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ON J-SYMMETRIC RESTRICTED SHIFTS[1]

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ABSTRACT. The restricted shift operators in proper left invariant subspaces of H^2 that are J-symmetric are characterized and the signature of the corresponding operator J is determined.

The fact that left shifts in vectorial H^2 spaces restricted to their left invariant subspaces can serve as models for the most general contractions in Hilbert space is well known [7], [13]. Thus the structural analysis of such operators contributes to the general structure theory. Of course in the general case we will have to consider shifts of infinite multiplicity.

For shifts of finite multiplicity more is known [7] and they have, besides their intrinsic mathematical interest, considerable importance due to their use as models for the generators in internal descriptions of linear time invariant dynamical systems. In fact most of finite dimensional system theory can be done, in an elegant way, using operator theoretic methods with an emphasis on Hankel and shift operators. We quote a few papers [1], [5], [6], [8] that can serve as a guide to the interested reader. This paper itself has been motivated by problems of system theory where the signature of the operator J is related to the residues of the transfer function of the system.

In a Hilbert space H we consider a bounded operator J satisfying $J = J^* = J^{-1}$. This implies there exist two orthogonal projections P_+ and P_- for which $I = P_+ + P_-$, $J = P_+ - P_-$ and $P_+P_- = 0$. Thus if $H_\pm = P_\pm H$ then clearly $H_\pm = \{x \in H | Jx = \pm x\}$. A bounded operator A is called J-symmetric if $A = JA^*J$. The J-symmetric operators have been widely studied and [10], [11] are some references to the literature. Our interest will be in a very special class of J-symmetric operators.

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Let S be the right shift in H^2 [9] i.e. the operator defined by $(Sf)(z) = zf(z)$. Let K be a proper left invariant subspace of H^2, that is, a subspace of H^2 invariant under S^*. By a theorem of Beurling [9], $K^\perp = \phi H^2$ for some inner function ϕ. Let $P_{\{\phi H^2\}^\perp}$ be the orthogonal projection of H^2 onto $K = \{\phi H^2\}^\perp$. We define an operator T_ϕ in $\{\phi H^2\}^\perp$ by $T_\phi f = P_{\{\phi H^2\}^\perp} Sf$ for all $f \in \{\phi H^2\}^\perp$ and then we have $T_\phi^* = S^* | \{\phi H^2\}^\perp$. Since T_ϕ is completely determined by the inner function ϕ we want to get the relation between the analytic properties of ϕ and the J-symmetry of T_ϕ. Inner functions are determined only up to a constant factor of modulus one, thus we will normalize the inner functions by requiring their first nonvanishing Taylor coefficient to be positive. For every $a \in H^\infty$ we let $\tilde{a}(z) = \overline{a(\bar{z})}$. Thus $a = \tilde{a}$ if and only if all coefficients in a power series expansion of a at zero are real.

For every inner function ϕ we define τ_ϕ on $\{\phi H^2\}^\perp$ by

$$(\tau_\phi f)(e^{it}) = e^{-it}\tilde{\phi}(e^{it}) f(e^{-it}). \tag{1}$$

τ_ϕ is a unitary map of $\{\phi H^2\}^\perp$ onto $\{\tilde{\phi} H^2\}^\perp$ for which $T_{\tilde{\phi}}\tau_\phi = \tau_\phi T_\phi^*$ [4]. Clearly $\tau_\phi^{-1} = \tau_\phi^* = \tau_{\tilde{\phi}}$. Thus if $\phi = \tilde{\phi}$ then τ_ϕ is a J operator in $\{\phi H^2\}^\perp$.

