left coprime with $\bar K$. Denote the skew-prime factors by $Q''$ and $\bar K_+$, i.e.*

$$\overline{K}Q'' = \overline{Q}''\overline{K}_+$$

*Then, if $X_{\overline{K}} = P^{-1}_{\hat K} - X_{\underline{K}}$, here exists a realization*

$$Q'' = \left(\begin{array}{c|c} \mathcal{A}_{Q''} & -\mathcal{Q}^{-1}_{Q''}\mathcal{C}_{Q''} \\ \hline \mathcal{C}_{Q''} & I \end{array}\right)$$

*with*

$$\mathcal{A}_{Q''} = \mathcal{A}_{\bar Q''} \qquad \mathcal{C}_{Q''} = \mathcal{C}_{\overline{Q}''} - B^*_{\hat K}X_{\overline{K}}P_{\hat K}Q_{\hat K\overline{Q}''}$$

*where $Q_{\hat K Q''}$ is the solution to*

$$A^*_{\hat K}Q_{\hat K\overline{Q}''} + Q_{\hat K\overline{Q}''}\mathcal{A}_{\overline{Q}''} + P^{-1}_{\hat K}B\mathcal{C}^*_{\overline{Q}''} = 0$$

**Proof:** Since $Q'$, in view of Lemma 3 applied to $\underline{Q'}$ and $\underline{K}$, is given by

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline x & x \end{array}\right)$$

where $\mathcal{A}_{Q'} = \mathcal{A}_{\underline{Q}'}$ and $\mathcal{B}_{Q'} = \mathcal{B}_{\underline{Q}'} - P_{\underline{Q}'\underline{K}}P^{-1}_{\underline{K}}B_{\underline{K}}$ and

$$\mathbf{P}_{H(\underline{K})}\xi^*(sI - \mathcal{A}_{\underline{Q}'})^{-1}\mathcal{B}_{\underline{Q}'} = \xi^* P_{\underline{Q}'\underline{K}}P^{-1}_{\underline{K}}(sI - A_{\underline{K}})^{-1}B_{\underline{K}}$$

But it is also, in view of Lemma 11

$$\mathbf{P}_{H_r(\underline{K})}\xi^*(sI - \mathcal{A}_{\underline{Q}'})^{-1}\mathcal{B}_{\underline{Q}'} = \xi^* P_{\underline{Q}'\hat K}X_{\underline{K}}(sI - A_{\hat K})^{-1}B_{\hat K}$$

Multiplying both expressions on the right by $s^n$ and taking the limit at infinity, we obtain

$$P_{\underline{Q}'\underline{K}}P^{-1}_{\underline{K}}B_{\underline{K}} = P_{\underline{Q}'\hat K}X_{\underline{K}}B_{\hat K}$$

which achieves the proof.

For the dual statement, let $Y_{\hat K}$ be the solution to (78). Then, in the same manner as before, using the dual statements in Lemmas 3 and 11 we get

$$\mathcal{C}_{Q''} = \mathcal{C}_{\overline{Q}''} - BP^{-1}_{\hat K}Y_{\hat K}Q_{\hat K\overline{Q}''}$$

Applying (80) we get the result. □

**Lemma 17:** *Let $(A, B)$ be a controllable pair such that $A$ has unmixing spectrum and let $\bar{K}$ and $\underline{K}$ be the inner such that $H_r(\overline{K}, \underline{K}) = \operatorname{span}\{(sI - A)^{-1}B\}$; let then $\hat{K} := \overline{K}\underline{K}$. Then*

(1) *$\mathcal{V}$ is an antistabilizable, controlled invariant subspace if and only if there exists*

$$\underline{Q}' = \left(\begin{array}{c|c} \mathcal{A}_{\underline{Q}'} & \mathcal{B}_{\underline{Q}'} \\ \hline -\mathcal{B}^*_{\underline{Q}'}\mathcal{P}^{-1}_{\underline{Q}'} & I \end{array}\right)$$

*such that*

$$\mathcal{V}_+ = I^{-1}_{A,B}\mathbf{P}_{H_r(\overline{K},\underline{K}^*)}H(\underline{Q}')\underline{K}^* \tag{87}$$

*If we set*

$$\mathcal{A}_{Q'} := \mathcal{A}_{\underline{Q}'} \qquad \mathcal{B}_{Q'} := \mathcal{B}_{\underline{Q}'} - P_{\underline{Q}\hat{K}}X_{\underline{K}}B_{\underline{K}}$$

*then we have $\mathcal{V}_+ = \operatorname{Im} P_{\hat{K}Q'}$ where $P_{\hat{K}Q'}$ is the solution to*

$$AP_{\hat{K}Q} + P_{\hat{K}Q}\mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} = 0 \tag{88}$$

(2) *$\mathcal{V}_-$ is a stabilizable, controlled invariant subspace if and only if there exists an inner function*

$$\bar{Q}'' = \left(\begin{array}{c|c} \mathcal{A}_{\bar{Q}''} & -\mathcal{Q}^{-1}_{\overline{Q}''}\mathcal{C}^*_{\overline{Q}''} \\ \hline \mathcal{C}_{\overline{Q}''} & I \end{array}\right)$$

*such that*

$$\mathcal{V}_- = I^{-1}_{A,B}\mathbf{P}_{H_r(\overline{K},\underline{K}^*)}H_r(\overline{Q}''^*)\overline{K} \tag{89}$$

*If we set*

$$\mathcal{A}_{Q''} := \mathcal{A}_{\overline{Q}''} \qquad \mathcal{C}_{Q''} := \mathcal{C}_{\overline{Q}''} - B^*_{\hat{K}}X_{\overline{K}}P_{\hat{K}}Q_{\hat{K}\overline{Q}''}$$

*then we have $\mathcal{V}_- = \operatorname{Im} Q_{\hat{K}Q''}$ where $Q_{\hat{K}Q''}$ is the solution to*

$$-AQ_{\hat{K}Q''} + Q_{\hat{K}Q''}\mathcal{A}_{Q''} + P^{-1}_{\hat{K}}B\mathcal{C}^*_{Q''} = 0 \tag{90}$$