Theorem 1. *T is a J-symmetric operator if and only if $\phi = \tilde{\phi}$. The corresponding J is given by $J = \pm \tau_\phi$.*

Before proceeding with the proof of the theorem we establish

Lemma 1. *The only unitary operators in $\{\phi H^2\}^\perp$ commuting with T_ϕ are multiplications by constants of modulus one.*

Proof. One part is trivial. So let us assume V is unitary in $\{\phi H^2\}^\perp$ and $T_\phi V = VT_\phi$. By Sarason's commutant theorem [12] we have $V = v(T)$ for some $v \in H^\infty$ satisfying $\|v\|_\infty = 1$. Since V is unitary we have $|v(e^{it})| = 1$, i.e. either v is inner ot a constant of modulus one. Assume v is inner, then $v^n f \perp \phi H^2$ for all $f \in \{\phi H^2\}^\perp$ and all $n \geq 0$. Now $f \in \{\phi H^2\}^\perp$ if and only if on the unit circle f has the factorization $f(e^{it}) = e^{-it}\phi(e^{it})\overline{h(e^{it})}$ for some $h \in H^2$ [3]. Thus our assumption implies $\bar{z} v^n \bar{h} \perp H^2$ for all $n \geq 0$. This implies $h \in \bigcap_{n=0}^\infty v^n H^2 = \{0\}$. Thus $h = 0$ and hence $f = 0$. Since f is arbitrary and ϕ nontrivial, this is impossible. Thus v is a constant of modulus one.

Proof of Theorem 1. If $\phi = \tilde{\phi}$ it follows from the remark following (1) that T_ϕ is J-symmetric with respect to τ_ϕ. Conversely let us assume T_ϕ

to be J-symmetric with respect to an arbitrary J. Hence $T_\phi^* = JT_\phi J$. This implies that the minimal inner functions [13] of T_ϕ and T_ϕ^* are the same, therefore $\phi = \tilde{\phi}$. This in turn implies, as above, that $T_\phi = \tau_\phi T_\phi^* \tau_\phi$ and hence $T_\phi = \tau_\phi(JT_\phi J)\tau_\phi = (\tau_\phi J)T_\phi(J\tau_\phi)$ or $T_\phi(J\tau_\phi) = (J\tau_\phi)T_\phi$. Thus $J\tau_\phi$, which is unitary, commutes with T_ϕ and hence, by Lemma 1, $J\tau_\phi = \alpha$ with α a constant of modulus one. Multiplying on the left by J we have $\tau_\phi = \alpha J$ and as τ_ϕ and J are selfadjoint it follows that α is real. Hence $\alpha = \pm 1$.

Next we want to find the dimensions of $K_+ = \{f \in K | \tau_\phi f = f\}$ and $K_- = \{f \in K | \tau_\phi f = -f\}$. Let us denote by k_0 and K_0 the functions defined by $k_0 = P_{\{\phi H^2\}^\perp} 1$ and $K_0 = P_{\{\phi H^2\}^\perp} \bar{z}\phi$ [2]. Since

$$T_\phi^n k_0 = P_{\{\phi H^2\}^\perp} z^n k_0 = P_{\{\phi H^2\}^\perp} z^n,$$

we have the set of vectors $\{T^n k_0 | n \geq 0\}$ spanning $\{\phi H^2\}^\perp$ or equivalently k_0 is a cyclic vector for T_ϕ. Now $\tau_\phi k_0 = \tilde{K}_0$ and $\tau_\phi K_0 = \tilde{k}_0$ which implies that $\tau_\phi T_\phi^{*n} K_0 = T_{\tilde{\phi}}^n \tilde{k}_0$ and hence, by the fact that τ_ϕ is unitary, K_0 is a cyclic vector for T_ϕ^*, or $\{T_\phi^{*n} K_0 | n \geq 0\}$ also span $\{\phi H^2\}^\perp$.

Theorem 2. *Let ϕ be inner satisfying $\phi = \tilde{\phi}$.*

a. $K_+ = \text{Clos}\{\alpha(T)k_0 + \alpha(T^*)K_0 |\ \alpha \in H^\infty\}$,
$K_- = \text{Clos}\{\alpha(T)k_0 - \alpha(T^*)K_0 | \alpha \in H^\infty\}$.

b. K_+ *is finite dimensional if and only if ϕ is a finite Blaschke product. The same holds for* K_-.

c. *If ϕ is a finite Blaschke product of n factors then* $\dim K_+ = [(n+1)/2]$ *and* $\dim K_- = [n/2]$.

Proof. a. Since $\phi = \tilde{\phi}$ we have $\tau_\phi T_\phi^* = T_\phi \tau_\phi$ and in this case $\tau_\phi k_0 = K_0$ and $\tau_\phi K_0 = k_0$. Thus

$$\tau_\phi\{\alpha(T)k_0 + (T^*)K_0\} = \{\alpha(T^*)K_0 + \alpha(T)k_0\}$$

and this implies that for all f in $K_+ = \text{Clos}\{\alpha(T)k_0 + \alpha(T^*)K_0 | \alpha \in H^\infty\}$ we have $\tau_\phi f = f$. Similarly $\tau_\phi g = -g$ for all g in $K_- = \text{Clos}\{\alpha(T)k_0 - \alpha(T^*)K_0 | \alpha \in H^\infty\}$. Clearly those subspaces are orthogonal for if $f \in K_+$ and $g \in K_-$ then

$$(f, g) = (\tau_\phi f, g) = (f, \tau_\phi g) = (f, -g) = -(f, g)$$

and hence $(f, g) = 0$. The direct sum of these two subspaces contains all vectors of the form $T^n k_0$, $n \geq 0$, and hence is all of $\{\phi H^2\}^\perp$.

b. $\{\phi H^2\}^{\perp}$ is finite dimensional if and only if ϕ is a Blaschke product with a finite number of factors. Thus in that case K_+ and K_- are finite dimensional. Assume now K_- is finite dimensional, in this case only a finite number of functions of the form $T^k k_0 - T^{*k} K_0$ are linearly independent. Thus for some integer n and, not all zero, α_j we have $\Sigma_{j=1}^n \alpha_j (T^j k_0 - T^{*j} K_0) = 0$. The functions $T^k k_0$ and $T^{*j} K_0$ are easily expressible in terms of the inner function ϕ and in general we have

$$(T^j k_0)(z) = z^j - \phi(z) \sum_{k=0}^{j} \bar{\phi}_k z^{j-k}$$

and

$$(T^{*j} K_0)(z) = z^{-(j+1)} \left[\phi(z) - \sum_{k=0}^{j} \phi_k z^k \right].$$

Thus if $\Sigma_{j=0}^n \alpha_j (T^j k_0 - T^{*j} K_0) = 0$ it follows, dropping bars as $\phi = \tilde{\phi}$, that

$$\sum_{j=0}^{n} \alpha_j z^j - \phi(z) \sum_{j=0}^{n} \alpha_j \sum_{k=0}^{j} \phi_k z^{j-k}$$

$$= z^{-(n+1)} \phi(z) \sum_{j=0}^{n} \alpha_j z^{h-j} - z^{-(n+1)} \sum_{j=0}^{n} \alpha_j z^{n-j} \sum_{k=0}^{j} \phi_k z^k.$$

Now

$$\sum_{j=0}^{n} \alpha_j \sum_{k=0}^{j} \phi_k z^{j-k} = \sum_{j=0}^{n} \left[\sum_{k=j}^{n} \alpha_k \phi_{k-j} \right] z^j$$

and

$$\sum_{j=0}^{n} \alpha_j z^{n-j} \sum_{k=0}^{j} \phi_k z^k = \sum_{j=0}^{n} \left[\sum_{k=0}^{j} \alpha_k \phi_{n+k-j} \right] z^j.$$

Therefore $p(z) = \phi(z) q(z)$ with p and q being $(2n+1)$-degree polynomials given by

$$p(z) = \sum_{j=0}^{n} \alpha_j z^{n+j+1} + \sum_{j=0}^{n} \left[\sum_{k=j}^{n} \alpha_k \phi_{k-j} \right] z^j$$

and

$$q(z) = \sum_{j=0}^{n} \alpha_j z^{n-j} + \sum_{j=0}^{n} \left[\sum_{k=j}^{n} \alpha_k \phi_{k-j} \right] z^{j+n+1}.$$