**Proof:** We have

$$\begin{aligned}
&\mathbf{P}_{H_r(\overline{K},\underline{K}^*)}(sI - \mathcal{A}_{\underline{Q}''})^{-1}\mathcal{B}_{\underline{Q}''}\underline{K}^* \\
&\quad = [\mathbf{P}_{H_r(\overline{K},\underline{K}^*)\underline{K}}(sI - \mathcal{A}_{\underline{Q}''})^{-1}\mathcal{B}_{\underline{Q}''}]\underline{K}^* \\
&\quad = [\mathbf{P}_{H_r(\hat{K})}(sI - \mathcal{A}_{\underline{Q}''})^{-1}\mathcal{B}_{\underline{Q}''}]\underline{K}^* \\
&\quad = P_{\underline{Q}'\hat{K}}P^{-1}_{\hat{K}}(sI - A_{\hat{K}})^{-1}B_{\hat{K}}\underline{K}^*
\end{aligned}$$

So, in view of Theorem 1, $\mathcal{V}_+ = \operatorname{Im} P_{\hat{K}\underline{Q}'}$ is an antistabilizable, controlled invariant subspace for $A_{\hat{K}}$, $B_{\hat{K}}$ where $P_{\hat{K}\underline{Q}'}$ satisfies the equality

$$A_{\hat{K}}P_{\hat{K}\underline{Q}'} + P_{\hat{K}\underline{Q}'}\mathcal{A}^*_{\underline{Q}'} + B_{\hat{K}}\mathcal{B}^*_{\underline{Q}'} = 0$$

Let $X_{\underline{K}}$ be the solution to HRE corresponding to $\underline{K}$. Then, in view of Lemma 15, we can choose $A_{\hat{K}}$, $B_{\hat{K}}$ such that $A_{\hat{K}} = A - BB^*X_{\underline{K}}$ and $B_{\hat{K}} = B$; thus we obtain

$$AP_{\hat{K}\underline{Q}'} + P_{\hat{K}\underline{Q}'}A^*_{\underline{Q}'} + B(\mathcal{B}^*_{\underline{Q}'} - B^*X_{\underline{K}}P_{\underline{Q}'\hat{K}}) = 0$$

that is $\mathcal{V}_+$ is controlled invariant for $A$, $B$ and $P_{\hat{K}\underline{Q}'} = P_{\hat{K}Q'}$. Substituting the term in parenthesis with $\mathcal{B}_{\underline{Q}'}$ yields (88).

Conversely, if $\mathcal{V}_+$ is antistabilizable controlled invariant, then letting $X_{\underline{K}}$ be the solution to the HRE corresponding to $\underline{K}$ we can set $A_{\hat{K}} = A - BB^*X_{\underline{K}}$ and $B_{\hat{K}} = B$ and $\mathcal{V}_+$ is antistabilizable, controlled invariant for $(A_{\hat{K}}$, $B_{\hat{K}})$ and we can use Theorem 1 again to get the conclusion.

For the dual statement, we can write again

$$\begin{aligned}
&\mathbf{P}_{H_r(\overline{K},\underline{K}^*)}(sI - \mathcal{A}_{\overline{Q}''})^{-1}\mathcal{B}_{\overline{Q}''}\overline{Q}''^*)\overline{K} \\
&\quad = [\mathbf{P}_{H_r(\overline{K},\underline{K}^*)\underline{K}}(sI - \mathcal{A}_{\overline{Q}''})^{-1}\mathcal{B}_{\overline{Q}''}\overline{Q}''^*)\overline{K}\underline{K}]\underline{K}^* \\
&\quad = [\mathbf{P}_{H_r(\hat{K})}(sI - \mathcal{A}_{\overline{Q}''})^{-1}\mathcal{B}_{\overline{Q}''}\overline{Q}''^*)\hat{K}]\underline{K}^* \\
&\quad = [\mathcal{P}_{\overline{Q}''}Q_{\hat{K}\overline{Q}''}(sI - A_{\hat{K}})^{-1}B]\underline{K}^*
\end{aligned}$$

So, in view of Theorem 1, $\mathcal{V}_- = \operatorname{Im} P_{\hat{K}}Q_{\hat{K}\bar{Q}''}$ is a stabilizable, controlled invariant subspace for $A_{\hat{K}}$, $B_{\hat{K}}$ where $Q_{\hat{K}\bar{Q}''}$ satisfies the equality

$$A^*_{\hat{K}}Q_{\hat{K}\overline{Q}''} + Q_{\hat{K}\overline{Q}''}\mathcal{A}_{\overline{Q}''} + P^{-1}_{\hat{K}}B_{\hat{K}}\mathcal{C}_{\overline{Q}''} = 0 \tag{91}$$

In view of Lemma 15, $B_{\hat{K}} = B$ and

$$\begin{aligned}
A_{\hat{K}} &= -P^{-1}_{\hat{K}}A^*_{\hat{K}}P^{-1}_{\hat{K}} - BB^*P^{-1}_{\hat{K}} \\
&= -P^{-1}_{\hat{K}}(A^* - X_{\underline{K}}BB^* + P^{-1}_{\hat{K}}BB^*)P^{-1}_{\hat{K}} \\
&= -P_{\hat{K}}(A^* + X_{\overline{K}}BB^*)P^{-1}_{\hat{K}}
\end{aligned}$$

and thus $A^*_{\hat{K}} = -P^{-1}_{\hat{K}}(A + BB^*X_{\hat{K}})P_{\hat{K}}$ so that (91) becomes

$$\begin{aligned}
P^{-1}_{\hat{K}}[-AP_{\hat{K}}Q_{\hat{K}\overline{Q}''} &+ P_{\hat{K}}Q_{\hat{K}\overline{Q}''}\mathcal{A}_{\overline{Q}''} \\
&+ B(\mathcal{C}_{\overline{Q}''} - B^*X_{\overline{K}}P_{\hat{K}}Q_{\hat{K}\overline{Q}''})] = 0
\end{aligned}$$

that is $\mathcal{V}_-$ is controlled invariant for $A$, $B$ and $Q_{\hat{K}\bar{Q}''} = Q_{\hat{K}Q''}$. Substituting the term in parenthesis with $\mathcal{C}_{Q''}$ yields (90).

Conversely, if $\mathcal{V}_-$ is antistabilizable controlled invariant, then letting $X_K$ be the solution to the HRE corresponding to $\bar{K}$, we can set $A_{\hat{K}} = A - BB^*X_{\underline{K}}$ and $B_{\hat{K}} = B$ so that $\mathcal{V}_-$ is stabilizable, controlled invariant for $(A_{\hat{K}}, B_{\hat{K}})$ and we can use Theorem 1 again to get the conclusion. □

Let now a proper rational function $W$ with no zeros are the imaginary axis be given; let $\underline{W} = W\underline{K}$ be its

Douglas–Shapiro–Shields factorization with $\underline{W} \in H^\infty_+$ and let $Q'$ be an inner function skew-prime with $\underline{K}$. We can then consider the skew-prime factorization

$$Q'\underline{K} = \underline{K}_-\underline{Q}' \tag{92}$$

Similarly, let $\overline{W} = W\underline{K}^*$ be its Douglas–Shapiro–Shields factorization with $\overline{W} \in H^\infty_-$ and let $\bar{Q}''$ be an inner function skew-prime with $\bar{K}$. We can then consider the skew-prime factorization

$$\overline{Q}''\overline{K}_+ = \overline{K}Q'' \tag{93}$$

**Definition 3:** We say that the inner function $Q'$ divides $W$ over $H^\infty_+$ if $W(Q')^* \in H^\infty_+\underline{K}^*_-$. If $Q'$ is maximal, we set $W_- := WQ'^*$. Similarly, we say that $Q''$ divides $W$ over $H^\infty_-$ if $WQ'' \in H^\infty_-\overline{K}_+$. If $Q''$ is maximal, we set $W_+ := WQ''$.

Figure 4 is an extension of figure 2.

We can then state the following theorem.

**Theorem 7:** *Let*

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

*be a transfer function with no zeros on the imaginary axis and unmixing poles and let $\underline{W} = W\underline{K}$ be its DSS factorization over $H^\infty_+$ and let $X_{\underline{K}}$ denote the solution to the HRE corresponding to $\underline{K}$. The following statements are equivalent:*

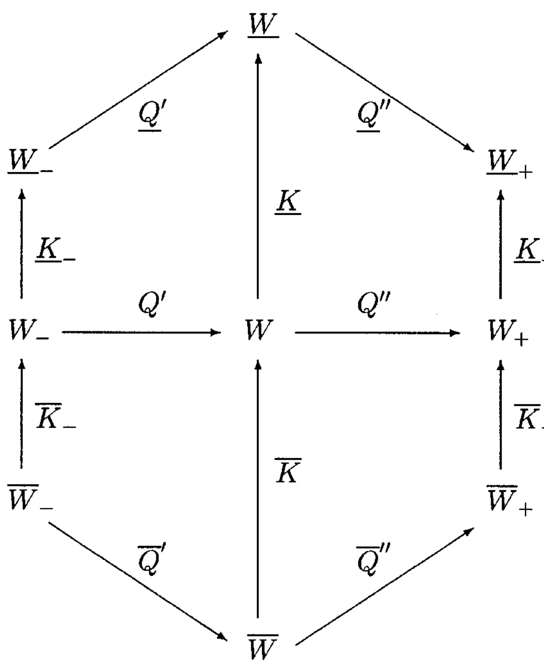


Figure 4.

- *$Q'$ and $\underline{K}$ are skew-prime and*

$$Q' = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline -\mathcal{B}^*_{Q'}\mathcal{P}^{-1}_{Q'} & I \end{array}\right)$$

*divides $W$ over $H^\infty_+$.*

- *$\mathcal{V} = \mathrm{Im}\ P_{\hat{K}Q'}$, where $P_{\hat{K}Q'}$ satisfies*

$$AP_{\hat{K}Q'} + P_{\hat{K}\hat{Q}'}\mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} = 0$$

*and $\mathcal{V} \in \mathrm{Ker}\ (C + D\mathcal{B}^*_{Q'}P^{\#}_{\hat{K}Q'})$ i.e. $\mathcal{V}$ is an antistabilizable, output nulling subspace.*

*Similarly, let $\bar{W} = W\bar{K}^*$ be the DSS factorization of $W$ over $H^\infty_-$ and let*

$$\hat{K} := \overline{K}\underline{K} = \left(\begin{array}{c|c} A_{\hat{K}} & B_{\hat{K}} \\ \hline -B^*_{\hat{K}}P^{-1}_{\hat{K}} & I \end{array}\right)$$

*and $X_{\bar{K}} := P^{-1}_{\hat{K}} - X_{\underline{K}}$. The following statements are equivalent:*

- *$Q''$ and $\bar{K}$ are skew-prime and*

$$Q'' = \left(\begin{array}{c|c} \mathcal{A}_{Q''} & -\mathcal{Q}^{-1}_{Q''}\mathcal{C}^*_{Q''} \\ \hline \mathcal{C}_{Q''} & I \end{array}\right)$$

*divides $W$ over $H^\infty_-$.*

- *$\mathcal{V} = \mathrm{Im}\ P_{\hat{K}}Q_{\hat{K}Q''}$, where $Q_{\hat{K}Q''}$ satisfies*

$$-AQ_{\hat{K}Q''} + Q_{\hat{K}Q''}\mathcal{A}_{Q''} + P^{-1}_{\hat{K}}B\mathcal{B}^*_{Q''}\mathcal{P}^{-1}_{Q''} = 0$$

*and $\mathcal{V} \in \mathrm{Ker}\ (C - D\mathcal{C}_{Q''}Q^{\#}_{\underline{KQ}''}P^{-1}_{\hat{K}})$, i.e. $\mathcal{V}$ is a stabilizable, output nulling subspace.*

**Proof:** As usual we can choose $\hat{A} := A - BB^*X_{\underline{K}}$ and $B_{\hat{K}} = B$; then we claim that

$$\underline{W} = \left(\begin{array}{c|c} \hat{A} & B \\ \hline C - DB^*X_{\underline{K}} & D \end{array}\right)$$