It is clear from the above that if p_i and q_i are the coefficients of p and q respectively then $p_i = q_{2n+1-i}$. It follows that ϕ is a Blaschke product of at most $2n+1$ factors. The same result follows from the assumption $\dim K_+ = n$.

c. Let us assume now that $\dim K = n$, i.e. ϕ is a Blaschke product of n factors. Let the zeroes of ϕ, repeated according to multiplicities, be λ_i, $i = 1, \dots, n$. Thus $\phi(z) = \Pi_{i=1}^n (z - \lambda_i)/(1 - \bar{\lambda}_i z)$. Let $m(z) = \Pi_{i=1}^n (z - \lambda_i)$. Thus m is the minimal polynomial of T_ϕ, and as $m = \tilde{m}$, also of T_ϕ^*. Clearly we have $\phi(e^{it}) = m(e^{it})/e^{int}m(e^{-it})$. Let $\tau = \tau_{z^n}$ be defined in $\{z^n H^2\}^\perp$ by (1). We will show that τ_ϕ and τ are similar and hence their signatures are the same. To this end we define a map Φ on $\{\phi H^2\}^\perp$ by

$$(\Phi f)(e^{it}) = e^{int} m(e^{-it}) f(e^{it}). \tag{2}$$

Let $p(e^{it}) = e^{int}m(e^{-it})$; then p is a polynomial of degree n and hence $\Phi f = pf$ and $pf \in H^2$ for all $f \in \{\phi H^2\}^\perp$. Now m being the minimal polynomial of T_ϕ^* implies $\overline{m} f \perp H^2$ for all $f \in \{\phi H^2\}^\perp$. Thus $z^n \overline{m} f = pf$ is a polynomial of degree $\leq n-1$. So Φ maps $\{\phi H^2\}^\perp$ into $\{z^n H^2\}^\perp$. Since $p \neq 0$ Φ is 1-1 and hence an invertible map of $\{\phi H^2\}^\perp$ onto $\{z^n H^2\}^\perp$. Now

$$\begin{aligned}(\Phi \tau_\phi f)(e^{it}) &= p(e^{it})(\tau_\phi f)(e^{it}) = p(e^{it})\phi(e^{it})f(e^{-it}) \\ &= e^{-it}e^{int}m(e^{-it})\phi(e^{it})f(e^{-it}) = e^{-it}m(e^{it})f(e^{-it}) \\ &= e^{-it}\cdot e^{int}\cdot e^{-int}m(e^{it})f(e^{-it}) = (\tau\Phi f)(e^{it}).\end{aligned}$$

Hence $\Phi\tau_\phi = \tau\Phi$ and this means that τ_ϕ and τ are similar. In particular $\tau_\phi f = f$ if and only if $\tau\Phi f = \Phi f$ and $\tau_\phi f = -f$ if and only if $\tau(\Phi f) = -(\Phi f)$. Thus the signatures of τ and τ_ϕ are the same. In terms of the natural orthonormal basis of $\{z^n H^2\}^\perp$ consisting of the functions $1, z, \dots, z^{n-1}$, τ has a matrix representation given by $T = (t_{ij})$, $t_{ij} = 0$ if $i + j \neq n$ and 1 if $i + j = n$. Clearly $\operatorname{tr}(T) = 0$ if n is even and 1 if it is odd, hence the result.

Corollary. *Let ϕ be inner. T_ϕ defined by* (1) *in* $\{\phi H^2\}^\perp$ *is selfadjoint if and only if* $\phi(z) = \alpha(z - \lambda)(1 - \lambda z)^{-1}$ *for some real* λ, $|\lambda| < 1$, *and* α *of modulus one.*

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