In fact

$$\underline{W} = W\underline{K} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} \hat{A} & B \\ \hline -B^*X_{\underline{K}} & I \end{array}\right) \tag{94}$$

$$= \left(\begin{array}{cc|c} \hat{A} & 0 & B \\ -BB^*X_{\underline{K}} & A & B \\ \hline -DB^*X_{\underline{K}} & C & D \end{array}\right)$$

$$= \left( \begin{array}{cc|c} \hat{A} & 0 & B \\ 0 & A & 0 \\ \hline C - DB^*X_{\overline{K}} & C & D \end{array} \right)$$ we have used the change of basis $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$

$$= \left( \begin{array}{c|c} \hat{A} & B \\ \hline C - DB^*X_{\underline{K}} & D \end{array} \right)$$

Now by definition, $Q$ divides $W$ if and only if $\underline{W}\underline{Q}^* \in H_+^\infty$ where

$$\underline{Q} = \left( \begin{array}{c|c} \mathcal{A}_{\underline{Q}} & \mathcal{B}_{\underline{Q}} \\ \hline -\mathcal{B}^*_{\underline{Q}}\mathcal{P}^{-1}_{\underline{Q}} & I \end{array} \right)$$

satisfies relation (92). But this is equivalent, in view of Lemma 4, to the fact that

$$\mathcal{V} \in \text{Ker } (C - DB^*X_{\underline{K}} + D\mathcal{B}^*_{\underline{Q}}P^{\#}_{\hat{K}\underline{Q}}) \tag{95}$$

where $P_{\hat{K}\underline{Q}}$ satisfies

$$(A - BB^*X_{\underline{K}})P_{\hat{K}\underline{Q}} + \mathcal{A}^*_{\underline{Q}}P_{\hat{K}\underline{Q}} + B\mathcal{B}^*_{\underline{Q}} = 0 \tag{96}$$

But, in view of Lemma 16, it is $\mathcal{A}_{Q'} = \mathcal{A}_{\underline{Q'}}$ and $\mathcal{B}_{Q'} = \mathcal{B}_{\underline{Q'}} - P_{\underline{Q}\hat{K}}X_{\underline{K}}B$. So, (96) can be written as

$$AP_{\hat{K}\underline{Q}} + \mathcal{A}^*_{Q'}P_{\hat{K}\underline{Q}} + B\mathcal{B}^*_{Q'} = 0$$

which entails $P_{\hat{K}Q'} = P_{\hat{K}\underline{Q'}}$; also substitution of $\mathcal{B}_{Q'}$ in (95) gives that $\mathcal{V} \subset \text{Ker } (\overline{C} + DB^*X_{\underline{K}}P^{\#}_{\hat{K}\underline{Q}})$ and yields the conclusion.

For the second statement, observe that it is

$$\overline{WQ}'' = W\overline{K}^*\overline{Q}'' = WQ''\overline{K}^*_+$$

and also $\overline{W} = \underline{W}\hat{K}^*$. So, $Q''$ divides $W$ over $H_-^\infty$ if and only if $\overline{WQ}'' \in H_-^\infty$. But, reasoning as above

$$\overline{W} = \underline{W}\hat{K} = \left( \begin{array}{c|c} A_{\hat{K}} & B \\ \hline C - DB^*X_{\underline{K}} & D \end{array} \right)$$

$$\hat{K} = \left( \begin{array}{c|c} -A^*_{\hat{K}} & P^{-1}_{\hat{K}}B \\ \hline DB^* + (C - DB^*X_{\underline{K}})P_{\hat{K}} & D \end{array} \right)$$

Thus, again in view of Lemma 4, we have that

$$\begin{aligned} \mathcal{V}_- &\subset \text{Ker}\,(C - DB^*X_{\underline{K}} - D(\mathcal{C}_{\overline{Q}''}Q^{\#}_{\overline{KQ}''} - B^*)P^{-1}_{\hat{K}}) \\ &= \text{Ker}\,(C - D(\mathcal{C}_{\overline{Q}''}Q^{\#}_{\overline{KQ}''} + B^*X_{\underline{K}}P_{\hat{K}} - B^*)P^{-1}_{\hat{K}} \\ &= \text{Ker}\,(C - D(\mathcal{C}_{\overline{Q}''}Q^{\#}_{\overline{KQ}''} - B^*X_{\overline{K}}P_{\hat{K}})P^{-1}_{\hat{K}}) \end{aligned}$$

But if $\xi \in \mathcal{V}_-$, then $\xi = P_{\hat{K}}Q_{\hat{K}\bar{Q}''}Q^{\#}_{\overline{KQ}''}P^{-1}_{\hat{K}}\xi$ and so

$$\begin{aligned} 0 &= (C - D(\mathcal{C}_{\overline{Q}''}Q^{\#}_{\overline{KQ}''} - B^*X_{\overline{K}}P_{\hat{K}})P^{-1}_{\hat{K}})\xi \\ &= C\xi - D(\mathcal{C}_{\overline{Q}''} - B^*X_{\overline{K}}P_{\hat{K}}Q_{\hat{K}\overline{Q}''})Q^{\#}_{\overline{KQ}''}P^{-1}_{\hat{K}}\xi \\ &= (C - D\mathcal{C}_{Q''}Q^{\#}_{\overline{KQ}''}P^{-1}_{\hat{K}})\xi \end{aligned}$$

□

We have therefore obtained the corresponding results of Lemma 4. We turn now our attention to the controllability subspace.

**Definition 4:** Let $\underline{K}$, $\overline{K}$, $\underline{Q}'$ and $\bar{Q}''$ be inner functions and $G_2$ a constant projection matrix on $\mathbb{C}^m$; set $\hat{K} := \overline{K}\underline{K}$

(1) We say that $\underline{Q}'$ **reduces** $H_r(\overline{K}, \underline{K}^*)$ **on the right (with respect to** $G_2$) if $\underline{Q}'$ reduces $\hat{K}$ on the right.

(2) Similarly, we say that $\bar{Q}''$ **reduces** $H_r(\overline{K}, \underline{K}^*)$ **on the left (with respect to** $G_2$) if $\bar{Q}''$ reduces $\hat{K}$ on the left.

The reduction is proper and minimal if the corresponding reduction for $\hat{K}$ is proper and minimal.

**Lemma 18:** *Let*

$$\underline{Q}' = \left( \begin{array}{c|c} \mathcal{A}_{\underline{Q}'} & \mathcal{B}_{\underline{Q}'} \\ \hline -\mathcal{B}^*_{\underline{Q}'}\mathcal{P}^{-1}_{\underline{Q}'} & I \end{array} \right), \quad \overline{Q}'' = \left( \begin{array}{c|c} \mathcal{A}_{\overline{Q}''} & \mathcal{B}_{\overline{Q}''} \\ \hline -\mathcal{B}^*_{\overline{Q}''}\mathcal{P}^{-1}_{\overline{Q}''} & I \end{array} \right)$$

*be inner functions and let* $H_r(\overline{K}, \underline{K}^*) = \text{span}\ \{\xi^*(sI - A)^{-1}B\}$*;* $\xi \in \mathbb{C}^n$*. Then*

(1) $\underline{Q}'$ *reduces* $H_r(\overline{K}, \underline{K}^*)$ *on the right (with respect to* $G_2$*) if and only if*

$$BG_2 = P_{\hat{K}\underline{Q}'}\mathcal{P}^{-1}\mathcal{B}_{\underline{Q}'}G_2 \tag{97}$$

*where* $P_{\hat{K}\underline{Q}'}$ *satisfies*

$$AP_{\hat{K}\underline{Q}'} + P_{\hat{K}\underline{Q}'}\mathcal{A}^*_{\underline{Q}'} + B(\mathcal{B}^*_{\underline{Q}'} - B^*X_{\underline{K}}P_{\hat{K}\underline{Q}'}) = 0.$$

(2) $\bar{Q}''$ *reduces* $H_r(\overline{K}, \underline{K}^*)$ *on the left (with respect to* $G_2$*) if and only if*

$$G_2B^*P^{-1} = G_2\mathcal{B}_{\overline{Q}''^*}\overline{Q}^*_{\hat{K}\overline{Q}''} \tag{98}$$

*where* $Q_{\hat{K}\bar{Q}''}$ *satisfies*

$$\begin{aligned} A^*Q_{\hat{K}\bar{Q}''} + Q_{\hat{K}\bar{Q}''}\mathcal{A} + P^{-1}B(\mathcal{B}^*_{\bar{Q}''*}\overline{P}^1_{\bar{Q}''*} \\ - B^*X_{\overline{K}}P_{\hat{K}\bar{Q}''}) = 0. \end{aligned}$$

**Proof:** The proof is the same as in the stable case, i.e. Lemma 5. □

**Theorem 8:** *Let* $H(\overline{K}, \underline{K}^*) = \text{span}\ \{\xi^*(sI - A)^{-1}B;\ \xi \in \mathbb{C}^n\}$*, where* $A$ *has unmixing spectrum, and let* $Z \subset X$ *and a projection matrix* $G_2$ *be given. The following are equivalent:*

(1) *There exists a matrix* $F$ *such that* $Z = I_{A,B}\mathcal{R}$ *where* $\mathcal{R} = \langle A + BF | BG_2 \rangle$*, i.e.* $\mathcal{R}$ *is a controllability subspace.*

(2) $Z = \mathbf{P}_{H_r(\overline{K}, \underline{K}*)} H_r(\underline{Q}')\underline{K}^*$ *and* $\underline{Q}'$ *reduces* $H_r(\overline{K}, \underline{K}^*)$ *on the right with respect to* $G_2$ *and the reduction is minimal and proper.*

(3) $Z = \mathbf{P}_{H_r(\overline{K}, \underline{K}*)} H_r(\bar{Q}''^*)\overline{K}$ *and* $\bar{Q}''$ *reduces* $H_r(\overline{K}, \underline{K}^*)$ *on the left with respect to* $G_2$ *and the reduction is minimal and proper.*

**Proof:** It is the same as in Theorem 3 where the result of Theorem 7 is used instead of Lemma 4, since in that proof the stability of $A$ is not used at all. □

**Corollary 2:** *Let*

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

*with unmixing poles be given and set* $\underline{W} := W\underline{K}$ *to be its DSS factorization over* $H_+^\infty$*. Then* $\mathcal{R}$ *is an output nulling controllability subspace for* $W$ *if and only if it has the same property with respect to* $\underline{W}$*.*

**Proof:** Follows directly from Theorem 7. □

**Corollary 3:** *Let* $W$ *be a spectral factor with unmixing poles admitting the factorizations* $W = W_-Q' = W_+Q''^*$ *and the DSS factorizations* $W = \overline{WK}^* = \underline{WK}$*. Set* $\hat{K} := \overline{K}\underline{K}$*. Then the supremal antistabilizable output nulling subspace for* $W$ *is given by* $\mathcal{V}_+^* = \operatorname{Im} P_{\hat{K}Q'}$ *where* $P_{\hat{K}Q'}$ *satisfies the equation*

$$AP_{\hat{K}Q'} + P_{\hat{K}Q'}\mathcal{A}_{Q'}^* + B\mathcal{B}_{Q''}^* = 0$$

*The supremal stabilizable, output nulling subspace for* $W$ *is given by* $\mathcal{V}_-^* = \operatorname{Im} P_{\hat{K}}Q_{\hat{K}Q''}$*, where* $Q_{\hat{K}Q''}$ *satisfies the equation*

$$-AQ_{\hat{K}Q''} + Q_{\hat{K}Q''}\mathcal{A}_{Q''} + P_{\hat{K}}^{-1}B\mathcal{C}_{Q'} = 0$$

*The supremal controllability subspace for* $W$ *is given by*

$$\mathcal{R}^* = \mathcal{V}^*{}_+ \cap \mathcal{V}_-^*$$

## 6. Algorithm

In view of the previous results, we can present a new algorithm for computing the supremal (anti)stabilizable, output nulling subspace ($\mathcal{V}_+^*$), $\mathcal{V}_-^*$ as well as the supremal output nulling reachability subspace $\mathcal{R}^*$.

Before stating the result, we need to state simple modifications of two previous results. The first is due to Chen and Francis (1989), and characterizes the existence of a one sided $H_+^\infty$ inverse.

**Proposition 6:** *Assume the rational function* $W$ *has minimal realization*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

*Then the following statements are equivalent*

(1) $W$ *has a left inverse in* $H_+^\infty$*.*

(2) $D$ *is injective and for some* $H$ *we have*

(a) $B + HD = 0$.

(b) $A + HC$ *is stable.*

*If* $H$ *is such that* (2) *is satisfied then a* $H_+^\infty$ *left inverse* $W^\sharp$ *of* $W$ *is given by*

$$W^\sharp = \left(\begin{array}{c|c} A + HC & H \\ \hline D^\sharp C & D^\sharp \end{array}\right)$$

*where* $D^\# = (D^*D)^{-1}D^*$.

The second result is adapted from Fuhrmann and Gombani (1998). Given an arbitrary, not necessarily stable, rational transfer function $W$, we can compute, by state space methods, its dual Lindquist–Picci pair.

**Theorem 9:** *Let*

$$W = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D & 0 \end{array}\right)$$

*be a* $p \times m$*, rational function with no zeros on the imaginary axis and unmixing poles. Assume without loss of generality that* $D$ *has full column rank. Let* $Q_-$ *and* $Q_+$ *be the maximal inner divisors of* $W$ *over* $H_+^\infty$ *and* $H_-^\infty$*; set*

$$W_- := WQ'^* \qquad W_+ := WQ''$$

*Then*

(1) *A minimal realization of* $W_-$ *is given by*

$$W_- = \left(\begin{array}{c|c} A & B_1 + \mathcal{X}_-C^*D(D^*D)^{-1} \\ \hline C & D \end{array}\right) \tag{99}$$

*where* $\mathcal{X}_- \geq 0$ *is the stabilizing solution of the Riccati equation*

$$(A - B_1(D^*D)^{-1}D^*C)\mathcal{X} + \mathcal{X}(A^* - C^*D(D^*D)^{-1}B_1^*) + B_2B_2^* - \mathcal{X}C^*D(D^*D)^{-2}D^*C\mathcal{X} = 0 \tag{100}$$

*i.e. the solution for which*

$$A - B_1(D^*D)^{-1}D^*C - \mathcal{X}_-C^*D(D^*D)^{-2}D^*C$$

*is stable. The Riccati equation can be rewritten as*

$$(A + H_-C)\mathcal{X}_- + \mathcal{X}_-(A^* + C^*H_-^*) + B_2B_2^* + \mathcal{X}_-C^*D(D^*D)^{-2}D^*C\mathcal{X}_- = 0 \tag{101}$$

(2) *If $\mathcal{X}$ is any solution of the Riccati equation* (100), *then*

$$\operatorname{Ker} B_2^* = \operatorname{Ker} \mathcal{X} \tag{102}$$

*Equivalently*

$$\operatorname{Im} B_2 \subset \operatorname{Im} \mathcal{X} \tag{103}$$

*There exists a linear map $\hat{C}$ for which*

$$B_2 = -\mathcal{X}\hat{C}^* \tag{104}$$

(3) *A minimal McMillan degree inner function $Q'$ satisfying $W_- Q' = W$ is given by*

$$Q' = \left(\begin{array}{c|cc} A + H_- C & -\mathcal{X}_- C^* D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1} D^* C & I & 0 \\ \hat{C}_- & 0 & I \end{array}\right) \tag{105}$$

*where $\hat{C}_-$ satisfies* (104). *The McMillan degree of $Q'$ is equal to* rank $\mathcal{X}_-$.

(4) *A minimal realization of $W_+$ is given by*

$$W_+ = \left(\begin{array}{c|c} A & B_1 + \mathcal{X}_+ C^* D(D^*D)^{-1} \\ \hline C & D \end{array}\right) \tag{106}$$

*where $\mathcal{X}_+$ is the antistabilizing solution of the Riccati equation*

$$(A - B_1(D^*D)^{-1}D^*C)\mathcal{X}_+ + \mathcal{X}_+(A^* - C^*D(D^*D)^{-1}B_1^*) + B_2B_2^* - \mathcal{X}_+C^*D(D^*D)^{-2}D^*C\mathcal{X}_+ = 0 \tag{107}$$

*i.e. the solution for which*

$$A - B_1(D^*D)^{-1}D^*C - \mathcal{X}_+C^*D(D^*D)^{-2}D^*C$$

*is antistable. The Riccati equation can be re-written as*

$$(A + H_+C)\mathcal{X}_+ + \mathcal{X}_+(A^* + C^*H_+^*) + B_2B_2^* + \mathcal{X}_+C^*D(D^*D)^{-2}D^*C\mathcal{X}_+ = 0 \tag{108}$$

(5) *A minimal McMillan degree inner function $Q''$ satisfying $WQ'' = W_+$ is given by*

$$Q'' = \left(\begin{array}{c|cc} -A^* - C^*H_+^* & C^*D(D^*D)^{-1} & \hat{C}_+^* \\ \hline (D^*D)^{-1}D^*C\mathcal{X}_+ & I & 0 \\ -B_2^* & 0 & I \end{array}\right) \tag{109}$$

*where $\hat{C}_+$ satisfies* (104). *The McMillan degree of $Q''$ is equal to* rank $\mathcal{X}_+$.

**Theorem 10:** *Let $W$ be a not necessarily stable transfer function with no zeros on the imaginary axis and unmixing poles. Let*

$$W = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D & 0 \end{array}\right)$$

*be a minimal realization. Assume w.l.o.g. that $D$ has full column rank. The following algorithm computes $\mathcal{V}_+^*$, $\mathcal{V}_-^*$ and $\mathcal{R}^*$.*

**Algorithm:**

I. Compute the maximal non-negative definite solution $\mathcal{X}_-$ of the algebraic Riccati equation

$$(A - B_1(D^*D)^{-1}D^*C)\mathcal{X} + \mathcal{X}(A^* - C^*D(D^*D)^{-1}B_1^*) + B_2B_2^* - \mathcal{X}C^*D(D^*D)^{-2}D^*C\mathcal{X} = 0 \tag{110}$$

II. Compute the maximal non-positive definite solution $\mathcal{X}_+$ of the algegraic Riccati equation (110).

III. Set

(1)

$$W_- = \left(\begin{array}{c|c} A & B_1 + \mathcal{X}_- C^* D(D^*D)^{-1} \\ \hline C & D \end{array}\right)$$

(2)

$$W_+ = \left(\begin{array}{c|c} A & B_1 + \mathcal{X}_+ C^* D(D^*D)^{-1} \\ \hline C & D \end{array}\right)$$

(3)

$$Q' = \left(\begin{array}{c|cc} A + H_- C & -\mathcal{X}_- C^* D(D^*D)^{-1} & B_2 \\ \hline (D^*D)^{-1} D^* C & I & 0 \\ \hat{C}_- & 0 & I \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A}_{Q'} & \mathcal{B}_{Q'} \\ \hline \mathcal{C}_{Q'} & I \end{array}\right)$$

(4)

$$Q'' = \left(\begin{array}{c|cc} -A^* - C^*H_+^* & C^*D(D^*D)^{-1} & \hat{C}_+^* \\ \hline (D^*D)^{-1}D^*C\mathcal{X}_+ & I & 0 \\ -B_2^* & 0 & I \end{array}\right) = \left(\begin{array}{c|c} \mathcal{A}_{Q''} & \mathcal{B}_{Q''} \\ \hline \mathcal{C}_{Q''} & I \end{array}\right)$$

IV. Find the maximal solutions $X_+$ and $X_-$ to the homogeneous Riccati equations

$$A^*X + XA + X(B_1B_1^* + B_2B_2^*)X = 0$$

and

$$A^*X + XA - X(B_1B_1^* + B_2B_2^*)X = 0$$

Set $P_{\hat{K}} := X_+ + X_-$.

V. Solve the Sylvester equations

$$AP_{\hat{K}Q'} + P_{\hat{K}Q'}\mathcal{A}^*_{Q'} + B\mathcal{B}^*_{Q'} = 0 \tag{111}$$

and

$$-AQ_{\hat{K}Q''} + Q_{\hat{K}Q''}\mathcal{A}_{Q''} + P_{\hat{K}}^{-1}B^*\mathcal{C}_{Q''} = 0 \tag{112}$$

VI. Compute

$$\mathcal{V}^*_+ = \mathrm{Im}\ P_{\hat{K}Q'} \tag{113}$$

and

$$\mathcal{V}^*_- = \mathrm{Im}\ P_{\hat{K}}Q_{\hat{K}Q''} \tag{114}$$

and

$$\mathcal{R}^* = \mathcal{V}^*_+ \cap \mathcal{V}^*_- = \mathrm{Im}\, P_{\hat{K}Q'} \cap \mathrm{Im}\, P_{\hat{K}}Q_{\hat{K}Q''} \tag{115}$$

## 7. Conclusions

We have presented a new approach to geometric control, based on the geometry of Hardy spaces occurring in stochastic realization theory for *non-full rank*, *non-square* spectral factors, which concludes the investigation initiated in Fuhrmann and Gombani (1998, 2000). The approach followed here for the study of output nulling subspaces is substantially different from the one in Lindquist *et al.* (1995), because it considers right zeros and the controllability subspace of $W$ is, in general, non-trivial. This allows for a complete description of controlled invariant subspaces, and of the output nulling and controllability subspaces of a given stable transfer function $W$. A new algorithm for the computation of output nulling and controllability subspaces based on the outer–inner factorization of $W$ has been introduced. The results are extended to an arbitrary $W$ (with the quite mild constraint that it has no zeros on the imaginary axis and unmixing poles). Some of these results are being extended to the case of $J$-spectral factorization, with applications to dissipative systems and robust control (see Gombani and Weiland 2000).

## Acknowledgements

P. A. Fuhrmann is the Earl Katz Family Chair in Algebraic System Theory. This work was partially supported by GIF under Grant No. I 184 and by CNR.

## References

Anderson, B. D. O., 1973, Algebraic properties of minimal degree spectral factors. *Automatica*, **9**, 491–500.

Basile, G., and Marro, G., 1968, Luoghi caratteristici dello spazio degli stati relativi al controllo dei sistemi lineari. *L'Elettronica*, **55**, 1–7.

Basile, G., and Marro, G., 1969, Controlled and conditioned invariant subspaces in linear system theory. *Journal of Optimization Theory and Applications*, **3**, 306–316.

Caines, P. E., and Delchamps, D., 1980, Splitting subspaces, spectral factorization and the positive real equation: structural features of the stochastic realization problem. *Proceedings of the IEEE Conference on Decision and Control*, Albuquerque, New Mexico, pp. 358–362.

Callier, F. M., 1985, On polynomial matrix spectral factorization by symmetric root extraction. *IEEE Transactions on Automatic Control*, **30**, 453–464.

Chen, T., and Francis, B. A., 1989, Spectral and inner–outer factorizations of rational matrices. *SIAM Journal on Matrix Analysis and Applications*, **10**(1), 1–17.

Douglas, R. G., Shapiro, H. S., and Shields, A. L., 1971, Cyclic vectors and invariant subspaces for the backward shift. *Ann. Inst. Fourier, Grenoble*, **20**, 37–76.

Faurre, P., Clerget, M., and Germain, F., 1979, *Operateurs Rationnels Positifs* (Paris: Dunod).

Ferrante, A., Michaletzky, G., and Pavon, M., 1993, Parametrization of all minimal square spectral factors. *Systems and Control Letters*, **21**, 249–254.

Finesso, L., and Picci, G., 1982, A characterization of minimal spectral factors. *IEEE Transactions on Automatic Control*, **27**, 122–127.

Fuhrmann, P. A., 1975, On Hankel operator ranges, meromorphic pseudo-continuation and factorization of operator values analytic functions. *Journal of the London Mathematical Society*, **13**, 323–327.

Fuhrmann, P. A., 1976, Algebraic system theory: An analyst's point of view. *Journal of the Franklin Institute*, **301**, 521–540.

Fuhrmann, P. A., 1981 a, Duality in polynomial models with some applications to geometric control theory. *IEEE Transactions on Automatic Control*, **26**, 284–295.

Fuhrmann, P. A., 1981 b, *Linear Systems and Operators in Hilbert Space* (New York: McGraw-Hill).

Fuhrmann, P. A., 1985, The algebraic Riccati equation – a polynomial approach. *Systems and Control Letters*, 369–376.

Fuhrmann, P. A., 1989, Elements of factorization theory from a polynomial point of view. In H. Nijmeijer and J. M. S. Schumacher (eds), *Three Decades of Mathematical System Theory. A Collection of Surveys at the Occasion of the Fiftieth Birthday of Jan C. Willems*. Lecture Notes in Control and Information Sciences, Vol. 135 (Berlin: Springer Verlag).

Fuhrmann, P. A., 1995, On the characterization and parametrization of minimal spectral factors. *Journal of Mathematical Systems, Estimation and Control*, No. 4, 383–444.

Fuhrmann, P. A., and Gombani, A., 1998, On a Hardy space approach to the analysis of spectral factors. *International Journal of Control*, **71**, 277–357.

Fuhrmann, P. A., and Gombani, A., 2000, Coinvariant subspaces, the Lyapunov equation and partial ordering of inner functions. *International Journal of Control*, **73**, 1129–1159.

Fuhrmann, P. A., and Ober, R., 1993, State space formulas for coprime factorizations. *T. Ando volume. Operator Theory Advances and Applications* (Basel: Birkhauser Verlag), pp. 39–75.

Fuhrmann, P. A., and Willems, J. C., 1979, Factorization indices at infinity for rational matrix functions. *Integral Equations and Operational Theory*, **2**, 287–301.

Fuhrmann, P. A., and Willems, J. C., 1980, A study of $(A, B)$-invariant subspaces via polynomial models. *International Journal of Control*, **31**, 467–494.

Genin, Y., Van Dooren, P., Kailath, T., Delosme, J. M., and Morf, M., 1983, On $\Sigma$-lossless transfer functions and related questions. *Liner Algebra Applications*, **50**, 251–275.

Gombani, A., and Weiland, S., 2000, On the parametrization of $J$-spectral factors. *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, December.

Halmos, P. R., 1961, Shifts on Hilbert spaces. *Journal of Reine Angew. Math.*, **208**, 102–112.

Hautus, M. L. J., 1980, $(A, B)$-invariant and stabilizability subspaces, a frequency domain description. *Automatica*, **16**, 703–707.

Hoffman, K., 1962, *Banach Spaces of Analytic Functions* (Englewood Cliffs: Prentice Hall).

Khargonekar, P. P., Georgiou, T. T., and Ozguler, A. B., 1983, Skew Prime Polynomial Matrices: the Polynomial Model Approach, Special Issue on Linear Control Theory, edited by R. W. Brockett and P. A. Fuhrmann. *Linear Algebra and Applications*, **50**, 403–435.

Lax, P. D., 1959, Translation invariant subspaces. *Acta Mathematica*, **101**, 163–178.

Lindquist, A., and Picci, G., 1979 a, On the stochastic realization problem. *SIAM Journal of Control and Optimization*, **17**, 365–389.

Lindquist, A., and Picci, G., 1979 b, Realization theory for multivariate stationary gaussian processes I: State space construction. *Proceedings of the 4th MTNS*, Delft, Holland, pp. 140–148.

Lindquist, A., and Picci, G., 1979 c, Realization theory for multivariate gaussian processes II: State space construction. *Proceedings of the 2nd Conference on Information Sciences and Systems*, Patras, Greece (Boston: Reidel), pp. 108–129.

Lindquist, A., and Picci, G., 1985 a, Realization theory for multivariable stationary Gaussian processes. *SIAM Journal of Control Optimization*, **23**, 809–857.

Lindquist, A., and Picci, G., 1985 b, Forward and backward semimartingale representations for stationary increment processes. *Stochastics*, **15**, 1–50.

Lindquist, A., and Picci, G., 1991, A geometric approach to modelling and estimation of linear stochastic systems. *Journal of Mathematical Systems, Estimation and Control*, **1**, 241–333.

Lindquist, A., Michaletzky, G., and Picci, G., 1995, Zeros of spectral factors, the geometry of splitting subspaces and the algebraic Riccati inequality. *SIAM Journal of Control and Optimization*.

Nikolskii, N. K., 1986, *Treatise on the Shift Operator* (Berlin: Springer-Verlag), p. 273.

Pavon, M., 1994, On the parametrization of nonsquare spectral factors. In U. Helmke, R. Mennicken and J. Saurer (eds). *Systems and Networks: Mathematical Theory and Applications*, vol. II, Mathematical Research vol. 79 (Berlin, Akademie Verlag), pp. 413–416.

Picci, G., and Pinzoni, S., 1994, Acausal models of stationary processes. *Linear Algebra Applications*, 205–206, 997–1043.

Rozanov, Yu. A., 1967, *Stationary Random Processes* (San Francisco: Holden-Day).

Ruckebusch, G., 1975, Representations markoviennes de processus Gaussiens Stationaires. Thèse de 3[e] cycle, University of Paris VI.

Ruckebusch, G., 1980 a, Théorie geometrique de la representations Markovienne. Thèse de doctorat d'etat, University of Paris VI.

Ruckebusch, G., 1980 b, Théorie geometrique de la representation Markovienne. *Annals of the Institute Henri Poincaré*, **XVI**, 225–297.

Scherer, C., 1991, The solution set of the Algebraic Riccati Equation and the Algebraic Riccati Inequality. *Linear Algebra and its Applications*, **153**, 99–122.

Schrader, C. B., and Sain, M., 1989, Research on system zeros: a survey. *International Journal of Control*, **50**, 1407–1433.

Schumacher, J. M., 1981, Dynamic feedback in finite- and infinite-dimensional linear systems. PhD thesis, Free University of Amsterdam, The Netherlands.

Shamir, T., and Fuhrmann, P. A., 1985, Minimal factorizations of rational matrix functions in terms of polynomial models. *Linear Algebra and its Applications*, **68**, 67–91.

Shayman, M. A., 1983, Geometry of the algebraic Riccati equation, Part I. *SIAM Journal of Control Optimization*, **21**, 375–394.

Willems, J. C., 1971, Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control*, **16**, 621–634.

Wimmer, H., 1979, A Jordan factorization theorem for polynomial matrices. *Proceedings of the American Mathematical Society*, **75**, 201–206.

Wonham, W. M., 1979, *Linear Multivariable Control*, 2nd edn (New York: Springer Verlag).